ON ALMOST PERIODIC SOLUTIONS OF THE DIFFERENTIAL EQUATION $x''(t) = Ax(t)$ IN HILBERT SPACES

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ABSTRACT. We prove almost periodicity of solutions of the equation $x''(t) = Ax(t)$ when the linear operator $A$ satisfies an inequality of the form $\text{Re}(Ax,x) \geq 0$.

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1. Introduction. Let $H$ be a Hilbert space equipped with norm $\| \cdot \|$ and scalar product $(\cdot, \cdot)$. Almost periodic functions (in Bochner’s sense) are continuous functions $f : \mathbb{R} \to H$ such that for every $\epsilon > 0$, there exists a positive real number $l$ such that every interval $[a, a+l]$ contains at least a point $\tau$ such that

$$\sup_{t \in \mathbb{R}} \| f(t+\tau) - f(t) \| < \epsilon.$$  \hfill (1.1)

The Bochner’s criterion (cf. [1, 3, 4]) states that a function $f : \mathbb{R} \to H$ is almost periodic if and only if for every sequence of real numbers $(\sigma_n)_{n=1}^{\infty}$ there exists a subsequence $(s_n)_{n=1}^{\infty}$ such that $(f(t+s_n))_{n=1}^{\infty}$ is uniformly convergent in $t \in \mathbb{R}$.

We proved in [2] that if $A = A_+ + A_-$, where $A_+$ is a symmetric linear operator and $A_-$ is a skew-symmetric linear operator such that $\text{Re}(A_+,x,A_-x) \geq -c \| A_+x \|^2$ for every $x \in H$, then every solution of $x'(t) = Ax(t), t \in \mathbb{R}$, with a relatively compact range in $H$ is almost periodic.

In this note, we use the technique described in [2] to prove similar results for some classes of linear differential equation of second order $x''(t) = Ax(t)$.

2. Main results

THEOREM 2.1. Assume that the linear operator $A$ satisfies the inequality of the form $\text{Re}(Ax,x) \geq 0$, for any $x \in H$. Then solutions of the differential equation

$$x''(t) = Ax(t), \quad t \in \mathbb{R},$$  \hfill (2.1)

(that are functions $x(t) \in C^2(\mathbb{R},H)$) with relatively compact ranges in $H$, are almost periodic.

PROOF. Consider $x(t)$ a solution of (2.1) with a relatively compact range in $H$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(t) = \| x(t) \|^2$. Then $\phi$ is a bounded function over $\mathbb{R}$.
Moreover, for every \( t \in \mathbb{R} \), we have
\[
\begin{align*}
\phi'(t) &= (x'(t), x(t)) + (x(t), x'(t)), \\
\phi''(t) &= 2[[|x'(t)|^2 + \text{Re}(Ax(t), x(t))]] \\
&\geq 0,
\end{align*}
\]
which shows that \( \phi \) is a convex function over \( \mathbb{R} \), therefore it is constant
\[
\phi(t) = \phi(0), \quad \forall t \in \mathbb{R},
\]
or
\[
||x(t)|| = ||x(0)||, \quad \forall t \in \mathbb{R}.
\]
We fix \( s \in \mathbb{R} \) and consider the function \( \gamma_s(\cdot) : \mathbb{R} \to H \) defined by
\[
\gamma_s(t) = x(t + s).
\]
Then \( \gamma_s(t) \) obviously satisfies (2.1). Now fix \( s_1 \) and \( s_2 \) in \( \mathbb{R} \). Then \( \gamma_{s_1}(t) - \gamma_{s_2}(t) \) also satisfies (2.1); therefore we have
\[
||\gamma_{s_1}(t) - \gamma_{s_2}(t)|| = ||\gamma_{s_1}(0) - \gamma_{s_2}(0)||, \quad \forall t \in \mathbb{R},
\]
which gives
\[
||x(t + s_1) - x(t + s_2)|| = ||x(s_1) - x(s_2)||, \quad \forall t \in \mathbb{R}.
\]
Let \( (\sigma_n)_{n=1}^{\infty} \) be a sequence of real numbers. Then by relative compactness of \( x(t) \), there exists a subsequence \( (s_n)_{n=1}^{\infty} \subset (\sigma_n)_{n=1}^{\infty} \) such that \( (x(s_n))_{n=1}^{\infty} \) is Cauchy. Hence for any given \( \epsilon > 0 \), there exists \( N \) such that if \( n, m > N \), then
\[
||x(s_n) - x(s_m)|| < \epsilon.
\]
Consequently,
\[
\sup_{t \in \mathbb{R}} ||x(t + s_n) - x(t + s_m)|| < \epsilon.
\]
We conclude that \( x(t) \) is almost periodic by the Bochner’s criterion.

**Remark 2.2.** Examples of such problem occur when \( A \) is a positive or monotone linear operator.

**References**


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