ON THE EXTENDED HARDY’S INEQUALITY

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ABSTRACT. We generalize a strengthened version of Hardy’s inequality and give a new simpler proof.

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In the recent paper [4], Hardy’s inequality was generalized. In this note, the results given in [4] are further generalized and a new much simpler proof is given. The following Hardy’s inequality is well known [1, Theorem 349].

THEOREM 1 (Hardy’s inequality). Let \( \lambda_n > 0 \), \( A_n = \sum_{k=1}^{n} \lambda_k \), \( a_n \geq 0 \) \((n \in \mathbb{N})\), \( 0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty \), then

\[
\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1}
\]

Recently, [4] gave an improvement of Theorem 1, and the following result was proved.

THEOREM 2. Let \( 0 < \lambda_{n+1} \leq \lambda_n \), \( A_n = \sum_{k=1}^{n} \lambda_k \), \( a_n \geq 0 \) \((n \in \mathbb{N})\), \( 0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty \), then

\[
\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left( 1 - \frac{\lambda_n}{2(A_n + \lambda_n)} \right) \lambda_n a_n. \tag{2}
\]

In this note, we will prove the following theorem.

THEOREM 3. Let \( 0 < \lambda_{n+1} \leq \lambda_n \), \( A_n = \sum_{k=1}^{n} \lambda_k \), \( a_n \geq 0 \) \((n \in \mathbb{N})\), \( 0 < \sum_{n=1}^{\infty} \lambda_n a_n < +\infty \), then

\[
\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} < e \sum_{n=1}^{\infty} \left( 1 + \frac{5\lambda_n}{5A_n + \lambda_n} \right)^{-1/2} \lambda_n a_n. \tag{3}
\]

To prove Theorem 3, we introduce some lemmas.

LEMMA 4. For \( x > 0 \), then

\[
e\left( 1 - \frac{1}{2x+1} \right) < \left( 1 + \frac{1}{x} \right)^x < e \left( 1 + \frac{5}{5x+1} \right)^{-1/2}. \tag{4}
\]
**Proof.** (i) Define \( f(x) \) as

\[
f(x) = x \ln \left( 1 + \frac{1}{x} \right) + \frac{1}{2} \ln \left( 1 + \frac{5}{5x+1} \right), \quad x \in (0, +\infty).
\]  

(5)

It is obvious that when \( x > 0 \), the inequality

\[
\left( 1 + \frac{1}{x} \right)^x < e \left( 1 + \frac{5}{5x+1} \right)^{-1/2}
\]  

(6)

is equivalent to \( f(x) < 1 \). It is easy to see that

\[
f'(x) = -\frac{1}{x+1} + \ln \left( 1 + \frac{1}{x} \right) - \frac{25}{2(5x+6)(5x+1)}
\]  

(7)

and for \( x \in (0, +\infty) \), it can be shown that

\[
f''(x) = \frac{1}{(x+1)^2} - \frac{1}{x(x+1)} + \frac{25}{2(5x+6)^2} - \frac{25}{2(5x+6)^2}
\]

\[
= \frac{-125x^2 - 50x^2 + 35x - 72}{2x(x+1)^2(5x+1)^2(5x+6)^2} < 0.
\]  

(8)

Hence \( f'(x) \) is decreasing on \((0, +\infty)\). Then for any \( x \in (0, +\infty) \), we have \( f'(x) > \lim_{x \to +\infty} f'(x) = 0 \), thus, \( f(x) \) is increasing on \((0, +\infty)\), and \( f(x) < \lim_{x \to +\infty} f(x) = 1 \) for \( x \in (0, +\infty) \). The inequality (6) is valid.

(ii) Define \( g(x) \) as

\[
g(x) = x \ln \left( 1 + \frac{1}{x} \right) - \ln \left( 1 - \frac{1}{2x+1} \right), \quad x \in (0, +\infty).
\]  

(9)

When \( x > 0 \), the inequality

\[
e \left( 1 - \frac{1}{2x+1} \right)^x < \left( 1 + \frac{1}{x} \right)^x
\]  

(10)

is equivalent to \( g(x) > 1 \). For \( x \in (0, +\infty) \), it can be shown that

\[
g'(x) = -\frac{1}{x+1} + \ln \left( 1 + \frac{1}{x} \right) - \frac{1}{x(2x+1)},
\]  

\[
g''(x) = \frac{5x^2 + 5x + 1}{x^2(2x+1)^2} > 0.
\]  

(11)

Hence, \( g'(x) \) is increasing on \((0, +\infty)\). Then for any \( x \in (0, +\infty) \), we have \( g'(x) < \lim_{x \to +\infty} g'(x) = 0 \), therefore, \( g(x) \) is decreasing on \((0, +\infty)\) and \( g(x) > \lim_{x \to +\infty} g(x) = 1 \) for \( x \in (0, +\infty) \). Inequality (10) is valid.

By virtue of (6) and (10), inequalities (4) are valid. This proves Lemma 4. \( \square \)

**Remark 5.** By a direct calculation, we have

\[
\left( 1 + \frac{5}{5x+1} \right)^{-1/2} < 1 - \frac{1}{2(x+19/20)} \quad (x > 0).
\]  

(12)
Then by (4) and (12), we have
\[ e \left(1 - \frac{1}{2x + 1}\right) < (1 + \frac{1}{x})^x < e \left[1 - \frac{1}{2(x + 19/20)}\right] \quad (x > 0). \] (13)

Inequality (13) is equivalent to
\[ \frac{e}{2(x + 19/20)} < e - \left(1 + \frac{1}{x}\right)^x < \frac{e}{2x + 1} \quad (x > 0). \] (14)

Thus, [1, Lemma 2] is contained in Lemma 4. Inequalities (4) and (14) are the new inequalities on the constant \( e \) (cf. [3, Theorem 3.8.26]; and [2, page 358]).

**Lemma 6** (see [1, Theorem 9]). Let \( g_m > 0, \alpha_m \geq 0 \ (m = 1, 2, \ldots, n), \sum_{m=1}^{n} g_m = 1, \) then
\[ \alpha_1^{g_1} \alpha_2^{g_2} \cdots \alpha_n^{g_n} \leq \sum_{m=1}^{n} g_m \alpha_m. \] (15)

**Proof of Theorem 3.** Setting \( c_m > 0, g_m = \lambda_m/A_n, \alpha_m = c_m\alpha_m \ (m = 1, 2, \ldots, n), \) by Lemma 6, we have
\[ (c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n} \leq \frac{1}{A_n} \sum_{m=1}^{n} \lambda_m c_m a_m. \] (16)

Then we find that
\[ \sum_{n=1}^{\infty} \lambda_n+1 (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} = \sum_{n=1}^{\infty} \lambda_n+1 (c_1 a_1)^{\lambda_1/A_1} (c_2 a_2)^{\lambda_2/A_2} \cdots (c_n a_n)^{\lambda_n/A_n} \]
\[ \leq \sum_{n=1}^{\infty} \left(\frac{\lambda_n+1}{c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}}\right)^{1/A_n} \frac{1}{A_n} \sum_{m=1}^{n} \lambda_m c_m a_m \]
\[ = \sum_{m=1}^{\infty} \lambda_m a_m c_m \sum_{n=m}^{\infty} \frac{\lambda_n+1}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}}. \] (17)

Define \( c_m = ((A_m+1)/A_m)^{\lambda_m/A_n} a_m \ (m = 1, 2, \ldots) \) and \( A_0 = 0. \) Because \( 0 < \lambda_{n+1} \leq \lambda_n \ (n = 1, 2, \ldots), \) we have
\[ c_m^\lambda_m = \frac{(A_m+1)^{\lambda_m}}{A_m^{\lambda_m-1}}; \quad (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n} = A_n+1 \ (n \in \mathbb{N}); \]
\[ c_m \sum_{n=m}^{\infty} \frac{\lambda_n+1}{A_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/A_n}} = \left(1 + \frac{\lambda_{m+1}}{A_m}\right)^{\lambda_m/A_n} \sum_{n=m}^{\infty} \frac{\lambda_n+1}{A_n A_n^{\lambda_m+1}} \]
\[ = (1 + \frac{\lambda_{m+1}}{A_m})^{\lambda_m/A_n} A_m \sum_{n=m}^{\infty} \left(\frac{1}{A_n} - \frac{1}{A_n+1}\right) \]
\[ \leq (1 + \frac{\lambda_{m+1}}{A_m})^{\lambda_m/A_n} \cdot \] (18)
Then by (4) and (17), we obtain that
\[ \sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{1/A_n} \leq \sum_{m=1}^{\infty} \left( 1 + \frac{\lambda_m}{A_m} \right)^{A_m/\lambda_m} \lambda_m a_m \]
\[ \leq e \sum_{m=1}^{\infty} \left( 1 + \frac{5\lambda_m}{5A_m + \lambda_m} \right)^{-1/2} \lambda_m a_m. \]  
(19)

Hence inequality (3) is valid, and Theorem 3 is proved.

Remark 7. With inequality (12), Theorem 3 is obviously an improvement and an extension of [4, Theorem 1].

Setting \( \lambda_n \equiv 1 \), (3) changes into
\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left( 1 + \frac{5}{5n+1} \right)^{-1/2} a_n. \]  
(20)

By inequality (12), we have
\[ \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left[ 1 - \frac{1}{2(n+19/20)} \right] a_n. \]  
(21)

Thus, inequalities (20) and (21) are obviously an improvement and extension of [5, Theorem 3.1].

References


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