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HIGHER ORBITAL INTEGRALS, SHALIKA GERMS, AND THE HOCHSCHILD HOMOLOGY OF HECKE ALGEBRAS

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ABSTRACT. We give a detailed calculation of the Hochschild and cyclic homology of the algebra $\mathscr{C}_c^\infty(G)$ of locally constant, compactly supported functions on a reductive p-adic group G. We use these calculations to extend to arbitrary elements the definition of the higher orbital integrals introduced by Blanc and Brylinski (1992) for regular semi-simple elements. Then we extend to higher orbital integrals some results of Shalika (1972). We also investigate the effect of the "induction morphism" on Hochschild homology.

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1. Introduction. Orbital integrals play a central role in the harmonic analysis of reductive p-adic groups; they are, for instance, one of the main ingredients in the Arthur-Selberg trace formula. An orbital integral on a unimodular group G is an important particular case of an invariant distribution on G. Invariant distributions have been used in [3] to prove the irreducibility of certain induced representations of GL_n over a p-adic field.

Let G be a locally compact, totally disconnected topological group. We denote by $\mathscr{C}_c^\infty(G)$ the space of compactly supported, locally constant, and complex valued functions on G. The choice of a Haar measure on G makes $\mathscr{C}_c^\infty(G)$ an algebra with respect to the convolution product. We refer to $\mathscr{C}_c^\infty(G)$ with the convolution product as the (full) Hecke algebra of G. If G is unimodular, then any invariant distribution on G defines a trace on $\mathscr{C}_c^\infty(G)$, and conversely, any trace on $\mathscr{C}_c^\infty(G)$ is obtained in this way (this well-known fact follows also from Lemma 3.1). Since the space of traces on an algebra A identifies naturally with the first (i.e., 0th) Hochschild cohomology group of that algebra A, it is natural to ask what are all the Hochschild cohomology groups of $\mathscr{C}_c^\infty(G)$. The Hochschild cohomology and homology groups of an algebra A are denoted in this paper by $\mathrm{HH}^q(A)$ and, respectively, by $\mathrm{HH}_q(A)$. Since $\mathrm{HH}^q(\mathscr{C}_c^\infty(G))$ is the algebraic dual of $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, it is enough to determine the Hochschild homology groups of $\mathscr{C}_c^\infty(G)$.

In this paper, G is typically a p-adic group, which, we recall, means that G is the set of $\mathbb{F}$-rational points of a linear algebraic group $\mathbb{G}$ defined over a finite extension $\mathbb{F}$ of the field $\mathbb{Q}_p$ of p-adic numbers, p being a fixed prime number. The group $\mathbb{G}$ does not have to be reductive, although this is certainly the most interesting case. When we assume $\mathbb{G}$ (or G, by abuse of language) to be reductive, we state this explicitly. For us, the most important topology to consider on G is the locally compact, totally

disconnected topology induced from an embedding of $G \subset \mathrm{GL}_n(\mathbb{F})$. Nevertheless, the Zariski topology on G, which is induced from the Zariski topology of $\mathbb{G}$, also plays a role in our study.

To state the main result of this paper on the Hochschild homology of the algebra $\mathscr{C}_c^\infty(G)$, we need to introduce first the concepts of a "standard subgroup" and of a "relatively regular element" of a standard subgroup. For any group G and any subset $A \subset G$, we denote

$$C_G(A) := \{g \in G,\ ga = ag,\ \forall a \in A\}, \qquad N_G(A) := \{g \in G,\ gA = Ag\}, \tag{1.1}$$

$W_G(A) := N_G(A)/C_G(A)$, and $Z(A) := A \cap C_G(A)$. This latter notation is used only when A is a subgroup of G. The subscript G is dropped from the notation whenever the group G is understood. A commutative subgroup S of G is called standard if S is the group of semi-simple elements of the center of $C(s)$ for some semi-simple element $s \in G$. An element $s \in S$ with this property is called regular relative to S, or S-regular. The set of S-regular elements is denoted by S^{reg}.

We fix from now on a p-adic group G. Our results are stated in terms of standard subgroups of G. We denote by H_u the set of unipotent elements of a subgroup H. Sometimes, the set $C(S)_u$ is also denoted by $\mathcal{U}_S$, in order to avoid having too many parentheses in our formulae. Let $\Delta_{C(S)}$ denote the modular function of the group $C(S)$ and let

$$\mathscr{C}_c^\infty(\mathcal{U}_S)_\delta := \mathscr{C}_c^\infty(C(S)_u) \otimes \Delta_{C(S)} \tag{1.2}$$

be $\mathscr{C}_c^\infty(\mathcal{U}_S)$ as a vector space, but with the product $C(S)$-module structure, that is, $\gamma(f)(u) = \Delta_{C(S)}(\gamma) f(\gamma^{-1} u \gamma)$, for all $\gamma \in C(S)$, $f \in \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta$, and $u \in \mathcal{U}_S$.

One of the main results of this paper, namely Theorem 3.6, identifies the groups $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))$ in terms of the following data:

(1) the set Σ of conjugacy classes of standard subgroups S of G;
(2) the subsets $S^{\mathrm{reg}} \subset S$ of S-regular elements;
(3) the actions of the Weyl groups $W(S)$ on $\mathscr{C}_c^\infty(S)$; and
(4) the continuous cohomology of the $C(S)$-modules $\mathscr{C}_c^\infty(\mathcal{U}_S)_\delta$;

where S ranges through a set of representatives of Σ. More precisely, if G is a p-adic group defined over a field of characteristic zero, as before, then Theorem 3.6 states the existence of an isomorphism

$$\mathrm{HH}_q(\mathscr{C}_c^\infty(G)) \simeq \bigoplus_{S \in \Sigma} \mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta). \tag{1.3}$$

This isomorphism is obtained by identifying the E^∞-term of an implicit, convergent spectral sequence, and hence it is not natural. This isomorphism can be made natural by using a generalization of the Shalika germs. The isomorphism (1.3) is in the spirit of the results of Karoubi [13] and Burghelea [8]. See also [10].

It is important to relate the determination of the Hochschild homology in (1.3) with the periodic cyclic homology groups of $\mathscr{C}_c^\infty(G)$. Let $\mathrm{HH}_{[q]} := \oplus_{k \in \mathbb{Z}} \mathrm{HH}_{q+2k}$. Recall that an element $\gamma \in G$ is called compact, by definition, if it belongs to a compact subgroup of G. The set of compact elements of G is open and closed and is clearly G-invariant for the action of G on itself by conjugation. Also, we denote by $\mathrm{HH}_{[q]}(\mathscr{C}_c^\infty(G))_{\mathrm{comp}}$ the

localization of the homology group $\mathrm{HH}_{[q]}(\mathscr{C}_c^\infty(G))$ to the set of compact elements of G (see [5] or [18]). Then, the periodic cyclic homology of the Hecke algebra $\mathscr{C}_c^\infty(G)$ is related to its Hochschild homology by

$$HP_q(\mathscr{C}_c^\infty(G)) \simeq \mathrm{HH}_{[q]}(\mathscr{C}_c^\infty(G))_{\mathrm{comp}}. \tag{1.4}$$

This relation is implicit in [12]. Consequently, the results of this paper complement the results on the cyclic homology of p-adic groups in [12, 23]. More precisely, let S_{comp} be the set of compact elements of a standard subgroup S and let $\mathrm{H}_{[q]} := \oplus_{k\in\mathbb{Z}} \mathrm{H}_{q+2k}$, then

$$HP_q(\mathscr{C}_c^\infty(G)) \simeq \bigoplus_{S\in\Sigma} \mathscr{C}_c^\infty(S_{\mathrm{comp}}^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_{[q]}(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta). \tag{1.5}$$

It is interesting to remark that $HP_*(\mathscr{C}_c^\infty(G))$ can also be related to the admissible dual (or spectrum) of G, see [15], and hence our results have significance for the representation theory of p-adic groups. (See also [18] for similar results on the groups of real points of algebraic groups defined over $\mathbb{R}$.) These periodic cyclic cohomology groups are seen to be isomorphic to $K_*(C_r^*(G)) \otimes \mathbb{C}$, by combining results from [1, 16] to those of [12]. See also [15].

Assume for the moment that G is reductive. Then, in order to better understand the role played by the groups $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))$ and $\mathrm{H}_*(G, \mathscr{C}_c^\infty(G_u))$ in the representation theory of G, we relate $\mathrm{H}_*(G, \mathscr{C}_c^\infty(G_u))$ to the analogous cohomology groups, $\mathrm{H}_*(P, \mathscr{C}_c^\infty(P_u)_\delta)$ and $\mathrm{H}_*(M, \mathscr{C}_c^\infty(M_u)_\delta)$, associated to parabolic subgroups P of G and to their Levi components M. In particular, we define morphisms between these Hochschild homology groups that are analogous to the induction and inflation morphisms that play such a prominent role in the representation theory of p-adic groups. These morphisms are induced by morphisms of algebras.

In [5], Blanc and Brylinski have introduced higher orbital integrals associated to regular semi-simple elements by proving first that

$$\mathrm{HH}_q(\mathscr{C}_c^\infty(G)) \simeq \mathrm{H}_q(G, \mathscr{C}_c^\infty(G)_\delta), \tag{1.6}$$

a result which they called "the MacLane isomorphism." (Actually, they did not have to twist with the modular function, because they worked only with unimodular groups G, see Lemma 3.1 for the slightly more general version needed in this paper.) Our approach also starts from the MacLane isomorphism, but after that we rely more on filtrations of the G-module $\mathscr{C}_c^\infty(G)$ than on localization. This allows us to define higher orbital integrals at arbitrary elements. Then, we study the properties of these orbital integrals and we obtain, in particular, a proof of the existence of abstract Shalika germs for the higher orbital integrals. Actually, the existence of Shalika germs turns out to be a consequence of some general homological properties of the ring $R^\infty(G)$ of (conjugacy) invariant, locally constant functions on the group G. We also use the techniques developed in [18] in the framework of real algebraic groups. It would be interesting to relate the results of this paper to those of [2] on the periodic cyclic homology of Iwahori-Hecke algebras and those of [14] on invariant distributions.

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2. Standard subgroups. Our description of the Hochschild homology of Hecke algebras is in terms of "standard" subgroups, a class of commutative groups that we define and study below. The main role of the standard subgroups is to define a stratification of G by sets invariant with respect to inner automorphisms. This section is devoted to establishing the basic facts about standard subgroups. We begin by fixing notation.

If G is a group and $A \subset G$ is a subset, we denote by $C_G(A)$ the centralizer of A, that is, the set of elements of G that commute with every element of A. When G is understood, we omit it from notation. Also, we denote by $N_G(A)$ the normalizer of A in G, that is, the set of elements $g \in G$ such that $gAg^{-1} = A$. We then set $W_G(A) := N_G(A)/C_G(A)$ and $Z(A) := A \cap C_G(A)$. By $Z = Z(G) = C_G(G)$, we denote the center of G. Again, we omit G if the group is understood.

Let $\mathbb{G}$ be a linear algebraic group defined over a totally disconnected, locally compact field $\mathbb{F}$ of characteristic zero. Thus $\mathbb{F}$ is a finite algebraic extension of $\mathbb{Q}_p$, the field of p-adic numbers. The set $\mathbb{G}(\mathbb{F})$ of $\mathbb{F}$-rational points of $\mathbb{G}$ is called a p-adic group and is denoted simply by G. It is known [6] that $G = \mathbb{G}(\mathbb{F})$ identifies with a closed subgroup of $\mathrm{GL}_n(\mathbb{F})$, and hence it has a natural locally compact topology that makes it a totally disconnected space.

Definition 2.1. A commutative subgroup $S \subset G$ is called standard if and only if, there exists a semi-simple element $s_0 \in G$ such that S is the group of semi-simple elements of the center of $C(s_0)$, the centralizer of s_0 in G. A semi-simple element $s_0 \in S$ with this property will be called regular relative to S or, simply, S-regular. The set of S-regular elements $s \in S$ is denoted by S^{reg}.

Clearly, every standard subgroup is commutative. More properties of standard subgroups are summarized in Proposition 2.2.

We denote by H_{ss} the subset of semi-simple elements of a group H.

Proposition 2.2. *Let S be a standard subgroup and $s_0 \in S^{\mathrm{reg}}$.*

(i) *The group S is the set of $\mathbb{F}$-rational points of a subgroup $\mathbb{S} \subset \mathbb{G}$ defined over the field $\mathbb{F}$.*

(ii) *We have that $C(S) = C(s_0)$, so $S = Z(C(s_0))_{ss}$ and $N(C(S)) = N(S)$.*

(iii) *Every semi-simple element $\gamma \in G$ is S-regular for one, and only one, standard subgroup S.*

(iv) *The set S^{reg} is a Zariski open subset of S.*

Proof. (i) We identify the group $\mathbb{G}$ with its set of $\bar{\mathbb{F}}$-rational points, for some algebraically closed extension of $\mathbb{F}$. Let Γ be the Galois group of $\bar{\mathbb{F}}$ over $\mathbb{F}$. Then Γ acts on $\mathbb{G}$ and G can be identified with the set of fixed points of this action because $\mathbb{F}$ is a perfect field.

Let s_0 be a semi-simple element of G. From the above identification, we easily obtain that $C_{\mathbb{G}}(s_0)$, the centralizer of s_0 in $\mathbb{G}$, is invariant with respect to Γ. From this it follows that $C_{\mathbb{G}}(s_0)$ is defined over $\mathbb{F}$ and $C(s_0) := C_G(s_0)$ is the set of $\mathbb{F}$-rational points of $C_{\mathbb{G}}(s_0)$. Let $\mathbb{S}$ be the center of $C_{\mathbb{G}}(s_0)$. Then we see, using the same reasoning, that $\mathbb{S}$ is defined over $\mathbb{F}$ and that S is the set of its $\mathbb{F}$-rational points.

(ii) We have $C(s_0) \supset C(S)$ because $s_0 \in S$. But, $S \subset Z(C(s_0))$, so $C(S) \subset C(s_0)$, too. By definition, $S = Z(C(s_0))$, so $S = Z(C(S))$. The last part follows because $N(H) \subset N(Z(H))$ and $N(H) \subset N(C(H))$, for any group H.

(iii) Let $\gamma \in G$ be a semi-simple element. Define $S(\gamma) := Z(C(\gamma))_{ss}$. Then $S(\gamma)$ is a standard subgroup, by definition, and γ is regular relative to $S(\gamma)$. Clearly, if γ is S-regular, then $S = S(\gamma)$.

(iv) Let $S \subset G$. We may assume that $\mathbb{G} \subset \mathrm{GL}_n(\bar{\mathbb{F}})$ and that $G = \mathbb{G} \cap \mathrm{GL}_n(\mathbb{F})$. The statement is obvious if $\mathbb{G} = \mathrm{GL}_n(\mathbb{F})$. In general, the result follows because $C_G(s) = C_{\mathrm{GL}_n(\mathbb{F})}(s) \cap G$. □

For any p-adic group H, we denote by H_u the set of unipotent elements of H, and call it the unipotent variety of H. In the particular case of $H = C(S)$, where $S \subset G$ is a standard subgroup, we also denote $C(S)_u = \mathcal{U}_S$.

We now define a natural Ad_G-invariant stratification of G, called the standard stratification of G.

Let $\mathfrak{g}$ be the Lie algebra of $\mathbb{G}$ in the sense of linear algebraic groups. Denote by $a_i(g)$ the coefficients of the polynomial $\det(t+1-\mathrm{Ad}_g)$,

$$\det\left(t+1-\mathrm{Ad}_g\right) = \sum_{i=0}^{m} a_i(g)t^i \in \mathbb{F}[t]. \tag{2.1}$$

Let a_r be the first nonzero coefficient a_i, and define

$$V_k = \{g \in G,\ a_r(g) = a_{r+1}(g) = \cdots = a_{r+k-1}(g) = 0\}. \tag{2.2}$$

Thus $V_0 = G$, by convention, and $G \setminus V_1 = G'$, the set of regular elements of G if G is reductive. Also, $V_{m+1} = \emptyset$ because $a_m = 1$. We observe that the functions $a_i(g)$ are G-invariant polynomial functions on G, and that they depend only on the semi-simple part of g.

In order to proceed further, recall that the Jordan decomposition of an element $g \in G$ is $g = g_s g_u$, where g_s is semi-simple, g_u is unipotent, and $g_s g_u = g_u g_s$. This decomposition is unique [6]. Let $S \subset G$ be a standard subgroup. If $g = g_s g_u$ is the Jordan decomposition of $g \in G$ and if $g_s \in S^{\mathrm{reg}}$, then $g_u \in \mathcal{U}_S := C(S)_u$, by definition, and hence $g \in S^{\mathrm{reg}}\mathcal{U}_S$.

Fix now a standard subgroup $S \subset G$, and let

$$F_S = \mathrm{Ad}_G\left(S^{\mathrm{reg}}\right), \qquad F_S^u = \mathrm{Ad}_G\left(S^{\mathrm{reg}}\mathcal{U}_S\right) \tag{2.3}$$

be the set of semi-simple elements of G conjugated to an element of S^{reg} and, respectively, the set of elements $g \in G$ conjugated to an element of $S^{\mathrm{reg}}\mathcal{U}_S$ (i.e., the set of elements $g \in S$ such whose semi-simple part is in F_S).

Also, let $N(S) := \{g \in G,\ gSg^{-1} = S\}$ be the normalizer of S and $W(S) = N(S)/C(S)$. Since $N(S)$ leaves S^{reg} invariant and is actually the normalizer of this set, it follows that the quotient $W(S)$ can be identified with a set of automorphisms of S. Since $N(S)$ is the set of $\mathbb{F}$-rational points of an algebraic group, the rigidity of tori (see [6, page 117]) shows that $W(S)$ is finite.

The natural map $(g,s) \to gsg^{-1}$ descends to a map

$$\phi_S : \frac{(G \times S^{\mathrm{reg}})}{N(S)} \ni (g,s) \longrightarrow gsg^{-1} \in F_S. \tag{2.4}$$

Similarly, we obtain a map

$$\phi_S^u : \frac{(G \times S^{\mathrm{reg}} \mathcal{U}_S)}{N(S)} \ni (g, su) \longrightarrow gsug^{-1} \in F_S^u. \tag{2.5}$$

In Proposition 2.3 consider the locally compact (and Hausdorff) topology of G, and not the Zariski topology. Recall that we denote by G_{ss} the set of semi-simple elements of G.

Proposition 2.3. *Let S be a standard subgroup of G. Using the above notation, we have*

(i) *The set F_S is an analytic submanifold of G, and the maps ϕ_S and ϕ_S^u are homeomorphisms.*

(ii) *For each k, the set $V_k \smallsetminus V_{k+1}$ is the disjoint union of the sets F_S^u that have a nonempty intersection with $V_k \smallsetminus V_{k+1}$, and each $F_S^u \subset V_k \smallsetminus V_{k+1}$ is an open subset of $V_k \smallsetminus V_{k+1}$.*

(iii) *Similarly, the set $G_{ss} \cap (V_k \smallsetminus V_{k+1})$ is a disjoint union of the sets F_S that have a nonempty intersection with $V_k \smallsetminus V_{k+1}$, and each $F_S \subset V_k \smallsetminus V_{k+1}$ is an open subset of $G_{ss} \cap (V_k \smallsetminus V_{k+1})$.*

Proof. (i) First we check that ϕ_S and ϕ_S^u are injective. Indeed, assume that $g_1 s_1 g_1^{-1} = g_2 s_2 g_2^{-1}$, for some $s_1, s_2 \in S^{\mathrm{reg}}$. Then, if $g = g_2^{-1} g_1$, we have

$$gC(s_1)g^{-1} = C(s_2) \Longrightarrow gC(S)g^{-1} = C(S) \Longrightarrow gSg^{-1} = S, \tag{2.6}$$

and hence $g \in N(S)$. Consequently, we have $(g_1, s_1) = (g_2 g, g^{-1} s_2 g) = g^{-1}(g_2, s_2)$, with $g \in N(S)$, as desired. The same argument shows that if F_S and $F_{S'}$ have a point in common, then the standard subgroups S and S' are conjugated in G.

The injectivity of ϕ_S^u follows from the injectivity of ϕ_S, indeed, if $g_1(s_1 u_1)g_1^{-1} = g_2(s_2 u_2)g_2^{-1}$, let $g = g_2^{-1} g_1$ as above, and conclude that $g s_1 g^{-1} = s_2$, by the uniqueness of the Jordan decomposition. As above, this implies that $g \in N(S)$.

Since the differential $d\phi_S$ is a linear isomorphism onto its image (i.e., it is injective) and ϕ_S is injective, it follows that ϕ_S is a local homeomorphism onto its image (for the locally compact topologies), and that its image is an analytic submanifold (see [22, Theorem 2.3, page 38]). The set $G_{ss} \cap (V_k \smallsetminus V_{k+1})$ is an algebraic variety on which G acts with orbits of the same dimension, and hence ϕ_S is proper [6]. This proves that ϕ_S is a homeomorphism. Using an inverse for ϕ_S, we obtain that ϕ_S^u is also a homeomorphism.

Now to prove (ii), consider a standard subgroup $S \subset G$, and let d be the dimension of $C(S)$. Then $a_0 = a_1 = \cdots = a_{d-1} = 0$ on S, and S^{reg} is an open component of $S \cap \{a_d \neq 0\}$. It follows that, if $s \in (V_k \smallsetminus V_{k+1}) \cap S^{\mathrm{reg}}$, then $F_S \subset V_k \smallsetminus V_{k+1}$. This shows that $G_{ss} \cap (V_k \smallsetminus V_{k+1})$ is a union of sets of the form F_S. This must then be a disjoint union because the sets F_S are either equal or disjoint, as proved above.

Now, if $g \in V_k \smallsetminus V_{k+1}$ has semi-simple part s, then $s \in F_S \subset G_{ss} \cap (V_k \smallsetminus V_{k+1})$, for some standard subgroup S, and hence $g \in F_S^u \subset V_k \smallsetminus V_{k+1}$. The sets F_S^u are open in the induced topology because the map $V_k \smallsetminus V_{k+1} \to G_{ss} \cap (V_k \smallsetminus V_{k+1})$ is continuous. See also [27]. $\square$

3. Homology of Hecke algebras. In this section, we obtain a first identification of the Hochschild homology groups of Hecke algebras of p-adic groups. To this end, we use several general results on Hochschild homology of algebras, on algebraic groups, and on the continuous cohomology of totally disconnected groups. Good references are [6, 7, 17], for the general theory, and [15] for questions related to Hochschild homology.

Let G be a p-adic group on which we fix a Haar measure dg. Consider now the space $\mathscr{C}_c^\infty(G)$ of compactly supported, locally constant functions on G. Fix a Haar measure dh on G. Then the convolution product, denoted $*$, is defined by

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)\, dh. \tag{3.1}$$

The convolution product makes $\mathscr{C}_c^\infty(G)$ an algebra, called the Hecke algebra of G. It is important in representation theory to determine the (Ad_G-)invariant linear functionals on $\mathscr{C}_c^\infty(G)$. If G is unimodular, the space of invariant linear functionals on $\mathscr{C}_c^\infty(G)$ coincides with the space of traces on $\mathscr{C}_c^\infty(G)$. The space of traces of $\mathscr{C}_c^\infty(G)$ identifies with $\mathrm{HH}^0(\mathscr{C}_c^\infty(G))$, the first Hochschild cohomology group of $\mathscr{C}_c^\infty(G)$. It is reasonable then to ask, what are all Hochschild cohomology groups of $\mathscr{C}_c^\infty(G)$? Since Hochschild cohomology is the algebraic dual of Hochschild homology, it is enough to concentrate on the latter.

We first recall the definition of the Hochschild homology groups of the algebra $\mathscr{C}_c^\infty(G)$. Let

$$\mathscr{C}_c^\infty(G^{q+1}) = \mathscr{C}_c^\infty(G) \otimes \mathscr{C}_c^\infty(G) \otimes \cdots \otimes \mathscr{C}_c^\infty(G), \tag{3.2}$$

$(q+1)$-times, be the usual (algebraic) tensor product of vector spaces. The Hochschild differential $b : \mathscr{C}_c^\infty(G^{q+2}) \to \mathscr{C}_c^\infty(G^{q+1})$ is given by

$$\begin{aligned}(bf)(g_0, g_1, \dots, g_q) = {} & \sum_{j=0}^{q} (-1)^j \int_G f(g_0, \dots, g_{j-1}, \gamma, \gamma^{-1} g_j, g_{j+1}, \dots, g_q)\, d\gamma \\ & + (-1)^{q+1} \int_G f(\gamma^{-1} g_0, g_1, \dots, g_q, \gamma)\, d\gamma.\end{aligned} \tag{3.3}$$

By definition, the qth Hochschild homology group of $\mathscr{C}_c^\infty(G)$, denoted by $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, is the qth homology group of the complex $(\mathscr{C}_c^\infty(G^{q+1}), b)$. Hochschild homology can be defined for any algebra. Our definition takes into account the particular structure of $\mathscr{C}_c^\infty(G)$, in particular, that it is an inductive limit of unital algebras, so there is no need to first adjoin a unit in order to define Hochschild homology. The computation of the groups $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$ is the main purpose of this paper.

The group G acts by conjugation on $\mathscr{C}_c^\infty(G)$, and we denote by $\mathscr{C}_c^\infty(G)_{\mathrm{ad}}$ the G-module defined by this action. Also, let Δ_G denote the modular function of G, which, we recall, is defined by the relation

$$\Delta_G(h) \int_G f(gh)\, dg = \int_G f(g)\, dg. \tag{3.4}$$

We are especially interested in the G-module $\mathscr{C}_c^\infty(G)_\delta$ obtained from $\mathscr{C}_c^\infty(G)_{\mathrm{ad}}$ by twisting it with the modular function. More precisely, let $\mathscr{C}_c^\infty(G)_\delta = \mathscr{C}_c^\infty(G)$ as vector spaces,

and let the action of G on functions be given by the formula

$$(\gamma\cdot f)(g)=\Delta_G(\gamma)f(\gamma^{-1}g\gamma),\quad f\in\mathscr{C}_c^\infty(G)_\delta. \tag{3.5}$$

The reason for this twisting is that, for G nonunimodular, the traces of $\mathscr{C}_c^\infty(G)$ are the G-invariant functionals on $\mathscr{C}_c^\infty(G)_\delta$, not on $\mathscr{C}_c^\infty(G)$ (this is an immediate consequence of Lemma 3.1). More generally, our approach to the Hochschild homology of $\mathscr{C}_c^\infty(G)$ is based on Lemma 3.1.

Before stating and proving Lemma 3.1, we need to introduce some notation. First, if M is an arbitrary G-module, we denote by $M\otimes\Delta_G$ the tensor product of the G-modules M and $\mathbb{C}$, where the action on $\mathbb{C}$ is given by the multiplication with the modular function of G. (In particular, $\mathscr{C}_c^\infty(G)_\delta=\mathscr{C}_c^\infty(G)\otimes\Delta_G$.)

If M is a right G-module and M' is a left G-module, then $M\otimes_G M'$ is defined as the quotient of $M\otimes M'$ by the submodule generated by $mg\otimes m'-m\otimes gm'$. For example, if $H\subset G$ is a closed subgroup and if X is a left H-space, then we have an isomorphism of G-spaces

$$\mathscr{C}_c^\infty(G)\otimes_H\left(\mathscr{C}_c^\infty(X)\otimes\Delta_H\right)\simeq\mathscr{C}_c^\infty(G\times_H X), \tag{3.6}$$

where $G\times X$ is the quotient $(G\times X)/H$ for the action $h(g,x)=(gh^{-1},hx)$. This isomorphism is obtained by observing that the natural map

$$\begin{aligned} t_X:\mathscr{C}_c^\infty(G)\otimes\mathscr{C}_c^\infty(X)=\mathscr{C}_c^\infty(G\times X)&\longrightarrow\mathscr{C}_c^\infty(G\times_H X),\\ t_X(f)(\overline{(g,x)})&=\int_H f(gh,h^{-1}x)\,dh, \end{aligned} \tag{3.7}$$

passes to the quotient to give the desired isomorphism. Sometimes it will be convenient to regard a left G-module as a right G-module by replacing g with g^{-1}. Equation (3.6) is one of the main reasons why we need to consider the modular function.

Also, recall that a G-module M is smooth if and only if the stabilizer of each element of M is open in G. The continuous homology groups of G with coefficients in the smooth module M, denoted $\mathrm{H}_k(G,M)$, can be defined using tensor products as follows. Let $\mathscr{B}_q(G)=\mathscr{C}_c^\infty(G^{q+1})$, $q=0,1,\dots$, be the Bar complex of the group G, with differential

$$(df)(g_0,g_1,\dots,g_q)=\sum_{j=0}^{q+1}(-1)^j\int_G f(g_0,\dots,g_{j-1},\gamma,g_j,\dots,g_q)\,d\gamma. \tag{3.8}$$

Then the complex $(\mathscr{B}_q,d)$ gives a resolution of $\mathbb{C}$ with projective $\mathscr{C}_c^\infty(G)$-modules, and the complex

$$\mathscr{B}_q(G)\otimes_G M \tag{3.9}$$

computes $\mathrm{H}_q(G,M)$. See [4, 7].

We need the following extension of a result from [5].

Lemma 3.1. *Let $\mathscr{C}_c^\infty(G)_\delta=\mathscr{C}_c^\infty(G)\otimes\Delta_G$ be the G-module obtained by twisting the adjoint action of G on $\mathscr{C}_c^\infty(G)$ by the modular function. Then we have a natural isomorphism*

$$\mathrm{HH}_q\left(\mathscr{C}_c^\infty(G)\right)\simeq\mathrm{H}_q\left(G,\mathscr{C}_c^\infty(G)_\delta\right). \tag{3.10}$$

Proof. Consider the complex (3.9), which computes the continuous cohomology of $M = \mathscr{C}_c^\infty(G)_\delta$, and let

$$h_G : \mathscr{B}_q(G) \otimes \mathscr{C}_c^\infty(G)_\delta \simeq \mathscr{C}_c^\infty(G) \otimes \mathscr{C}_c^\infty(G^{q+1}) = \mathscr{C}_c^\infty(G^{q+2}) \longrightarrow \mathscr{C}_c^\infty(G)^{\otimes q+1} \tag{3.11}$$

be the map

$$h_G(f)(g_0, g_1, \dots, g_q) = \int_G f(g^{-1}hg, g^{-1}g_0, g^{-1}g_0g_1, \dots, g^{-1}g_0g_1\cdots g_q)\, dg, \qquad h = g_0g_1\cdots g_q. \tag{3.12}$$

As in (3.6), the map h_G descends to the quotient to induce an isomorphism

$$\tilde{h}_G : \mathscr{B}_q(G) \otimes_G \mathscr{C}_c^\infty(G)_\delta \simeq \mathscr{C}_c^\infty(G^{q+2}) \otimes_G \mathbb{C} \simeq \mathscr{C}_c^\infty(G)^{\otimes q+1} \tag{3.13}$$

of complexes, that is, $\tilde{h}_G \circ (d \otimes_G 1) = b \circ \tilde{h}_G$, which establishes the isomorphism $\mathrm{H}_q(G, \mathscr{C}_c^\infty(G)_\delta) \simeq \mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, as desired. □

To better justify the twisting of the module $\mathscr{C}_c^\infty(G)$ by the modular function in Lemma 3.1, note that the trivial representation of G gives rise to an obvious morphism $\pi_0 : \mathscr{C}_c^\infty(G) \to \mathbb{C}$, by $\pi_0(f) = \int_G f(g)\, dg$, which hence defines a trace on $\mathscr{C}_c^\infty(G)$. However, π_0 is not G-invariant for the usual action of G, but is invariant if we twist the adjoint action of G by the modular function, as indicated.

We proceed now to a detailed study of the G-module $\mathscr{C}_c^\infty(G)_\delta$ using the standard stratification introduced in the previous section.

Let $R^\infty(G)$ be the ring of locally constant Ad_G-invariant functions on G with the pointwise product, which we regard as a subset of the set of endomorphisms of the G-module $\mathscr{C}_c^\infty(G)_\delta = \mathscr{C}_c^\infty(G) \otimes \Delta_G$. Let $\det(t + 1 - \mathrm{Ad}_g) = \sum_{i=0}^m a_i(g)t^i$, as before. For each $k \geq 1$, denote by $I_k \subset R^\infty(G)$ the ideal generated by functions $f : G \to \mathbb{C}$ of the form

$$f = \phi(a_r, a_{r+1}, \dots, a_{r+k-1}), \tag{3.14}$$

where ϕ is a locally constant function $\phi : \mathbb{F}^k \to \mathbb{C}$ such that $\phi(0, 0, \dots, 0) = 0$. (Recall that each of the polynomials $a_0, \dots, a_{r-1}$ is the zero polynomial.) By convention, we set $I_0 = (0)$; also, it follows that $I_{m+1} = R^\infty(G)$.

Fix now k, and let $\phi_n : \mathbb{F}^k \to \mathbb{C}$ be 1 on the set

$$\{\xi = (\xi_0, \dots, \xi_{k-1}) \in \mathbb{F}^k,\ \max|\xi_i| \geq q^{-n}\}, \tag{3.15}$$

and vanishes outside this set. (Here q is the number of elements of the residual field of $\mathbb{F}$, and the non-Archimedean norm "$|\ |$" is normalized such that its range is $\{0\} \cup \{q^n,\ n \in \mathbb{Z}\}$.) Also, let $p_n = \phi_n(a_r, a_{r+1}, \dots, a_{r+k-1}) \in I_k$. Then $p_n = p_n^2 = p_n p_{n+1}$ and $I_k = \cup p_n R^\infty(G)$.

For further reference, we state as a lemma a basic property of the constructions we have introduced.

Lemma 3.2. *If M is an $R^\infty(G)$-module, then $I_k M = \cup p_n M$.*

As a consequence of the above lemma, we obtain the following result.

Corollary 3.3. *Consider the G-modules,*

$$M_k = \left(\frac{I_{k+1}}{I_k}\right) \otimes_{R^\infty(G)} \mathscr{C}_c^\infty(G)_\delta \simeq \frac{I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta}. \tag{3.16}$$

Then, for each $q \geq 0$, we have an isomorphism

$$\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right) \simeq \bigoplus_{k=0}^{m} \mathrm{H}_q\left(G, M_k\right) \tag{3.17}$$

of vector spaces.

Proof. There exists a (not natural) isomorphism

$$\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right) \simeq \frac{\bigoplus_{k=0}^{m} I_{k+1}\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)}{I_k\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)} \tag{3.18}$$

of vector spaces.

By Lemma 3.2, the inclusion of $I_k\mathscr{C}_c^\infty(G)_\delta \to \mathscr{C}_c^\infty(G)_\delta$ of G-modules induces natural isomorphisms

$$\begin{aligned} \mathrm{H}_q\left(G, I_k\mathscr{C}_c^\infty(G)_\delta\right) &\simeq \mathrm{H}_q\left(G, \varinjlim p_n\mathscr{C}_c^\infty(G)_\delta\right) \\ &\simeq \varinjlim p_n\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right) \\ &\simeq I_k\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right), \end{aligned} \tag{3.19}$$

because the functor H_q is compatible with inductive limits and with direct sums.

The naturality of these isomorphisms and Lemma 6.1 shows that

$$\mathrm{H}_q \frac{G, I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta} \simeq \frac{I_{k+1}\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)}{I_k\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)}. \tag{3.20}$$

This is enough to complete the proof. □

If X is a totally disconnected, locally compact space X, we denote by $\mathscr{C}_c^\infty(X)$ the space of compactly supported, locally constant, complex valued functions on X. Recall that, if $U \subset X$ is an open subset of X as above, then restriction defines an isomorphism

$$\frac{\mathscr{C}_c^\infty(X)}{\mathscr{C}_c^\infty(U)} \simeq \mathscr{C}_c^\infty(X \smallsetminus U). \tag{3.21}$$

We now study the homology of the subquotients

$$M_k = \frac{I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta} \simeq \mathscr{C}_c^\infty(V_k \smallsetminus V_{k+1}) \tag{3.22}$$

by identifying them with induced modules. Let Σ_k be a set of representatives of conjugacy classes of standard subgroups S such that $F_S \subset V_k \smallsetminus V_{k+1}$ (or, equivalently, $F_S^u \subset V_k \smallsetminus V_{k+1}$).

Lemma 3.4. *Using the above notation, we have $I_k\mathscr{C}_c^\infty(G) = \mathscr{C}_c^\infty(G \smallsetminus V_k)$ and*

$$\frac{I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta} \simeq \bigoplus_{S\in\Sigma_k} \mathscr{C}_c^\infty(F_S^u). \tag{3.23}$$

Proof. It follows from the definition of I_k that, if $f \in I_k\mathscr{C}_c^\infty(G)_\delta$, then f vanishes in a neighborhood of V_k. Conversely, if f is in $\mathscr{C}_c^\infty(G \smallsetminus V_k)$, then we can find some polynomial a_i, with $i \le r+k-1$, such that $|a_i|$ is bounded from below on the support of f by, say, q^{-n}, then $p_n f = f$. The second isomorphism follows from the first isomorphism using (3.21) and Lemma 3.2. □

If $H \subset G$ is a closed subgroup and M is a smooth (left) H-module (i.e., the stabilizer of each $m \in M$ is an open subgroup of M), we denote

$$\operatorname{ind}_H^G(M) = \mathscr{C}_c^\infty(G) \otimes_H M = \frac{\mathscr{C}_c^\infty(G) \otimes M}{\{fh \otimes m - f \otimes hm,\ h \in H\}}, \tag{3.24}$$

where the right H-module structure on $\mathscr{C}_c^\infty(G)$ is $(fh)(g) = f(gh^{-1})$. Then Shapiro's lemma, see [9], states that

$$\mathrm{H}_k\left(G, \operatorname{ind}_H^G(M) \otimes \Delta_G\right) \simeq \mathrm{H}_k(H, M). \tag{3.25}$$

(A proof of Shapiro's lemma for nonunimodular groups is contained in the proof of Theorem 6.2.)

The basic examples of induced modules are obtained from H-spaces. If X is an H-space (we agree that H acts on X from the left), then

$$\mathscr{C}_c^\infty\left(\frac{G \times X}{H}\right) \simeq \operatorname{ind}_H^G\left(\mathscr{C}_c^\infty(X) \otimes \Delta_H\right) \simeq \operatorname{ind}_H^G\left(\mathscr{C}_c^\infty(X)_\delta\right) \tag{3.26}$$

as G-modules, where H acts on $G \times X$ by $h(g,x) = (gh^{-1}, hx)$. For example, Proposition 2.3 identifies $\mathscr{C}_c^\infty(F_S^u)$ with an induced module:

$$\mathscr{C}_c^\infty(F_S^u) \simeq \operatorname{ind}_{N(S)}^G\left(\mathscr{C}_c^\infty(S^{\mathrm{reg}}\mathcal{U}_S) \otimes \Delta_{N(S)}\right) = \operatorname{ind}_{N(S)}^G\left(\mathscr{C}_c^\infty(S^{\mathrm{reg}}\mathcal{U}_S)_\delta\right). \tag{3.27}$$

Shapiro's lemma is an easy consequence of the Serre-Hochschild spectral sequence, see [9], which states the following. Let M be a smooth G-module and $H \subset G$ be a normal subgroup. Then the action of G on $\mathrm{H}_q(H,M)$ descends to an action of G/H, and there exists a spectral sequence with $E^2_{p,q} = \mathrm{H}_p(G/H, \mathrm{H}_q(H,M))$, convergent to $\mathrm{H}_{p+q}(G,M)$.

Let $M_k = I_{k+1}\mathscr{C}_c^\infty(G)_\delta / I_k\mathscr{C}_c^\infty(G)_\delta$, as before.

Proposition 3.5. *Using the above notation, we have*

$$\mathrm{H}_q(G, M_k) \simeq \bigoplus_{\langle S\rangle \in \Sigma_k} \mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_q\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right), \tag{3.28}$$

a natural isomorphism of $R^\infty(G)$-modules.

Proof. Let S be a standard subgroup of G. Recall first that $W(S) = N(S)/C(S)$ is a finite group that acts freely on S^{reg}, which gives an $N(S)$-equivariant isomorphism

$$\mathscr{C}_c^\infty(S^{\mathrm{reg}}\mathcal{U}_S)_\delta = \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta \otimes \mathscr{C}_c^\infty(S^{\mathrm{reg}}). \tag{3.29}$$

Let M be a smooth $N(S)$-module. The Hochschild-Serre spectral sequence applied to the module M and the normal subgroup $C(S) \subset N(S)$ gives natural isomorphisms

$$\mathrm{H}_q(N(S), M) \simeq \mathrm{H}_0\left(W(S), \mathrm{H}_q(C(S), M)\right) \simeq \mathrm{H}_q(C(S), M)^{W(S)}. \tag{3.30}$$

Combining these two isomorphisms, we obtain

$$\begin{aligned}\mathrm{H}_k\left(G,\mathscr{C}_c^\infty\left(F_S^u\right)_\delta\right) &\simeq \mathrm{H}_k\left(G,\mathrm{ind}_{N(S)}^G\left(\mathscr{C}_c^\infty\left(S^{\mathrm{reg}}\mathcal{U}_S\right)_\delta\right)_\delta\right)\\ &\simeq \mathrm{H}_k\left(N(S),\mathscr{C}_c^\infty\left(S^{\mathrm{reg}}\mathcal{U}_S\right)_\delta\right)\\ &\simeq \left(\mathrm{H}_k\left(C(S),\mathscr{C}_c^\infty\left(\mathcal{U}_S\right)_\delta\right)\otimes\mathscr{C}_c^\infty\left(S^{\mathrm{reg}}\right)\right)^{W(S)}\\ &\simeq \mathscr{C}_c^\infty\left(S^{\mathrm{reg}}\right)^{W(S)}\otimes\mathrm{H}_k\left(C(S),\mathscr{C}_c^\infty\left(\mathcal{U}_S\right)_\delta\right).\end{aligned} \tag{3.31}$$

The result then follows from Lemma 3.4, which implies directly that

$$M_k \simeq \oplus_{S\in\Sigma_k}\mathscr{C}_c^\infty\left(F_S^u\right)_\delta. \tag{3.32}$$

The proof is now complete. □

Combining Proposition 3.5 with Corollary 3.3, we obtain the main result of this section. Recall that a p-adic group $G=\mathbb{G}(\mathbb{F})$ is the set of $\mathbb{F}$-rational points of a linear algebraic group $\mathbb{G}$ defined over a non-Archimedean, nondiscrete, locally compact field $\mathbb{F}$ of characteristic zero. Also, recall that $\mathcal{U}_S$ is the set of unipotent elements commuting with the standard subgroup S, and that the action of $C(S)$ on $\mathscr{C}_c^\infty(\mathcal{U}_S)$ is twisted by the modular function of $C(S)$, yielding the module $\mathscr{C}_c^\infty(\mathcal{U}_S)_\delta=\mathscr{C}_c^\infty(\mathcal{U}_S)\otimes\Delta_{C(S)}$.

Theorem 3.6. *Let G be a p-adic group. Let Σ be a set of representative of conjugacy classes of standard subgroups of $S\subset G$ and $W(S)=N(S)/C(S)$, then we have an isomorphism*

$$\mathrm{HH}_q\left(\mathscr{C}_c^\infty(G)\right)\simeq\bigoplus_{S\in\Sigma}\mathscr{C}_c^\infty\left(S^{\mathrm{reg}}\right)^{W(S)}\otimes\mathrm{H}_q\left(C(S),\mathscr{C}_c^\infty\left(\mathcal{U}_S\right)_\delta\right). \tag{3.33}$$

Remark 3.7. The isomorphism of Theorem 3.6 is not natural. A natural description of $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$ will be obtained in one of the following sections by considering higher orbital integrals and their Shalika germs.

4. Higher orbital integrals and their Shalika germs. Proposition 3.5 allows us to determine the structure of the localized cohomology groups $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))_{\mathfrak{m}}$, where $\mathfrak{m}$ is a maximal ideal of $R^\infty(G)$. This will lead to an extension of the higher orbital integrals introduced by Blanc and Brylinski in [5] and to a generalization of some results of Shalika [24] to higher orbital integrals, all discussed in this section. In this way, we also obtain a new, more natural description of the groups $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$.

First recall the following result.

Proposition 4.1. *Let G be a reductive p-adic group over a field of characteristic 0, and let $S\subset G$ be a standard subgroup, and $\gamma\in S^{\mathrm{reg}}$ (i.e., γ is a semi-simple element such that $C(S)=C(\gamma)$). Then there exists an $N(S)$-invariant closed and open neighborhood U of γ in $C(S)$ such that*

$$G\times U\ni(g,h)\longrightarrow ghg^{-1}\in G \tag{4.1}$$

defines a homeomorphism of $G\times_{N(S)}U:=(G\times U)/N(S)$ onto a G-invariant, closed, and open subset $V\subset G$ containing γ.

Proof. The result follows from Luna's lemma. For p-adic groups, Luna's lemma is proved in [19, page 109, Properties "C" and "D"]. □

For any maximal ideal $\mathfrak{m} \subset R^\infty(G)$ and any $R^\infty(G)$-module M, we denote by $M_\mathfrak{m}$ the localization of M at $\mathfrak{m}$, that is, $M_\mathfrak{m} \simeq S^{-1}M$, where S is the multiplicative subset $R^\infty(G) \setminus \mathfrak{m}$.

From Proposition 4.1 we obtain the following consequences for the ring $R^\infty(G)$.

Corollary 4.2. *Let $\gamma \in S$, U, and V be as in Proposition 4.1.*

(i) *The ring $R^\infty(G)$ decomposes as the direct sum $\mathscr{C}^\infty(V)^G \oplus \mathscr{C}^\infty(V^c)^G$, and $\mathscr{C}^\infty(V)^G \simeq \mathscr{C}^\infty(U)^{N(S)} \subset \mathscr{C}^\infty(C(S))^{N(S)} = R^\infty(C(S))^{W(S)}$. (Here V^c is the complement of V in G.)*

(ii) *For any two semi-simple elements $\gamma, \gamma' \in G$, if $\phi(\gamma) = \phi(\gamma')$ for all functions $\phi \in R^\infty(G)$, then γ and γ' are conjugated in G.*

(iii) *Let $\mathfrak{m} \in R^\infty(G)$ be the maximal ideal consisting of functions that vanish at a semi-simple element $\gamma \in G$. Then $\mathfrak{p}$ is generated by an increasing sequence of projections, and $M_\mathfrak{m} \simeq M/\mathfrak{m}M$, for any $R^\infty(G)$-module M.*

Proof. (i) is an immediate consequence of Proposition 4.1.

(ii) follows from [19, Proposition 2.5].

To prove (iii), observe that the maximal ideal $\mathfrak{m}$ is generated by an increasing sequence of projections p_n, that is, $\mathfrak{m} = \cup p_n R^\infty(G)$, with $p_n^2 = p_n$ and $p_{n+1}p_n = p_n$.

We know from Proposition 2.5 of [19] that $R^\infty(G)$ is isomorphic to $C^\infty(X)$, for some locally compact, totally disconnected topological space X. Moreover, if M is a $C^\infty(X)$-module and $\mathfrak{m}$ is the maximal ideal of functions vanishing at x_0, for some fixed point $x_0 \in X$, then $\mathscr{C}^\infty(X)_\mathfrak{m} \simeq \mathscr{C}^\infty(X)/\mathfrak{m}\mathscr{C}^\infty(X)$, and hence

$$M_\mathfrak{m} = M \otimes_{\mathscr{C}^\infty(X)} \mathscr{C}^\infty(X)_\mathfrak{m} \simeq \frac{M \otimes_{\mathscr{C}^\infty(X)} \mathscr{C}^\infty(X)}{\mathfrak{m}\mathscr{C}^\infty(X)} \simeq \frac{M}{\mathfrak{m}M}. \tag{4.2}$$

Since X is metrizable, we can choose a basis V_n of compact open neighborhoods of x_0 in X. If we let p_n to be the characteristic function of V_n^c, then p_n are projections generating $\mathfrak{m}$. By choosing V_n to be decreasing, we obtain an increasing sequence of projections p_n. □

We now consider for each maximal ideal $\mathfrak{m} \subset R^\infty = R^\infty(G)$ the localization $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))_\mathfrak{m}$.

Proposition 4.3. *Let $\mathfrak{m}$ be a maximal ideal of $R^\infty(G)$. If $\mathfrak{m}$ consists of the functions that vanish at the semi-simple element $\gamma \in G$ and $S \subset G$ is a standard subgroup such that $\gamma \in S^{\mathrm{reg}}$, then*

$$\mathrm{HH}_q\left(\mathscr{C}_c^\infty(G)\right)_\mathfrak{m} \simeq \mathrm{H}_q\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right). \tag{4.3}$$

For all other maximal ideals $\mathfrak{m} \subset R^\infty(G)$, we have $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))_\mathfrak{m} = 0$.

Note that $\mathrm{H}_q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)) \simeq \mathrm{H}_q(C(\gamma), \mathscr{C}_c^\infty(C(\gamma)_u))$.

Proof. Let $\mathfrak{m}_\gamma := \{f \in R^\infty(G),\ f(\gamma) = 0\}$. The vanishing of $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))_\mathfrak{m}$ in the last part of Proposition 4.4 because $\mathscr{C}_c^\infty(G)_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ that are not of the form $\mathfrak{m}_\gamma$, for some semi-simple element $\gamma \in G$.

Assume now that $\mathfrak{m} = \mathfrak{m}_\gamma$. The localization functor $V \to V_\mathfrak{m}$ is exact by standard homological algebra. Let $(0) = I_0 \subset I_1 \subset \cdots \subset I_m \subset I_{m+1} = R^\infty(G)$ be the sequence of

ideals introduced shortly after Lemma 3.1. The sequence of ideals $(I_k)_{\mathfrak{m}}$ is an increasing sequence satisfying $(I_0)_{\mathfrak{m}} \simeq 0$ and $(I_{m+1})_{\mathfrak{m}} \simeq \mathbb{C}$. Choose k such that $(I_k)_{\mathfrak{m}} \simeq 0$ and $(I_{k+1})_{\mathfrak{m}} \simeq \mathbb{C}$. (This happens if and only if $\gamma \in V_k \smallsetminus V_{k+1}$.) It follows that

$$\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)_{\mathfrak{m}} \simeq \mathrm{H}_q\left(\frac{G, I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta}\right)_{\mathfrak{m}}. \tag{4.4}$$

Since all the isomorphisms of Proposition 3.5 are compatible with the localization functor, we obtain that

$$\begin{aligned}\mathrm{HH}_q\left(\mathscr{C}_c^\infty(G)\right)_{\mathfrak{m}} &\simeq \mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)_{\mathfrak{m}} \\ &\simeq \bigoplus_{\langle S\rangle \in \Sigma_k} \mathrm{H}_q\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right) \otimes \left(\frac{\mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)}}{\mathfrak{m}\mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)}}\right).\end{aligned} \tag{4.5}$$

The only quotient $\mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)}/\mathfrak{m}\mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)}$ that does not vanish is the one containing (a conjugate of) γ, and then it is isomorphic to $\mathbb{C}$. This completes the proof. □

An alternative proof of Proposition 4.3 can be obtained by writing

$$\mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_\delta\right)_{\mathfrak{m}} \simeq \mathrm{H}_q\left(G, \mathscr{C}_c^\infty(G)_{\delta\mathfrak{m}}\right) \simeq \mathrm{H}_q\left(\frac{G, \mathscr{C}_c^\infty(G)_\delta}{\mathfrak{m}\mathscr{C}_c^\infty(G)_\delta}\right), \tag{4.6}$$

and then observing that $\mathscr{C}_c^\infty(G)_\delta/\mathfrak{m}\mathscr{C}_c^\infty(G)_\delta \simeq \mathscr{C}_c^\infty(\mathcal{U}_S)$, by Corollary 4.2(iii). However our first proof is more convenient when dealing with orbital integrals. See also [18], which was first circulated in 1990 as a preprint of the Mathematical Institute of the Romanian Academy (INCREST) No. 18, March 1990, and where the localization techniques were first introduced.

We now extend the definition of higher orbital integrals introduced by Blanc and Brylinski [5] to cover nonregular semi-simple elements also. Fix a standard subgroup $S \subset G$, and let k be such that $S^{\mathrm{reg}} \subset V_k \smallsetminus V_{k+1}$. As in the above proof, Proposition 3.5 gives a natural $R^\infty(G)$-linear, degree preserving, surjective morphism

$$\mathrm{H}_*\left(\frac{G, I_{k+1}\mathscr{C}_c^\infty(G)_\delta}{I_k\mathscr{C}_c^\infty(G)_\delta}\right) \longrightarrow \mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right), \tag{4.7}$$

and hence a linear map

$$I_{k+1}\mathrm{HH}_*\left(\mathscr{C}_c^\infty(G)\right) = \mathrm{H}_*\left(G, I_{k+1}\mathscr{C}_c^\infty(G)_\delta\right) \longrightarrow \mathscr{C}_c^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right). \tag{4.8}$$

Fix $c \in \mathrm{H}^q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta)$ and $\gamma \in S^{\mathrm{reg}}$, and let

$$\mathcal{O}_{\gamma,c} = \mathcal{O}_{\gamma,c}^S : I_{k+1}\mathrm{HH}_q\left(\mathscr{C}_c^\infty(G)\right) \longrightarrow \mathbb{C} \tag{4.9}$$

be the evaluation of the map at γ and c in (4.8). We obtain, in particular, that for any $f \in I_{k+1}\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, the function $\gamma \to \mathcal{O}_{\gamma,c}(f)$ is a locally constant, compactly supported function on S^{reg}. The function $\mathcal{O}_{\gamma,c}$ can then be extended to the whole group $\mathrm{HH}_q(\mathscr{C}_c^\infty(G))$ using a simple observation as follows. By Lemma 3.2, we know that for

any $\gamma \in S^{\mathrm{reg}}$ there exists a locally constant function $\phi \in I_{k+1}$ such that $\phi(\gamma) = 1$. Then let

$$\mathbb{O}_{\gamma,c}(f) := \mathbb{O}_{\gamma,c}(\phi f), \tag{4.10}$$

which is independent of ϕ. It follows from the definition of $\mathbb{O}_{\gamma,c}$ that, for any $f \in \mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, the function $\gamma \to \mathbb{O}_{\gamma,c}(f)$ is a locally constant function on S^{reg}, but not necessarily compactly supported. We thus obtain the following result.

Proposition 4.4. *Let $S \subset G$ be a standard subgroup. Then there exists a degree-preserving, $R^\infty(G)$-linear map*

$$\mathbb{O}^S : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathscr{C}^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_*(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta), \tag{4.11}$$

which is an isomorphism when localized at any maximal ideal $\mathfrak{m} = \mathfrak{m}_\gamma \subset R^\infty(G)$, consisting of functions vanishing at some element $\gamma \in S^{\mathrm{reg}}$.

We call the maps $\mathbb{O}^S$ and $\mathbb{O}_{\gamma,c} = \mathbb{O}^S_{\gamma,c}$ "higher orbital integrals" because they generalize the usual notion of orbital integral. (If c is a cocycle of dimension q, we call $\mathbb{O}_{\gamma,c}$ an order q higher orbital integral.) Indeed, assume that G and $C(S)$ are unimodular. Let $c_0 = 1 \in \mathrm{H}^0(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta)$ be the evaluation at the identity element $e \in G$, and let $f \in \mathscr{C}_c^\infty(G) = \mathrm{HH}_0(\mathscr{C}_c^\infty(G))$. Then

$$\mathbb{O}_{\gamma,c_0}(f) = \mathbb{O}_{\gamma,1}(f) = \int_{G/C(S)} f(g\gamma g^{-1})\, d\bar{g}, \tag{4.12}$$

where $d\bar{g}$ is the induced measure on $G/C(S)$.

If $\gamma \in G$ is a semi-simple element and S is a standard subgroup of G such that $C(\gamma) = C(S)$, (i.e., $\gamma \in S^{\mathrm{reg}}$), then restriction at γ defines a map

$$\mathbb{O}_\gamma = \mathbb{O}^S_\gamma : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathrm{H}_*(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta) \tag{4.13}$$

such that $c(\mathbb{O}_\gamma(f)) = \mathbb{O}_{\gamma,c}(f)$, for all $c \in \mathrm{H}^q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta)$. The localization of $\mathbb{O}_\gamma$ at γ yields the isomorphism of Proposition 4.4.

A word on notation, whenever we write $\mathbb{O}^S_{\gamma,c}$ or $\mathbb{O}^S_\gamma$, we assume that $\gamma \in S^{\mathrm{reg}}$, which actually determines S. This means that we can omit S from notation. However, if we want to write that $\mathbb{O}_{\gamma,c} = \mathbb{O}^S_{\gamma,c}$ is obtained by evaluating

$$\mathbb{O}^S : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathscr{C}^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_*(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta) \tag{4.14}$$

to a point $\gamma \in S^{\mathrm{reg}}$ and then by pairing with c, that is,

$$\mathbb{O}^S_{\gamma,c}(f) = \langle \mathbb{O}^S(f)(\gamma), c \rangle, \tag{4.15}$$

then it is obviously better to include S in the notation.

Let $\gamma \in G$ be a semi-simple element. We want now to investigate the behavior orbital integrals $\mathbb{O}_{g,c}$ with g in a small neighborhood of γ. Fix a standard subgroup $S \subset G$ such that γ is in the closure of $\mathrm{Ad}_G(S^{\mathrm{reg}})$, but is not in $\mathrm{Ad}_G(S^{\mathrm{reg}})$, and a class $c \in \mathrm{H}^q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S))$. More precisely, we want to study the germ of the function $g \to \mathbb{O}_{g,c}(f)$ at an element γ, where $f \in \mathrm{HH}_q(\mathscr{C}_c^\infty(G))$ is arbitrary. The germ of a function h at γ will be denoted by h_γ.

The following theorem extends one of the basic properties of Shalika germs from usual orbital integrals to higher orbital integrals.

Theorem 4.5. *Let $S \in G$ be a standard subgroup and let $\gamma \in S$ be an element in the closure of S^{reg}, such that $\gamma \notin S^{\mathrm{reg}}$. Then there exists a degree preserving linear map*

$$\sigma_\gamma^S : \mathrm{H}_*\left(C(\gamma), \mathscr{C}_c^\infty(C(\gamma)_u)_\delta\right) \longrightarrow \mathscr{C}^\infty(S^{\mathrm{reg}})_\gamma^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right), \tag{4.16}$$

such that

$$\mathbb{O}^S(f)_\gamma = \sigma_\gamma^S(\mathbb{O}_\gamma(f)), \tag{4.17}$$

for all $f \in \mathrm{HH}_*(\mathscr{C}_c^\infty(G))$.

Note that, in the notation for the maps σ_γ^S, the standard subgroup S is no longer determined by γ.

Proof. By the definition of the localization of a module, the map

$$\mathbb{O}^S : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(G)\right) \longrightarrow \mathscr{C}^\infty(S^{\mathrm{reg}})_\gamma^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right) \tag{4.18}$$

factors through a map

$$F : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(G)\right)_\gamma \longrightarrow \mathscr{C}^\infty(S^{\mathrm{reg}})_\gamma^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta\right). \tag{4.19}$$

Since $\mathbb{O}_\gamma : \mathrm{HH}_*(\mathscr{C}_c^\infty(G))_\gamma \to \mathrm{H}_*(C(\gamma), \mathscr{C}_c^\infty(C(\gamma)_u)_\delta)$ is an isomorphism, by Proposition 4.3, we may define

$$\sigma_\gamma^S = F \circ \mathbb{O}_\gamma^{-1}, \tag{4.20}$$

and all desired properties for σ_γ^S will be satisfied. □

Let $\gamma \in S \setminus S^{\mathrm{reg}}$ be such that γ is in the closure of S^{reg}, as above, and also let $c \in \mathrm{H}^q(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)_\delta)$. Then a consequence of Theorem 4.5 is that the germ at γ of the higher orbital integrals $\mathbb{O}_{g,c}^S$ depends only on $\mathbb{O}_\gamma$. More precisely, if $g \in S^{\mathrm{reg}}$, $f \in \mathrm{HH}_q(\mathscr{C}_c^\infty(G))$, and we regard $\mathbb{O}_{g,c}^S(f)$ as a function of g, then its germ at γ, denoted $\mathbb{O}_{g,c}^S(f)_\gamma$, is given by

$$\mathbb{O}_{g,c}^S(f)_\gamma = \langle \sigma_\gamma^S(\mathbb{O}_\gamma(f)), c\rangle. \tag{4.21}$$

This observation allows us to relate Theorem 4.5 with results of Shalika [24] and Vigneras [27]. So assume now that G is reductive and let $\xi_i \in \mathrm{H}_0(C(\gamma), \mathscr{C}_c^\infty(C(\gamma)_u)_\delta)$ be the basis dual to the basis of $\mathrm{H}^0(C(\gamma), \mathscr{C}_c^\infty(C(\gamma)_u)_\delta)$ given by the orbital integrals over the orbits of γu, for u nilpotent in $C(\gamma)$. If we let $F_i^S = \sigma_\gamma^S(\xi_i)$, then we recover the usual definition of Shalika germs. Due to this fact, we call the maps σ_γ^S, introduced in Theorem 4.5, the higher Shalika germs.

We can now characterize the range of the higher orbital integrals. Combining all higher orbital integrals for $S \subset G$ ranging through a set Σ of representatives of standard subgroups of G, we obtain a map

$$\mathbb{O} : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(G)\right) \longrightarrow \bigoplus_{S \in \Sigma} \mathscr{C}^\infty(S^{\mathrm{reg}})^{W(S)} \otimes \mathrm{H}_*\left(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S)\right). \tag{4.22}$$

Theorem 4.6. *Let Σ be a set of representatives of standard subgroups of G and σ_γ^S be the maps introduced in Theorem 4.5 for $\gamma \in \bar{S}^{\mathrm{reg}} \setminus S^{\mathrm{reg}}$. Also, let*

$$\mathcal{F} \subset \bigoplus_{S\in\Sigma} \mathcal{C}^\infty(S^{\mathrm{reg}}) \otimes \mathrm{H}_*\left(C(S), \mathcal{C}_c^\infty(\mathcal{U}_S)_\delta\right) \tag{4.23}$$

be the space of sections ξ satisfying $\xi_\gamma = \sigma_\gamma^S(\xi(\gamma))$ for all standard subgroups S and all $\gamma \in \bar{S}^{\mathrm{reg}} \setminus S^{\mathrm{reg}}$. Then $\mathbb{O}$ establishes an $R^\infty(G)$-linear isomorphism

$$\mathbb{O} : \mathrm{HH}_*\left(\mathcal{C}_c^\infty(G)\right) \longrightarrow \mathcal{F}. \tag{4.24}$$

Proof. Note first that the map $\mathbb{O}$ is well defined, that is, its range is contained in $\mathcal{F}$, by Theorem 4.5.

To prove that $\mathbb{O}$ is an isomorphism, filter both $\mathrm{HH}_*(\mathcal{C}_c^\infty(G))$ and $\mathcal{F}$ by the subgroups $I_k \mathrm{HH}_*(\mathcal{C}_c^\infty(G))$ and, respectively, by $I_k\mathcal{F}$, using the ideals I_k introduced in Section 3. Since $\mathbb{O}$ is $R^\infty(G)$-linear, it preserves this filtration and induces maps

$$\frac{I_{k+1}\mathrm{HH}_*\left(\mathcal{C}_c^\infty(G)\right)}{I_k\mathrm{HH}_*\left(\mathcal{C}_c^\infty(G)\right)} \longrightarrow \frac{I_{k+1}\mathcal{F}}{I_k\mathcal{F}}. \tag{4.25}$$

These maps are, by construction, exactly the isomorphisms of Proposition 3.5. Standard homological algebra then implies that $\mathbb{O}$ itself is an isomorphism, as desired. □

A consequence of the above result is the following "density" corollary.

Corollary 4.7. *Let $a \in \mathrm{HH}_q(\mathcal{C}_c^\infty(G))$. If all order q, higher orbital integrals of a vanish, then $a = 0$.*

We also need certain specific cocycles below. Let τ_0 be the trace $\tau_0(f) = f(e)$ on $\mathcal{C}_c^\infty(G)$, G unimodular, obtained by evaluating f at the identity e of G. Let G_0 be the kernel of all characters of G that are equal to 1 on all compact subgroups of G. Then $G/G_0 \simeq \mathbb{Z}^r$, where r is the rank of a split component of G. Let $p_j : G \to \mathbb{Z}$ be the morphisms obtained by considering the jth component of $\mathbb{Z}^r$. Then

$$\delta_j(f)(g) = p_j(g)f(g) \tag{4.26}$$

defines a derivation of $\mathcal{C}_c^\infty(G)$. Moreover, we can identify $\mathrm{H}^*(G)$ with $\Lambda^*\mathbb{C}^r$, the exterior algebra with generators $\delta_1,\dots,\delta_r$. Fix $c \in \mathrm{H}^*(G)$. It is enough to assume that $c = \delta_1 \wedge \cdots \wedge \delta_q$, and then we define the map $D_c : \mathcal{C}_c^\infty(G)^{\otimes q+1} \to \mathcal{C}_c^\infty(G)$ by the formula

$$D_c(f_0,\dots,f_q) = (q!)^{-1} \sum_{\sigma\in S_q} \epsilon(\sigma) f_0 \delta_{\sigma(1)}(f_1)\delta_{\sigma(2)}(f_2)\cdots\delta_{\sigma(q)}(f_q). \tag{4.27}$$

Then $\tau_c = \tau_0 \circ D_c(f_0,\dots,f_q)$ defines a Hochschild q cocycle on $\mathcal{C}_c^\infty(G)$, and

$$\tau_c = \mathbb{O}_{e,c}, \tag{4.28}$$

if we naturally identify c with an element of the cohomology group $\mathrm{H}^*(G, \mathcal{C}_c^\infty(G_u))$.

5. The cohomology of the unipotent variety. It follows from the main result of Section 3, Theorem 3.6, that in order to obtain a more precise description of the Hochschild homology of $\mathscr{C}_c^\infty(G)$, we need to understand the continuous cohomology of the H-module $\mathscr{C}_c^\infty(H_u)_\delta$, where H ranges through the set of centralizers of standard subgroups of G and H_u is the variety of unipotent elements in H. We call the variety H_u the unipotent variety of H, as it is usually customary. Since the cohomology groups $\mathrm{H}_k(H,\mathscr{C}_c^\infty(H_u)_\delta)$ depend only on H (i.e., they do not depend on G), it is enough to consider the case $H = G$. Motivated by this, in this section, we gather some results on the groups $\mathrm{H}_q(G,\mathscr{C}_c^\infty(G_u)_\delta)$.

We first need to recall the computation of the groups $\mathrm{H}_*(G) = \mathrm{H}_*(G,\mathbb{C})$, see for example [7]. More generally, we also need to compute $\mathrm{H}_*(G,\mathbb{C}_\chi)$, where $\chi : G \to \mathbb{C}^*$ is a character of G and $\mathbb{C}_\chi = \mathbb{C}$ as a vector space, but with G-action given by the character χ.

Assume first that $G = S$ is a commutative p-adic group, and let S_0 be the union of all compact-open subgroups of S. Then S_0 is a subgroup of S and S/S_0 is a free Abelian subgroup, whose rank we denote by $\mathrm{rk}(S)$. For this group, we then have

$$\mathrm{H}_k(S) \simeq \mathrm{H}_k\left(\frac{S}{S_0},\mathbb{C}\right) \simeq \Lambda^k\mathbb{C}^{\mathrm{rk}(S)}. \tag{5.1}$$

Moreover, $\mathrm{H}_k(S,\mathbb{C}_\chi) = 0$ if $\chi : S \to \mathbb{C}^*$ is a nontrivial character of S.

For an arbitrary p-adic group G, we may identify the cohomology groups $\mathrm{H}_q(G)$ with those of a commutative p-adic group. Indeed, if G^0 is the connected component of G (in the sense of algebraic groups) then G/G^0 is finite, and hence $\mathrm{H}_q(G) \simeq \mathrm{H}_q(G^0)$, by the Hochschild-Serre spectral sequence. This tells us that we may assume G to be connected as an algebraic group. Choose then a Levi decomposition $G = MN$, where N is the unipotent radical of G, M is a reductive subgroup, uniquely determined up to conjugation, and the product MN is a semi-direct product. Since $\mathrm{H}_q(N) = 0$ for $q > 0$, it follows that $\mathrm{H}_q(G) \simeq \mathrm{H}_q(M)$ by another application of the Serre-Hochschild spectral sequence.

Let $M_1 \subset M$ be the commutator subgroup of M, which is also a p-adic group, see [6]. The cohomology groups $\mathrm{H}_q(M_1)$ were computed in [5, Proposition 6.1, page 316], or [7] and they also vanish for $q > 0$ (recall that the crucial idea of this proof is that the fundamental domain of the building of M_1 is a simplex). All in all, we obtain that

$$\mathrm{H}_q(G) \simeq \mathrm{H}_q(M) \simeq \mathrm{H}_q\left(M^{\mathrm{ab}}\right), \tag{5.2}$$

where $M^{\mathrm{ab}} = M/M_1$ is the abelianization of M.

We summarize the above discussion in the following well-known statement.

Lemma 5.1. *Let G be a p-adic group, not necessarily reductive, and let r be the rank of a split component of the reductive quotient of G. Then*

$$\mathrm{H}_q(G) = \mathrm{H}_q(G,\mathbb{C}) \simeq \Lambda^q\mathbb{C}^r. \tag{5.3}$$

Moreover, $\mathrm{H}_q(G,\mathbb{C}_\chi) = 0$, if χ is a nontrivial character of G.

We continue by discussing first a few elementary properties of $\mathrm{H}_k(G,\mathscr{C}_c^\infty(G_u)_\delta)$.

Remark 5.2. If $G_1 \to G$ is a surjective morphism with finite kernel F, then there exists a natural homeomorphism $G_{1u} \simeq G_u$ of the unipotent varieties of the two groups. Since the kernel F acts trivially on G_{1u}, using the Hochschild-Serre spectral sequence we obtain an isomorphism

$$\mathrm{H}_k\left(G_1, \mathscr{C}_c^\infty(G_{1u})_\delta\right) \simeq \mathrm{H}_k\left(G_1, \mathscr{C}_c^\infty(G_u)\right) \simeq \mathrm{H}_k\left(G, \mathscr{C}_c^\infty(G_u)\right). \tag{5.4}$$

Remark 5.3. If $G \subset G_1$ is a normal p-adic subgroup with $F \simeq G_1/G$ finite, then we again have a natural homeomorphism $G_{1u} = G_u$. This gives

$$\mathrm{H}_k\left(G_1, \mathscr{C}_c^\infty(G_{1u})_\delta\right) \simeq \mathrm{H}_k\left(G_1, \mathscr{C}_c^\infty(G_u)_\delta\right) \simeq \mathrm{H}_k\left(G, \mathscr{C}_c^\infty(G_u)_\delta\right)^F, \tag{5.5}$$

using once again the Hochschild-Serre spectral sequence. In particular, if the characteristic morphism $F \to \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is trivial, then we get a natural isomorphism $\mathrm{H}_k(G, \mathscr{C}_c^\infty(G_u)_\delta) \simeq \mathrm{H}_k(G_1, \mathscr{C}_c^\infty(G_{1u})_\delta)$.

Remark 5.4. If $G = G' \times G''$, then $G_u = G'_u \times G''_u$ naturally, and hence $\mathscr{C}_c^\infty(G_u) \simeq \mathscr{C}_c^\infty(G'_u) \otimes \mathscr{C}_c^\infty(G''_u)$. This gives

$$\mathrm{H}_k\left(G, \mathscr{C}_c^\infty(G_u)_\delta\right) \simeq \bigoplus_{i+j=k} \mathrm{H}_i\left(G', \mathscr{C}_c^\infty(G'_u)\right) \otimes \mathrm{H}_j\left(G'', \mathscr{C}_c^\infty(G''_u)\right). \tag{5.6}$$

Remark 5.5. If Z is a commutative p-adic group of split rank r, then

$$\mathrm{H}_k\left(Z, \mathscr{C}_c^\infty(Z_u)_\delta\right) \simeq \mathscr{C}_c^\infty(Z_u) \otimes \Lambda^k \mathbb{C}^r. \tag{5.7}$$

Remark 5.6. The above isomorphisms reduce the computation of $\mathrm{H}_k(G, \mathscr{C}_c^\infty(G_u)_\delta)$ for G reductive, to the computation of the cohomology groups corresponding to its semi-simple quotient $H := G/Z(G)$:

$$\mathrm{H}_k\left(G, \mathscr{C}_c^\infty(G_u)_\delta\right) = \mathrm{H}_k\left(G, \mathscr{C}_c^\infty(G_u)\right) \simeq \bigoplus_{i+j=k} \mathrm{H}_i\left(H, \mathscr{C}_c^\infty(H_u)_\delta\right) \otimes \Lambda^j \mathbb{C}^r, \tag{5.8}$$

where r is the rank of a split component of G. Let τ_0 be the trace obtained by evaluating at the identity. Using τ_0, we obtain an injection $H^j(G) \ni c \to \tau_0 \otimes c \in H^j(G, \mathscr{C}_c^\infty(G_u))$.

In order to obtain more precise results on $\mathrm{H}_*(G, \mathscr{C}_c^\infty(G_u)_\delta)$, we need to take a closer look at the structure of $\mathscr{C}_c^\infty(G_u)$ as a G-module. For a G-space X, we denote by $\langle X \rangle$ the quotient space X/G with the induced topology, which may be non-Hausdorff. Thus $\langle G_u \rangle$ is the set of unipotent conjugacy classes of G.

Assume now that $\langle G_u \rangle$ is a finite set. (This happens, for example, if G is reductive, because the ground field $\mathbb{F}$ has characteristic zero.) Then the space G_u can be written as an increasing union of open G-invariant sets $U_l \subset G_u$, $U_{-1} = \varnothing$, such that each difference set $U_l \setminus U_{l-1}$ is a disjoint union of open and closed G-orbits,

$$U_l \setminus U_{l-1} = \cup X_{l,j}. \tag{5.9}$$

A filtration U_l with these properties will be called "nice." There may be several nice filtrations of G_u.

A nice filtration of G_u, as above, gives rise, by standard arguments, to a spectral sequence converging to $\mathrm{H}_k(G,\mathscr{C}_c^\infty(G_u)_\delta)$, as follows. First, let $\langle g\rangle \in \langle G_u\rangle$ be the orbit through an element $g \in G_u$. Also, let $C(g)$ denote the centralizer of $g \in G_u$ and r_g denote the rank of a split component of $C(g)$ if $C(g)$ is unimodular, $r_g = 0$ otherwise. This definition of r_g is such that $\mathrm{H}_k(C(g),\Delta_{C(g)}) \simeq \Lambda^k\mathbb{C}^{r_g}$.

Proposition 5.7. *Let G be a p-adic group with finitely many unipotent orbits (i.e., $\langle G_u\rangle$ is finite). Then, for any nice filtration (U_l) of G_u by open G-invariant subsets, there exists a natural spectral sequence with*

$$E^2_{p,q} = \bigoplus_{\langle u\rangle\in\langle U_p\smallsetminus U_{p-1}\rangle} \Lambda^{p+q}\mathbb{C}^{r_u}, \tag{5.10}$$

convergent to $\mathrm{H}_{p+q}(G,\mathscr{C}_c^\infty(G_u)_\delta)$.

Proof. The argument is standard and goes as follows. Recall first that any filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_N = \mathscr{C}_c^\infty(G_u)_\delta$ by G-submodules gives rise to a spectral sequence with $E^1_{p,q} = \mathrm{H}_{p+q}(G,F_p/F_{p-1})$, convergent to $\mathrm{H}_{p+q}(G,\mathscr{C}_c^\infty(G_u)_\delta)$.

Now, associated to the open sets U_l of a nice filtration, there exists an increasing filtration $F_l = \mathscr{C}_c^\infty(U_l)_\delta \subset \mathscr{C}_c^\infty(G_u)_\delta$ by G-submodules such that

$$\frac{\mathscr{C}_c^\infty(U_l)_\delta}{\mathscr{C}_c^\infty(U_{l-1})_\delta} \simeq \bigoplus_j \mathscr{C}_c^\infty(X_{l,j})_\delta, \tag{5.11}$$

where each $X_{l,j}$ is the orbit of a unipotent element (because $U_l\smallsetminus U_{l-1}$ has the topology given by the disjoint union of the orbits $X_{l,j}$). Fix l and j, and let u be a unipotent element in $X_{l,j}$ (so that then $X_{l,j}$ is the orbit through u), which implies that $\mathscr{C}_c^\infty(X_{l,j}) \simeq \mathrm{ind}_{C(u)}^G(\Delta_{C(u)})$. Finally, from Shapiro's lemma we obtain that

$$\mathrm{H}_k\left(G,\mathscr{C}_c^\infty(X_{l,j})_\delta\right) \simeq \mathrm{H}_k\left(C(u),\Delta_{C(u)}\right) \simeq \Lambda^k\mathbb{C}^{r_u}, \tag{5.12}$$

and this completes the proof. □

We expect this spectral sequence to converge for G reductive. This is the case, for example, for $G = \mathrm{GL}_n(\mathbb{F})$ and for $\mathrm{SL}_n(\mathbb{F})$. See Section 7. The convergence of the spectral sequence implies, in particular, the convergence of the orbital integrals of unipotent elements in reductive groups (which is a well-known fact due to Deligne and Rao [20]). In general, the convergence of the spectral sequence of Proposition 5.7 can be interpreted as the convergence of "higher orbital integrals."

6. Induction and the unipotent variety. We assume from now on in this section that G is reductive. We fix a parabolic subgroup $P \subset G$, $P \neq G$, and a Levi subgroup $M \subset P$, so that $P = MN$, where N is the unipotent radical of P, and the product is a semi-direct product. In this section, we relate the groups $\mathrm{H}_*(G,\mathscr{C}_c^\infty(G_u)_\delta)$ to the groups, $\mathrm{H}_*(P,\mathscr{C}_c^\infty(P_u)_\delta)$ and $\mathrm{H}_*(M,\mathscr{C}_c^\infty(M_u)_\delta)$. Since P is nonunimodular, this justifies the consideration of such groups in the previous sections.

Let K be a "good" maximal compact subgroup of G (see [11, Theorem 5]), so that $G = KP$. This decomposition shows that the map

$$K\times P \ni (k,p) \longrightarrow kpk^{-1} \in G \tag{6.1}$$

is proper, and hence the map $G\times_P P := (G\times P)/P \ni (g,p) \to gpg^{-1} \in G$ is also proper. This gives a map

$$\mathscr{C}_c^\infty(G)_\delta = \mathscr{C}_c^\infty(G) \longrightarrow \mathscr{C}_c^\infty(G\times_P P) \simeq \operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)\otimes\Delta_P) = \operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta) \tag{6.2}$$

of G-modules. This map of G-modules and the standard identification of Hochschild homology with continuous cohomology, equation (3.10), then give a morphism

$$\operatorname{ind}_P^G : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathrm{HH}_*(\mathscr{C}_c^\infty(P)), \tag{6.3}$$

defined as the composition of the following sequence of morphisms:

$$\begin{aligned}\mathrm{HH}_*(\mathscr{C}_c^\infty(G)) &\simeq \mathrm{H}_*(G,\mathscr{C}_c^\infty(G)_\delta) \longrightarrow \mathrm{H}_*(G,\operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta)\otimes\Delta_G)\\ &\simeq \mathrm{H}_k(P,\mathscr{C}_c^\infty(P)_\delta) \simeq \mathrm{HH}_*(\mathscr{C}_c^\infty(P))\end{aligned} \tag{6.4}$$

of Hochschild homology groups. The main result of this section states that ind_P^G is induced by a morphism of algebras, which we now proceed to define.

Let dk be the normalized Haar measure on the maximal compact subgroup K, normalized such that K has volume 1. The composition of kernels

$$T_1T_2(k_1,k_2) = \int_{G/P} T_1(k_1,k)T_2(k,k_2)\,dk \tag{6.5}$$

defines on $\mathscr{C}^\infty(K\times K)$ an algebra structure. Let

$$\phi_P^G : \mathscr{C}_c^\infty(G) \longrightarrow \mathscr{C}^\infty(K\times K)\otimes\mathscr{C}_c^\infty(P) \tag{6.6}$$

be defined by $\phi_P^G(f)(k_1,k_2,p) = f(k_1pk_2^{-1})$.

Recall [11] that the push-forward of the product $dp\,dk$ of Haar measure on $P\times K$, via the multiplication map $P\times K\ni(p,k)\to pk\in G$, is a left invariant measure on G, and hence a multiple λdg of the Haar measure dg on G. Suppose that the measure dk of K is the restriction of dg to K, and has total mass 1. Then the Haar measures on G and P are called compatible if $\lambda = 1$. We need the following result of Harish Chandra (implicitly stated in [25]).

Lemma 6.1. *Suppose the Haar measures on G and P are compatible. Then the linear map ϕ_P^G, defined in (6.6), is a morphism of algebras.*

Proof. The product on $\mathscr{C}^\infty(K\times K)\otimes\mathscr{C}_c^\infty(P) = \mathscr{C}_c^\infty(K\times K\times P)$ is given by the formula

$$(h_1h_2)(k_1,k_2,p) = \int_K\int_P h_1(k_1,k,q)h_2(k,k_2,q^{-1}p)\,dq\,dk. \tag{6.7}$$

Let $*$ denote the multiplication (i.e., convolution product) on $\mathscr{C}_c^\infty(G)$. Thus, we need to prove that

$$f_1*f_2(k_1pk_2^{-1}) = \int_K\int_P f_1(k_1qk^{-1})f_2(kq^{-1}pk_2^{-1})\,dq\,dk, \tag{6.8}$$

for all $f_1,f_2\in\mathscr{C}_c^\infty(G)$. Consider the map $P\times K\ni(q,k)\to g := qk^{-1}\in G$, and let $d\mu$ be the push-forward of the measure $dq\,dk$. Then the right-hand side of (6.8) becomes

$$\int_K\int_P f_1(k_1qk^{-1})f_2(kq^{-1}pk_2)\,dq\,dk = \int_G f_1(k_1g)f_2(g^{-1}pk_2^{-1})\,d\mu(g). \tag{6.9}$$

We know that $d\mu = dg$, by assumptions (see the discussion before the statement of this lemma), and then

$$\begin{aligned}\int_G f_1(k_1 g) f_2(g^{-1} p k_2^{-1})\, d\mu(g) &= \int_G f_1(g) f_2(g^{-1} k_1 p k_2^{-1})\, d\mu(g)\\ &= f_1 * f_2(k_1 p k_2^{-1}),\end{aligned} \tag{6.10}$$

by the invariance of the Haar measure. The lemma is proved. □

The trace $\mathscr{C}^\infty(K\times K) \to \mathbb{C}$ induces an isomorphism

$$\tau : \mathrm{HH}_*\left(\mathscr{C}^\infty(K\times K)\otimes \mathscr{C}_c^\infty(P)\right) \simeq \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right). \tag{6.11}$$

Explicitly, this isomorphism is given at the level of chains by

$$\begin{aligned}&\tau(f_0\otimes f_1\otimes\cdots\otimes f_q)(p_0,p_1,\dots,p_q)\\ &\quad := \int_{K^{q+1}} f_0(k_0,k_1,p_0) f_1(k_1,k_2,p_1)\cdots f_q(k_q,k_0,p_q)\, dk_0\cdots dk_q.\end{aligned} \tag{6.12}$$

This isomorphism combines with ϕ_P^G to give a morphism

$$(\phi_P^G)_* : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(G)\right) \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right). \tag{6.13}$$

Theorem 6.2. *Let P be a parabolic subgroup of a reductive p-adic group G. Consider the morphisms $(\phi_P^G)_*$ and $\mathrm{ind}_P^G : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \to \mathrm{HH}_*(\mathscr{C}_c^\infty(P))$, defined above (equations (6.3) and (6.13)). Then $(\phi_P^G)_* = \mathrm{ind}_P^G$.*

Proof. Let M_1 and M_2 be two left G-modules. We can regard M_1 as a right module, and then the tensor product $M_1\otimes_G M_2$ is the quotient of $M_1\otimes M_2$ by the group generated by the elements $gm_1\otimes gm_2 - m_1\otimes m_2$, as before. Alternatively, we can think of $M_1\otimes_G M_2$ as $(M_1\otimes M_2)\otimes_G \mathbb{C}$. This justifies the notation $f\otimes_G 1$ for a morphism $M_1\otimes_G M_2 \to M_1'\otimes_G M_2'$ induced by a morphism

$$f = f_1\otimes f_2 : M_1\otimes M_2 \longrightarrow M_1'\otimes M_2'. \tag{6.14}$$

We prove the theorem by an explicit computation. To this end, we use the results and notation (h_G and $\tilde{h}_G = h_G\otimes_G 1$) of Lemma 3.1.

By a direct computation using (6.12), we see that the morphism

$$\tau\circ\phi_P^G : \mathscr{C}_c^\infty(G)^{\otimes q+1} \longrightarrow \mathscr{C}_c^\infty(P)^{\otimes q+1} \tag{6.15}$$

between Hochschild complexes, is given by the formula

$$\begin{aligned}&\tau\circ\phi_P^G(f)(p_0,p_1,\dots,p_q)\\ &\quad = \int_{K^{q+1}} f(k_0 p_0 k_1^{-1}, k_1 p_1 k_2^{-1},\dots,k_q p_q k_0^{-1})\, dk_0\, dk_1\cdots dk_q.\end{aligned} \tag{6.16}$$

We now want to realize the map $\mathrm{ind}_P^G : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \to \mathrm{HH}_*(\mathscr{C}_c^\infty(P))$, at the level of complexes. In the process, it is convenient to identify the smooth G-module $\mathscr{C}_c^\infty((G\times P)/P) \simeq \mathrm{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta)$ with a subspace of the space of functions on $G\times P$, using the projection $G\times P \to (G\times P)/P$.

Consider the G-morphism

$$l : \mathcal{B}_q(G) \otimes \mathcal{C}_c^\infty(G) \longrightarrow \mathcal{B}_q(G) \otimes \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right) \tag{6.17}$$

induced by the morphism

$$\mathcal{C}_c^\infty(G) \longrightarrow \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right) \subset \mathcal{C}^\infty(G \times P). \tag{6.18}$$

Explicitly,

$$l(f)(g_0, g_1, \dots, g_q, g, p) = f(g_0, g_1, \dots, g_q, gpg^{-1}). \tag{6.19}$$

Then the resulting morphism

$$l \otimes_G 1 : \mathrm{H}_q\left(G, \mathcal{C}_c^\infty(G)_\delta\right) = \mathrm{H}_q\left(G, \mathcal{C}_c^\infty(G)_\delta\right) \longrightarrow \mathrm{H}_q\left(G, \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right)\right) \tag{6.20}$$

is the morphism $\mathrm{H}_q(G, \mathcal{C}_c^\infty(G)_\delta) \to \mathrm{H}_q(G, \operatorname{ind}_P^G(\mathcal{C}_c^\infty(P)_\delta))$ on homology corresponding to the G-morphism $\mathcal{C}_c^\infty(G) \to \operatorname{ind}_P^G(\mathcal{C}_c^\infty(P)_\delta)$.

The G-morphism

$$\begin{aligned} r : \mathcal{B}_q(G) \otimes \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right) &\longrightarrow \operatorname{ind}_P^G\left(\mathcal{B}_q(P) \otimes \mathcal{C}_c^\infty(P)_\delta\right) \\ &= \mathcal{C}_c^\infty(G) \otimes_P \left(\mathcal{B}_q(P) \otimes \mathcal{C}_c^\infty(P)_\delta\right) \subset \mathcal{C}^\infty\left(G \times P^{q+2}\right) \end{aligned} \tag{6.21}$$

given by the formula

$$r(f)(g, p_0, p_1, \dots, p_q, p) = \int_{K^{q+1}} f(gp_0k_1^{-1}, gp_1k_2^{-1}, \dots, gp_nk_0^{-1}, g, p)\, dk, \tag{6.22}$$

($dk = dk_0 \cdots dk_q$) is well defined and commutes with the differentials of the two complexes. Moreover, it induces an isomorphism in homology, because the only nonzero homology groups are in dimension 0, and they are both isomorphic to $\operatorname{ind}_P^G(\mathcal{C}_c^\infty(P)_\delta)$. We have an isomorphism

$$\chi : \operatorname{ind}_P^G\left(\mathcal{B}_q(P) \otimes \mathcal{C}_c^\infty(P)_\delta\right) \otimes_G \mathbb{C} \longrightarrow \left(\mathcal{B}_q(P) \otimes \mathcal{C}_c^\infty(P)_\delta\right) \otimes_P \mathbb{C} \tag{6.23}$$

of complexes. This shows that the homology of the complex $\operatorname{ind}_P^G(\mathcal{C}_c^\infty(P)_\delta) \otimes_G \mathbb{C}$ is isomorphic to $\mathrm{H}_q(P, \mathcal{C}_c^\infty(P)_\delta)$, and that the map induced on homology, that is,

$$\chi(r \otimes_G 1) : \mathrm{H}_q\left(G, \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right)\right) \longrightarrow \mathrm{H}_q\left(P, \mathcal{C}_c^\infty(P)_\delta\right) \tag{6.24}$$

is also an isomorphism (the Shapiro isomorphism).

Below, where convenient, we drop the composition sign $\circ$, for example, we write rl instead of $r \circ l$.

Recall now that the isomorphism $\mathrm{H}_q(G, \mathcal{C}_c^\infty(G)_\delta) \simeq \mathrm{HH}_q(\mathcal{C}_c^\infty(G))$ is induced by the morphism of complexes $\tilde{h}_q$ defined in Lemma 3.1, equation (3.10). From the definition of the morphism $\operatorname{ind}_P^G : \mathrm{HH}_*(\mathcal{C}_c^\infty(G)) \to \mathrm{HH}_*(\mathcal{C}_c^\infty(P))$ and the above discussion, we obtain the equality of the morphisms $\mathrm{H}_q(G, \mathcal{C}_c^\infty(G)_\delta) \to \mathrm{H}_q(P, \mathcal{C}_c^\infty(P)_\delta)$ induced by $\chi \circ (rl \otimes_G 1)$ and $\tilde{h}_P^{-1} \circ \operatorname{ind}_P^G \circ \tilde{h}_G$. Thus, in order to complete the proof, it would be enough to check that $\tilde{h}_P \circ \chi \circ (rl \otimes_G 1) = \tau \circ \phi_P^G \circ \tilde{h}_G$ at the level of complexes. Let

$$t : \mathcal{B}_q(G) \otimes \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right) \longrightarrow \mathcal{B}_q(G) \otimes_G \operatorname{ind}_P^G\left(\mathcal{C}_c^\infty(P)_\delta\right) \tag{6.25}$$

be the natural projection. Since the map h_G is surjective, it is also enough to check that $\tilde{h}_P \circ \chi \circ (r \otimes_G 1)tl = \tau \circ \phi_P^G \circ h_G$.

Let

$$r'(f)(p_0, p_1, \dots, p_q, p) = \int_{K \times K^{q+1}} f(k' p_0 k_1^{-1}, k' p_1 k_2^{-1}, \dots, k' p_n k_0^{-1}, k', p)\, dk'\, dk, \quad (6.26)$$

where $dk = dk_0 \cdots dk_q$, as before. Then r' induces a morphism

$$r' : \mathcal{B}_q(G) \otimes \operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta) \longrightarrow \mathcal{B}_q(P) \otimes \mathscr{C}_c^\infty(P)_\delta \quad (6.27)$$

of complexes satisfying $h_P \circ r' = \tilde{h}_P \circ \chi \circ (r \otimes_G 1)t$. Directly from the definitions we obtain then that $h_P \circ r' \circ l = \tau \circ \phi_P^G \circ h_G$. This completes the proof. □

For simplicity, we have stated and proved the above result only for G reductive, however, it extends to arbitrary G and P such that G/P is compact, by including the modular function of G, where appropriate.

In order to better understand the effect of the morphism

$$\operatorname{ind}_P^G = (\phi_P^G)_* : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathrm{HH}_*(\mathscr{C}_c^\infty(P)), \quad (6.28)$$

it is sometimes useful to look at its action on the geometric fibers of the group $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))$ regarded as an $R^\infty(G)$-module. This is especially useful because the action on the geometric fibers also recovers some classical results on the characters of induced representations.

First we observe that restriction defines a morphism $\rho_P^G : R^\infty(G) \to R^\infty(P)$. Let M be a Levi component of the parabolic group P. Because the group G is reductive, we also have $R^\infty(P) \simeq R^\infty(M)$.

Lemma 6.3. *Let P be a parabolic subgroup of a reductive p-adic group G, and let $\rho_P^G : R^\infty(G) \to R^\infty(P)$ be the morphism induced by restriction, used to define an $R^\infty(G)$-module structure on $\mathrm{HH}_*(\mathscr{C}_c^\infty(P))$. Then*

$$\operatorname{ind}_P^G : \mathrm{HH}_*(\mathscr{C}_c^\infty(G)) \longrightarrow \mathrm{HH}_*(\mathscr{C}_c^\infty(P)) \quad (6.29)$$

is $R^\infty(G)$-linear, in the sense that $\operatorname{ind}_P^G(f\xi) = \rho_P^G(f)\operatorname{ind}_P^G(\xi)$, for all $f \in R^\infty(G)$ and all $\xi \in \mathrm{HH}_*(\mathscr{C}_c^\infty(G))$.

Proof. The result of the lemma follows from the fact that the map

$$\mathscr{C}_c^\infty(G) \longrightarrow \operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta) \quad (6.30)$$

is $R^\infty(G)$-linear and the isomorphism of Shapiro's lemma,

$$\mathrm{H}_q(G, \operatorname{ind}_P^G(\mathscr{C}_c^\infty(P)_\delta)) \simeq \mathrm{H}_q(P, \mathscr{C}_c^\infty(P)_\delta), \quad (6.31)$$

is natural.

Alternatively, one can use the explicit formula of (6.16). □

If $\mathfrak{m} = \mathfrak{m}_\gamma \subset R^\infty(G)$ is the maximal ideal of functions vanishing at a semi-simple element $\gamma \in G$, then its image $(\rho_P^G)_*(\mathfrak{m}) := \rho_P^G(\mathfrak{m})R^\infty(P) \subset R^\infty(P) = R^\infty(M)$ is the

ideal of functions vanishing at all $g \in M$ that are conjugated to γ in G. If γ is elliptic, then $\mathfrak{m} = R^\infty(P)$. If $\gamma \in M$, then $(\rho_P^G)_*(\mathfrak{m})$ need not, in general, be maximal. Let $\gamma_1, \gamma_2, \dots, \gamma_l \in M$ be representatives of the conjugacy classes of M that are contained in the conjugacy class of γ. Then $(\rho_P^G)_*(\mathfrak{m}) = \mathfrak{m}_{\gamma_1} \cap \mathfrak{m}_{\gamma_2} \cap \cdots \cap \mathfrak{m}_{\gamma_l}$, and hence we obtain a morphism

$$(\rho_P^G)_\gamma : \mathbb{C} \simeq R^\infty(G)_\gamma = \frac{R^\infty(G)}{\mathfrak{m}} \longrightarrow \frac{R^\infty(M)}{(\rho_P^G)_*(\mathfrak{m})} \simeq \mathbb{C}^l. \tag{6.32}$$

We are ready now to study the morphisms

$$\begin{aligned}(\operatorname{ind}_P^G)_\gamma : \operatorname{HH}_q(\mathscr{C}_c^\infty(G))_\gamma &= \frac{\operatorname{HH}_q(\mathscr{C}_c^\infty(G))}{\mathfrak{m}\operatorname{HH}_q(\mathscr{C}_c^\infty(G))} \longrightarrow \operatorname{HH}_q(\mathscr{C}_c^\infty(P))_\gamma \\ &= \frac{\operatorname{HH}_q(\mathscr{C}_c^\infty(P))}{(\rho_P^G)_*(\mathfrak{m})\operatorname{HH}_q(\mathscr{C}_c^\infty(P))} \simeq \bigoplus_{j=1}^{l} \operatorname{HH}_q(\mathscr{C}_c^\infty(P))_{\gamma_j}.\end{aligned} \tag{6.33}$$

Let $C_P(\gamma_j)$ be the centralizer of γ_j in P and $C_G(\gamma_j) \simeq C_G(\gamma)$ be the centralizer of γ_j in G. Then $C_P(\gamma_j)_u$ identifies with a subspace of $C_G(\gamma)_u$, which gives rise to a continuous proper map $C_G(\gamma) \times_{C_P(\gamma)} C_P(\gamma)_u \to C_G(\gamma)_u$, and hence to a morphism

$$\mathscr{C}_c^\infty(C_G(\gamma)_u) \longrightarrow \operatorname{ind}_{C_P(\gamma_j)}^{C_G(\gamma)}(\mathscr{C}_c^\infty(C_P(\gamma_j)_u)_\delta) \tag{6.34}$$

of $C_G(\gamma)$-modules. Passing to cohomology, we obtain using Shapiro's lemma a morphism

$$\iota_{\gamma_j}^\gamma : \mathrm{H}_q(C_G(\gamma), \mathscr{C}_c^\infty(C_G(\gamma)_u)) \longrightarrow \mathrm{H}_q(C_P(\gamma), \mathscr{C}_c^\infty(C_P(\gamma_j)_u)_\delta). \tag{6.35}$$

Recall that Proposition 4.3 gives isomorphisms

$$\begin{aligned}\operatorname{HH}_q(\mathscr{C}_c^\infty(G))_\gamma &\simeq \mathrm{H}_q(C_G(\gamma), \mathscr{C}_c^\infty(C_G(\gamma)_u)), \\ \operatorname{HH}_q(\mathscr{C}_c^\infty(P))_{\gamma_j} &\simeq \mathrm{H}_q(C_P(\gamma_j), \mathscr{C}_c^\infty(C_P(\gamma_j)_u)_\delta).\end{aligned} \tag{6.36}$$

Proposition 6.4. *Let $\gamma \in G$ be a semi-simple element and $M \subset P$ be as above. If the conjugacy class of γ does not intersect M, then $\operatorname{HH}_*(\mathscr{C}_c^\infty(P))_\gamma = 0$, and hence $(\operatorname{ind}_P^G)_\gamma = 0$. Otherwise, using notation (6.35), we have*

$$(\operatorname{ind}_P^G)_\gamma = \oplus_{j=1}^l \iota_{\gamma_j}^\gamma : \operatorname{HH}_*(\mathscr{C}_c^\infty(G))_\gamma \longrightarrow \oplus_{j=1}^l \operatorname{HH}_*(\mathscr{C}_c^\infty(P))_{\gamma_j} \simeq \operatorname{HH}_*(\mathscr{C}_c^\infty(P))_\gamma. \tag{6.37}$$

Proof. This follows from definitions if we observe that, in the sequence of maps

$$\begin{aligned}G \times_P P \times_{C_P(\gamma_i)} (\gamma_i C_P(\gamma_i)_u) &\simeq G \times_{C_G(\gamma_i)} C_G(\gamma_i) \times_{C_P(\gamma_i)} (\gamma_i C_P(\gamma_i)_u) \\ &\longrightarrow G \times_{C_G(\gamma_i)} (\gamma_i C_G(\gamma_i)_u),\end{aligned} \tag{6.38}$$

the second map is induced by $C_G(\gamma_i) \times_{C_P(\gamma_i)} C_P(\gamma_i)_u \to C_G(\gamma_i)_u$ and their composition induces on homology the direct summand $\iota_{\gamma_j}^\gamma$ of the map $(\operatorname{ind}_P^G)_\gamma$. □

Another morphism that is likely to play an important role is the "inflation morphism," which we now define. Let $N \subset P$ be the unipotent radical of an algebraic p-adic group, and let $M = P/N$ be its reductive quotient. Then integration over N defines an algebra morphism

$$\psi_M^P : \mathscr{C}_c^\infty(P) \longrightarrow \mathscr{C}_c^\infty(M), \qquad \psi_M^P(f)(m) = \int_N f(mn)\,dn. \tag{6.39}$$

Integration over N also defines a G-morphism $\mathscr{C}_c^\infty(P)_\delta \to \mathscr{C}_c^\infty(M)$, and since N is a union of compact groups, we finally obtain morphisms

$$\begin{aligned}\mathrm{HH}_k\left(\mathscr{C}_c^\infty(P)\right) &\simeq \mathrm{H}_k\left(P,\mathscr{C}_c^\infty(P)_\delta\right) \longrightarrow \mathrm{H}_k\left(P,\mathscr{C}_c^\infty(M)\right)\\ &\simeq \mathrm{H}_k\left(M,\mathscr{C}_c^\infty(M)\right) \simeq \mathrm{HH}_k\left(\mathscr{C}_c^\infty(M)\right),\end{aligned} \tag{6.40}$$

whose composition we denote $\inf_M^P$.

Theorem 6.5. *If M is a Levi component of a p-adic group P, as above. Then we have*

$$\left(\psi_M^P\right)_* = \inf_M^P : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right) \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(M)\right). \tag{6.41}$$

Proof. Integration over N defines a morphism

$$f : \mathscr{B}(P)\otimes\mathscr{C}_c^\infty(P)_\delta \longrightarrow \mathscr{B}(M)\otimes\mathscr{C}_c^\infty(M), \tag{6.42}$$

which commutes with the action of P. Then $f\otimes_P 1$ coincides with the morphism of complexes induced by ψ_M^P.

Consider now the maps h_G defined in the proof of Lemma 3.1. Then $\psi_M^P\circ h_P = h_M\circ f$, and hence $\psi_M^P\circ \tilde{h}_P = \tilde{h}_M\circ(f\otimes_P 1)$, from which the result follows. □

We now want to proceed by analogy and establish the explicit form of the action of $\inf_M^P$ on the geometric fibers of the groups $\mathrm{HH}_*(\mathscr{C}_c^\infty(P))$ and $\mathrm{HH}_*(\mathscr{C}_c^\infty(M))$. Fix $\gamma\in M$. Integration over the nilpotent radical of $C_P(\gamma)$, the centralizer of γ in P, induces a morphism

$$\mathscr{C}_c^\infty\left(C_P(\gamma)_u\right)_\delta = \mathscr{C}_c^\infty\left(C_P(\gamma)_u\right)\otimes\Delta_{C_P(\gamma)} \longrightarrow \mathscr{C}_c^\infty\left(C_M(\gamma)_u\right) \tag{6.43}$$

of P-modules. Let

$$\begin{aligned} j_\gamma : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right)_\gamma &= \mathrm{H}_*\left(C_P(\gamma),\mathscr{C}_c^\infty\left(C_P(\gamma)_u\right)_\delta\right) \longrightarrow \mathrm{H}_*\left(C_P(\gamma),\mathscr{C}_c^\infty\left(C_M(\gamma)_u\right)\right)\\ &\simeq \mathrm{H}_*\left(C_M(\gamma),\mathscr{C}_c^\infty\left(C_M(\gamma)_u\right)\right)\\ &= \mathrm{HH}_*\left(\mathscr{C}_c^\infty(M)\right)_\gamma\end{aligned} \tag{6.44}$$

be the induced morphism.

Proposition 6.6. *Let P be a p-adic group, let $M\subset P$ be a Levi component, and $\gamma\in M$ be a semi-simple element. Let $d(\gamma)$ be the determinant of $\mathrm{Ad}_\gamma^{-1}-1$ acting on $\mathrm{Lie}(N)/\ker(\mathrm{Ad}_\gamma^{-1}-1)$. Then, using localization at the maximal ideal defined by γ in $R^\infty(G)=R^\infty(P)$ and notation (6.44), we have*

$$\left(\inf_M^P\right)_\gamma = |d(\gamma)|^{-1} j_\gamma : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right)_\gamma \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(M)\right)_\gamma. \tag{6.45}$$

Proof. Fix $\gamma\in G$, not necessarily semi-simple and let N_γ be the subgroup of elements of N commuting with γ. We choose a complement V_γ of $\mathrm{Lie}(N_\gamma)$ in $\mathrm{Lie}(N)$ and we use the exponential map to identify V_γ with a subset of N. Then the Jacobian of the map

$$V_\gamma\times N_\gamma \ni (n,n') \longrightarrow \gamma^{-1}n\gamma n^{-1}n' \in N = V_\gamma N_\gamma \tag{6.46}$$

is $d(\gamma)$, and from this the result follows. □

This result is compatible with the results of van Dijk on characters of induced representations, see [26].

7. Examples. The results of the previous sections can be used to obtain some explicit calculations of the groups $\mathrm{HH}_k(\mathscr{C}_c^\infty(G))$ for particular groups G.

Example 7.1. Let Z be a commutative p-adic group of split rank r (so that $\mathrm{H}_q(Z) \simeq \Lambda^q\mathbb{C}^r$, for all $q \geq 0$). Then

$$\mathrm{HH}_q\left(\mathscr{C}_c^\infty(Z)\right) \simeq \mathscr{C}_c^\infty(Z) \otimes \Lambda^q\mathbb{C}^r. \tag{7.1}$$

Example 7.2. Let P be the (parabolic) subgroup of upper triangular matrices in $\mathrm{SL}_2(\mathbb{F})$, and $A \subset P$ be the subgroup of diagonal matrices. Then inflation defines a morphism

$$\inf_A^P : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right) \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(A)\right) = \mathscr{C}_c^\infty(A) \otimes \Lambda^*\mathbb{C} \tag{7.2}$$

whose range is $\mathscr{C}_c^\infty(A) \oplus \mathscr{C}_c^\infty(A \smallsetminus \{\pm I\})$, with I the identity matrix of $\mathrm{SL}_2(\mathbb{F})$. (We see this by localizing at each $\gamma \in A$.) To describe the kernel of $\inf_A^P$, let

$$u_b = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}. \tag{7.3}$$

Then, if we choose b to range through Σ_u, a set of representative of $\mathbb{F}^*/\mathbb{F}^{*2}$, the set of elements u_b forms a set of representatives of the set of nontrivial conjugacy classes of unipotent elements of P (a unipotent element is nontrivial if it is different from the identity). Recall that $\mathbb{F}$ has characteristic zero, so Σ_u is a discrete set. Let $\mathbb{O}_{u_b}$ be the orbital integral associated to u_b, and let $\mathbb{O}_{-u_b}$ be the orbital integral associated to $-u_b$, then the two maps

$$F_\pm = \oplus_b \mathbb{O}_{\pm u_b} : \mathscr{C}_c^\infty(G) \longrightarrow \mathbb{C}^{\Sigma_u} \tag{7.4}$$

can be used to identify the kernel of $\inf_A^P$ as follows. The map

$$F_+ \oplus F_- : \ker\left(\inf_A^P\right) \longrightarrow \mathbb{C}^{\pm\Sigma_u} \tag{7.5}$$

is injective, and the range of each of the two morphisms $F_\pm$ is the set of elements with zero sum.

All in all, we consider the map $\Phi = \inf_A^P \oplus F_+ \oplus F_-$,

$$\Phi : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right) \longrightarrow \left(\mathscr{C}_c^\infty(A) \oplus \mathbb{C}^{\pm\Sigma_u}\right)_{(0)} \oplus \left(\mathscr{C}_c^\infty(A \smallsetminus \{\pm I\})\right)_{(1)}, \tag{7.6}$$

where the lower index (i) represents the degree. Then Φ is surjective in degree 1, and, in degree 0, its range consists of $(f, \lambda_{b,\epsilon})$, $f \in \mathscr{C}_c^\infty(A)$, $\lambda_{b,\epsilon} \in \mathbb{C}$, for $\epsilon \in \{\pm 1\}$ and $b \in \Sigma_u \simeq \mathbb{F}^*/\mathbb{F}^{*2}$, such that $\sum_b \lambda_{b,\epsilon} = f(\epsilon I)$, for $\epsilon = \pm 1$.

The following example is also discussed in [1, 5], but from a different perspective.

Example 7.3. Consider now the group $G = \mathrm{SL}_2(\mathbb{F})$, where $\mathbb{F}$ is a p-adic field of characteristic zero. Let F_q be the residual field of $\mathbb{F}$ (thus q denotes the number of elements of F_q and is the power of the prime number p). We choose ϵ in the valuation ring of $\mathbb{F}$, such that its image in F_q is not a square. Also, let τ be a generator of the (unique) maximal ideal of the valuation ring of $\mathbb{F}$. Fix $a_\theta \in \mathbb{F}$ not in the image of the norm map $N : \mathbb{F}[\theta]^* \to \mathbb{F}^*$. We use the notation of [21], so let $\theta \in \{\epsilon, \tau, \epsilon\tau\}$ and let T_θ and $T_\theta^\#$ be the elliptic tori defined by

$$T_\theta = \{[a_{ij}],\ a_{11} = a_{22},\ a_{21} = \theta a_{12}\}, \qquad T_\theta^\# = \{[a_{ij}],\ a_{11} = a_{22},\ a_{21} = \theta a_\theta^2 a_{12}\}. \tag{7.7}$$

We distinguish two cases, first the case where -1 is a square in $\mathbb{F}$ and then the case where it is not a square in $\mathbb{F}$. If -1 is a square, then the Weyl group of each of the tori T_θ or $T_\theta^\#$ has order 2 (and will be denoted by S_2). Otherwise $W(T) = \{1\}$, for each torus $T = T_\theta$ or $T = T_\theta^\#$, but T_θ and $T_\theta^\#$ are conjugate for each fixed θ.

Let $X = \cup_\theta (T_\theta/S_2 \cup T_\theta^\#/S_2)$, if -1 is a square, and $X = \cup_\theta T_\theta$ otherwise. We endow X with the induced topology. Then $X \smallsetminus \{\pm 1\}$ identifies with the set of elliptic conjugacy classes of $\mathrm{SL}_2(\mathbb{F})$. Denote by $A \subset \mathrm{SL}_2(\mathbb{F})$ the set of diagonal matrices in $\mathrm{SL}_2(\mathbb{F})$. Let $W(A) = S_2$ act on $\mathscr{C}_c^\infty(A) \otimes \Lambda^* \mathbb{C}$ by conjugation on $\mathscr{C}_c^\infty(A)$ and by the nontrivial character on $\mathbb{C}$.

Recall that the set u_b, $b \in \mathbb{F}/(\mathbb{F}^*)^2$, parameterizes the set of conjugacy classes of unipotent elements of $\mathrm{SL}_2(\mathbb{F})$. Consequently, the set u_b, $b \in \mathbb{F}^*/(\mathbb{F}^*)^2$, parameterizes the set of conjugacy classes of nontrivial unipotent elements of $\mathrm{SL}_2(\mathbb{F})$. Let l be the number of elements of $\mathbb{F}^*/(\mathbb{F}^*)^2$. Then we have the following proposition.

Proposition 7.4. *Let $P \subset \mathrm{SL}_2(\mathbb{F})$ be the subgroup of upper triangular matrices. The composition*

$$\phi := \inf_a A^P \circ \mathrm{ind}_P^G : \mathrm{HH}_*\left(\mathscr{C}_c^\infty\left(\mathrm{SL}_2(\mathbb{F})\right)\right) \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(A)\right) = \mathscr{C}_c^\infty(A) \otimes \Lambda^* \mathbb{C} \tag{7.8}$$

has range consisting of $W(A)$-invariant elements. The kernel of ϕ is isomorphic to $\mathscr{C}_c^\infty(X \smallsetminus \{\pm I\}) \oplus \mathbb{C}^{2l}$, via orbital integrals with respect to elliptic elements and orbital integrals with respect to $\pm u_b$, $b \in \mathbb{F}/(\mathbb{F}^*)^2$. The factor $\mathbb{C}^{2l}$ corresponds to the fact that there are $l+1$ conjugacy classes of unipotent elements of $\mathrm{SL}_2(\mathbb{F})$ but the orbital integral associated $\pm u_b$ satisfy $\sum_{b \in \mathbb{F}^*/(\mathbb{F}^*)^2} \mathcal{O}_{\epsilon u_b}(f) = \phi(f)(\epsilon)$, if $f \in \ker(\phi)$ and $\epsilon \in \{\pm 1\}$.

Proof. First of all, it is clear that the composition $\phi = \inf_A^P \circ \mathrm{ind}_P^G$ is invariant with respect to the Weyl group $W(A)$, and hence its range consists of $W(A)$-invariant elements.

The localization of ϕ at a regular, diagonal conjugacy class γ is onto by Proposition 4.3. Next, we know that every orbital integral extends to $\mathscr{C}_c^\infty(\mathrm{SL}_2(\mathbb{F}))$, and this implies directly that the spectral sequence of Proposition 5.7 collapses at the E^2 term. This proves that the localization of ϕ at $\gamma = 1$ is also onto, and hence ϕ is onto. The rest of the proposition follows also from Proposition 5.7 by localization. □

We also have the following alternative description of $\mathrm{HH}_*(\mathscr{C}_c^\infty(\mathrm{SL}_2(\mathbb{F})))$.

Corollary 7.5. *The morphism* $\mathrm{ind}_P^G : \mathrm{HH}_*(\mathscr{C}_c^\infty(\mathrm{SL}_2(\mathbb{F}))) \to \mathrm{HH}_*(\mathscr{C}_c^\infty(P))$ *has image consisting of those elements whose image through the morphism*

$$\inf_A^P : \mathrm{HH}_*\left(\mathscr{C}_c^\infty(P)\right) \longrightarrow \mathrm{HH}_*\left(\mathscr{C}_c^\infty(A)\right) \tag{7.9}$$

is $W(A)$*-invariant. The kernel of* ind_P^G *is isomorphic to* $\mathscr{C}^\infty(X)$.

Example 7.6. We end this section with a description of the ingredients entering in the formula (1.3) for the Hochschild homology of $\mathscr{C}_c^\infty(G)$, if $G = \mathrm{GL}_n(\mathbb{F})$. Let $\gamma \in G$ be a semi-simple element. The minimal polynomial Q_γ of γ decomposes as $Q_\gamma = p_1 p_2 \cdots p_r$ into irreducible polynomials with coefficients in $\mathbb{F}$. (We assume, for simplicity, that each polynomial p_j is a monic polynomial.) Also, let $P_\gamma = p_1^{l_1} p_2^{l_2} \cdots p_r^{l_r}$ be the characteristic polynomial of γ. Then the algebra generated by γ is

$$\mathbb{F}[\gamma] \simeq \mathbb{K}_1 \oplus \cdots \oplus \mathbb{K}_r, \tag{7.10}$$

where $\mathbb{K}_i = \mathbb{F}[t]/(p_i(t))$ are not necessarily distinct fields. The commutant $\{\gamma\}'$ of γ in $M_n(\mathbb{F})$ is the commutant of this algebra, and hence

$$\begin{gathered}
\{\gamma\}' \simeq M_{l_1}(\mathbb{K}_1) \oplus M_{l_2}(\mathbb{K}_2) \oplus \cdots \oplus M_{l_r}(\mathbb{K}_r), \\
C(\gamma) \simeq \prod_{i=1}^r \mathrm{GL}_{l_i}(\mathbb{K}_i), \qquad S := Z(C(\gamma)) \simeq \prod_{i=1}^r \mathbb{K}_i^*, \\
S^{\mathrm{reg}} \simeq \left\{(x_i) \in \prod_{i=1}^r \mathbb{K}_i^*,\ \mathbb{K}_i = \mathbb{F}[x_i] \text{ and the minimal polynomials of } x_i \text{ are distinct}\right\}.
\end{gathered} \tag{7.11}$$

By the Skolem-Noether theorem, the Weyl group $W(S) = N(S)/C(S)$ coincides with the group of algebra automorphisms of $\{\gamma\}'$. This group has as quotient a group isomorphic to the subgroup $\Pi \subset N(S)$ which permutes the algebras $M_{l_i}(\mathbb{K}_i)$. Then $\Pi \simeq S_{m_1} \times \cdots \times S_{m_t}$, that is, Π is a product of symmetric groups. We denote the kernel of this morphism by $W_0(S)$. It is isomorphic to $\prod_{i=1}^r \mathrm{Aut}_{\mathbb{F}}(\mathbb{K}_i)$ (again by the Skolem-Noether theorem). The group $W(S)$ is then the semi-direct product of $W_0(S)$ by Π. We hence obtain exact sequences

$$\begin{gathered}
1 \longrightarrow N_0(S) \longrightarrow N(S) \longrightarrow \Pi \longrightarrow 1, \\
1 \longrightarrow C(S) \longrightarrow N_0(S) \longrightarrow W_0(S) \longrightarrow 1.
\end{gathered} \tag{7.12}$$

According to (1.3), the only other ingredients necessary to compute $\mathrm{HH}_*(\mathscr{C}_c^\infty(G))$ are the groups $\mathrm{H}_*(C(S), \mathscr{C}_c^\infty(\mathcal{U}_S))$.

Now, the unipotent variety of $C(S)$ is the product of the unipotent varieties of $\mathrm{GL}_{l_i}(\mathbb{K}_i)$, $i = 1, r$, and the subgroup $C(S)$ preserves this product decomposition. We see then that in order to prove that the spectral sequence of Proposition 5.7 collapses (for any choice of open subsets U_i), it is enough to check this for the spectral sequence converging to the cohomology of $\mathscr{C}_c^\infty(\mathrm{GL}_n(\mathbb{K})_u)$, for an arbitrary characteristic zero p-adic field $\mathbb{K}$.

Fix a unipotent element $\gamma \in \mathrm{GL}_n(\mathbb{K})$. Define then $V_0 = 0$, $V_l = \ker(\gamma - 1)^l \subset \mathbb{K}^n$, if $l > 0$. Also, choose W_l such that $V_l = V_{l-1} \oplus W_l$, and define

$$P = \{\gamma \in \mathrm{GL}_n(\mathbb{K}),\ \gamma V_l \subset V_l\}, \qquad M = \{\gamma \in \mathrm{GL}_n(\mathbb{K}),\ \gamma W_l = W_l\}. \tag{7.13}$$

Then P is a parabolic subgroup with unipotent radical

$$N = \{\gamma \in \mathrm{GL}_n(\mathbb{K}),\ (\gamma - 1)V_l \subset V_{l-1}\}, \tag{7.14}$$

and M is a Levi component of P. It is easy to check, from definition, that the P-orbit of u in N is dense. The centralizer of u is then contained in P and has split rank less than or equal to the split rank of P. Fix a maximal split torus A in the centralizer of u. We can assume that this split torus is contained in M. From the definition and by direct inspection, the map $\mathrm{H}_*(A) \to \mathrm{H}_*(M)$ is injective, and hence the map

$$\mathrm{H}^*(M) \longrightarrow \mathrm{H}^*(A) = \mathrm{H}^*\left(C(u)\right) \tag{7.15}$$

is surjective.

Fix now a cohomology class $c_0 \in \mathrm{H}^q(C(\gamma)) \simeq \mathrm{H}^q(A)$ and choose a cohomology class $c \in \mathrm{H}^q(M)$ that maps to c_0 under the above restriction map. Also, let τ be the trace on $\tau_0(f) = f(e)$ on $\mathscr{C}_c^\infty(M)$ (obtained by evaluation at the identity e). Then the formula

$$\phi_0(f_0, \dots, f_q) = \tau_0(D_c(f_0, \dots, f_q)) \tag{7.16}$$

defines a Hochschild cyclic cocycle on $\mathscr{C}_c^\infty(M)$. Consequently,

$$\phi = \phi_0 \circ \inf_A^P \circ \mathrm{ind}_P^G \tag{7.17}$$

defines a Hochschild cocycle on $\mathscr{C}_c^\infty(G)$. For any filtration U_i of G_u by open, invariant open sets, such that each $U_l \smallsetminus U_{l-1}$ consists of a single orbit. Suppose that the orbit $U_l \smallsetminus U_{l-1}$ is the orbit of $\gamma \in \mathrm{GL}_n(\mathbb{F})$ considered above. Then the cocycle ϕ will vanish on $\mathscr{C}_c^\infty(U_l)$ and represent the cohomology class

$$c \in \mathrm{H}^q\left(C(\gamma)\right) \simeq \mathrm{H}^q\left(G, \mathscr{C}_c^\infty\left(U_l \smallsetminus U_{l-1}\right)\right). \tag{7.18}$$

From this it follows that the spectral sequence of Proposition 5.7 degenerates at E^2.

It is very likely that the above argument extends to arbitrary reductive G by choosing M and P as in [20].

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