ON FUZZY DOT SUBALGEBRAS OF BCH-ALGEBRAS

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ABSTRACT. We introduce the notion of fuzzy dot subalgebras in BCH-algebras, and study its various properties.

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1. Introduction. In [4], Hu and Li introduced the notion of BCH-algebras which are a generalization of BCK/BCI-algebras. In 1965, Zadeh [6] introduced the concept of fuzzy subsets. Since then several researchers have applied this notion to various mathematical disciplines. Jun [5] applied it to BCH-algebras, and he considered the fuzzification of ideals and filters in BCH-algebras. In this paper, we introduce the notion of a fuzzy dot subalgebra of a BCH-algebra as a generalization of a fuzzy subalgebra of a BCH-algebra, and then we investigate several basic properties related to fuzzy dot subalgebras.

2. Preliminaries. A BCH-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following conditions:

(i) \(x * x = 0\),

(ii) \(x * y = 0 = y * x\) implies \(x = y\),

(iii) \((x * y) * z = (x * z) * y\) for all \(x, y, z \in X\).

In any BCH-algebra \(X\), the following hold (see [2]):

(P1) \(x * 0 = x\),

(P2) \(x * 0 = 0\) implies \(x = 0\),

(P3) \(0 * (x * y) = (0 * x) * (0 * y)\).

A BCH-algebra \(X\) is said to be medial if \(x * (x * y) = y\) for all \(x, y \in X\). A nonempty subset \(S\) of a BCH-algebra \(X\) is called a subalgebra of \(X\) if \(x * y \in S\) whenever \(x, y \in S\). A map \(f\) from a BCH-algebra \(X\) to a BCH-algebra \(Y\) is called a homomorphism if \(f(x * y) = f(x) * f(y)\) for all \(x, y \in X\).

We now review some fuzzy logic concepts. A fuzzy subset of a set \(X\) is a function \(\mu : X \rightarrow [0, 1]\). For any fuzzy subsets \(\mu\) and \(\nu\) of a set \(X\), we define

\[
\mu \leq \nu \iff \mu(x) \leq \nu(x) \quad \forall x \in X,
\]

\[
(\mu \cap \nu)(x) = \min \{\mu(x), \nu(x)\} \quad \forall x \in X.
\]

Let \(f : X \rightarrow Y\) be a function from a set \(X\) to a set \(Y\) and let \(\mu\) be a fuzzy subset of \(X\).
The fuzzy subset \( \nu \) of \( Y \) defined by
\[
\nu(y) := \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\
0 & \text{otherwise},
\end{cases}
\]
is called the image of \( \mu \) under \( f \), denoted by \( f[\mu] \). If \( \nu \) is a fuzzy subset of \( Y \), the fuzzy subset \( \mu \) of \( X \) given by
\[
\mu(x) = \nu(f(x))
\]
for all \( x \in X \) is called the preimage of \( \nu \) under \( f \) and is denoted by \( f^{-1}[\nu] \).

A fuzzy relation \( \mu \) on a set \( X \) is a fuzzy subset of \( X \times X \), that is, a map \( \mu : X \times X \to [0,1] \). A fuzzy subset \( \mu \) of a BCH-algebra \( X \) is called a fuzzy subalgebra of \( X \) if
\[
\mu(x \ast y) \geq \min\{\mu(x),\mu(y)\}
\]
for all \( x,y \in X \).

3. Fuzzy product subalgebras. In what follows let \( X \) denote a BCH-algebra unless otherwise specified.

**Definition 3.1.** A fuzzy subset \( \mu \) of \( X \) is called a fuzzy dot subalgebra of \( X \) if
\[
\mu(x \ast y) \geq \mu(x) \cdot \mu(y)
\]
for all \( x,y \in X \).

**Example 3.2.** Consider a BCH-algebra \( X = \{0, a, b, c\} \) having the following Cayley table (see [1]):

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set \( \mu \) in \( X \) by \( \mu(0) = 0.5, \mu(a) = 0.6, \mu(b) = 0.4, \mu(c) = 0.3 \). It is easy to verify that \( \mu \) is a fuzzy dot subalgebra of \( X \).

Note that every fuzzy subalgebra is a fuzzy dot subalgebra, but the converse is not true. In fact, the fuzzy dot subalgebra \( \mu \) in Example 3.2 is not a fuzzy subalgebra since
\[
\mu(a \ast a) = \mu(0) = 0.5 < 0.6 = \mu(a) = \min\{\mu(a),\mu(a)\}.
\]

**Proposition 3.3.** If \( \mu \) is a fuzzy dot subalgebra of \( X \), then
\[
\mu(0) \geq (\mu(x))^2, \quad \mu(0^n \ast x) \geq (\mu(x))^{2n+1},
\]
for all \( x \in X \) and \( n \in \mathbb{N} \) where \( 0^n \ast x = 0 \ast (0 \ast (\cdots (0 \ast x) \cdots)) \) in which \( 0 \) occurs \( n \) times.

**Proof.** Since \( x \ast x = 0 \) for all \( x \in X \), it follows that
\[
\mu(0) = \mu(x \ast x) \geq \mu(x) \cdot \mu(x) = (\mu(x))^2
\]
for all \( x \in X \). The proof of the second part is by induction on \( n \). For \( n = 1 \), we have \( \mu(0 \ast x) \geq \mu(0) \cdot \mu(x) \geq (\mu(x))^3 \) for all \( x \in X \). Assume that \( \mu(0^k \ast x) \geq (\mu(x))^{2k+1} \) for...
all $x \in X$. Then
\[
\mu(0^{k+1} \ast x) = \mu(0 \ast (0^k \ast x)) \geq \mu(0) \cdot \mu(0^k \ast x) \\
\geq (\mu(x))^2 \cdot (\mu(x))^{2k+1+1} = (\mu(x))^{2(k+1)+1}.
\]
(3.4)

Hence $\mu(0^n \ast x) \geq (\mu(x))^{2n+1}$ for all $x \in X$ and $n \in \mathbb{N}$.

**Proposition 3.4.** Let $\mu$ be a fuzzy dot subalgebra of $X$. If there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} (\mu(x_n))^2 = 1$, then $\mu(0) = 1$.

**Proof.** According to Proposition 3.3, $\mu(0) \geq (\mu(x_n))^2$ for each $n \in \mathbb{N}$. Since $1 \geq \mu(0) \geq \lim_{n \to \infty} (\mu(x_n))^2 = 1$, it follows that $\mu(0) = 1$.

**Theorem 3.5.** If $\mu$ and $\nu$ are fuzzy dot subalgebras of $X$, then so is $\mu \cap \nu$.

**Proof.** Let $x, y \in X$, then
\[
(\mu \cap \nu)(x \ast y) = \min \{\mu(x \ast y), \nu(x \ast y)\} \\
\geq \min \{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\} \\
\geq (\min \{\mu(x), \nu(x)\}) \cdot (\min \{\mu(y), \nu(y)\}) \\
= ((\mu \cap \nu)(x)) \cdot ((\mu \cap \nu)(y)).
\]
(3.5)

Hence $\mu \cap \nu$ is a fuzzy dot subalgebra of $X$.

Note that a fuzzy subset $\mu$ of $X$ is a fuzzy subalgebra of $X$ if and only if a nonempty level subset
\[
U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}
\]
(3.6)
is a subalgebra of $X$ for every $t \in [0, 1]$. But, we know that if $\mu$ is a fuzzy dot subalgebra of $X$, then there exists $t \in [0, 1]$ such that
\[
U(\mu; t) := \{x \in X \mid \mu(x) \geq t\}
\]
(3.7)
is not a subalgebra of $X$. In fact, if $\mu$ is the fuzzy dot subalgebra of $X$ in Example 3.2, then
\[
U(\mu; 0.4) = \{x \in X \mid \mu(x) \geq 0.4\} = \{0, a, b\}
\]
(3.8)
is not a subalgebra of $X$ since $b \ast a = c \notin U(\mu; 0.4)$.

**Theorem 3.6.** If $\mu$ is a fuzzy dot subalgebra of $X$, then
\[
U(\mu; 1) := \{x \in X \mid \mu(x) = 1\}
\]
(3.9)
is either empty or is a subalgebra of $X$.

**Proof.** If $x$ and $y$ belong to $U(\mu; 1)$, then $\mu(x \ast y) \geq \mu(x) \cdot \mu(y) = 1$. Hence $\mu(x \ast y) = 1$ which implies $x \ast y \in U(\mu; 1)$. Consequently, $U(\mu; 1)$ is a subalgebra of $X$.
**Theorem 3.7.** Let $X$ be a medial BCH-algebra and let $\mu$ be a fuzzy subset of $X$ such that

$$\mu(0 \ast x) \geq \mu(x), \quad \mu(x \ast (0 \ast y)) \geq \mu(x) \cdot \mu(y),$$

(3.10)

for all $x, y \in X$. Then $\mu$ is a fuzzy dot subalgebra of $X$.

**Proof.** Since $X$ is medial, we have $0 \ast (0 \ast y) = y$ for all $y \in X$. Hence

$$\mu(x \ast y) = \mu(x \ast (0 \ast (0 \ast y))) \geq \mu(x) \cdot \mu(0 \ast y) \geq \mu(x) \cdot \mu(y)$$

(3.11)

for all $x, y \in X$. Therefore $\mu$ is a fuzzy dot subalgebra of $X$. \hfill \Box

**Theorem 3.8.** Let $g : X \to Y$ be a homomorphism of BCH-algebras. If $\nu$ is a fuzzy dot subalgebra of $Y$, then the preimage $g^{-1}[\nu]$ of $\nu$ under $g$ is a fuzzy dot subalgebra of $X$.

**Proof.** For any $x_1, x_2 \in X$, we have

$$g^{-1}[\nu](x_1 \ast x_2) = \nu(g(x_1 \ast x_2)) = \nu(g(x_1) \ast g(x_2))$$

$$\geq \nu(g(x_1)) \cdot \nu(g(x_2)) = g^{-1}[\nu](x_1) \cdot g^{-1}[\nu](x_2).$$

(3.12)

Thus $g^{-1}[\nu]$ is a fuzzy dot subalgebra of $X$. \hfill \Box

**Theorem 3.9.** Let $f : X \to Y$ be an onto homomorphism of BCH-algebras. If $\mu$ is a fuzzy dot subalgebra of $X$, then the image $f[\mu]$ of $\mu$ under $f$ is a fuzzy dot subalgebra of $Y$.

**Proof.** For any $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$, and $A_{12} = f^{-1}(y_1 \ast y_2)$. Consider the set

$$A_1 \ast A_2 := \{x \in X \mid x = a_1 \ast a_2 \text{ for some } a_1 \in A_1, a_2 \in A_2\}. \quad (3.13)$$

If $x \in A_1 \ast A_2$, then $x = x_1 \ast x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$$f(x) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2,$$

(3.14)

that is, $x \in f^{-1}(y_1 \ast y_2) = A_{12}$. Hence $A_1 \ast A_2 \subseteq A_{12}$. It follows that

$$f[\mu](y_1 \ast y_2) = \sup_{x \in f^{-1}(y_1 \ast y_2)} \mu(x) = \sup_{x \in A_{12}} \mu(x)$$

$$\geq \sup_{x \in A_1 \ast A_2} \mu(x) \geq \sup_{x_1 \in A_1, x_2 \in A_2} \mu(x_1 \ast x_2)$$

$$\geq \sup_{x_1 \in A_1, x_2 \in A_2} \mu(x_1) \cdot \mu(x_2).$$

(3.15)

Since $\cdot : [0,1] \times [0,1] \to [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\tilde{x}_1 \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\tilde{x}_2 \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$, then $\tilde{x}_1 \cdot \tilde{x}_2 \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \varepsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$. Then $f(a_1 \ast a_2) \geq f(a_1) \ast f(a_2) - \varepsilon$ and $f[a_1 \ast a_2] \geq f[a_1] \ast f[a_2] - \varepsilon$. Hence $f[\mu]$ is a fuzzy dot subalgebra of $Y$. \hfill \Box
we have

Consequently, and hence

\begin{equation}
\mu(a_1) \cdot \mu(a_2) \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \mu(x_2) - \varepsilon.
\end{equation}

(3.16)

Consequently,

\begin{align*}
f[\mu](y_1 \ast y_2) & \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \mu(x_2) \\
& \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \mu(x_2) \\
& = f[\mu](y_1) \cdot f[\mu](y_2),
\end{align*}

(3.17)

and hence \( f[\mu] \) is a fuzzy dot subalgebra of \( Y \).

\( \Box \)

**Definition 3.10.** Let \( \sigma \) be a fuzzy subset of \( X \). The **strongest fuzzy \( \sigma \)-relation** on \( X \) is the fuzzy subset \( \mu_\sigma \) of \( X \times X \) given by \( \mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y) \) for all \( x, y \in X \).

**Theorem 3.11.** Let \( \mu_\sigma \) be the strongest fuzzy \( \sigma \)-relation on \( X \), where \( \sigma \) is a fuzzy subset of \( X \). If \( \sigma \) is a fuzzy dot subalgebra of \( X \), then \( \mu_\sigma \) is a fuzzy dot subalgebra of \( X \times X \).

**Proof.** Assume that \( \sigma \) is a fuzzy dot subalgebra of \( X \). For any \( x_1, x_2, y_1, y_2 \in X \), we have

\[
\mu_\sigma((x_1, y_1) \ast (x_2, y_2)) = \mu_\sigma(x_1 \ast x_2, y_1 \ast y_2)
\]

\[
= \sigma(x_1 \ast x_2) \cdot \sigma(y_1 \ast y_2)
\]

\[
\geq (\sigma(x_1) \cdot \sigma(x_2)) \cdot (\sigma(y_1) \cdot \sigma(y_2))
\]

(3.18)

and so \( \mu_\sigma \) is a fuzzy dot subalgebra of \( X \times X \).

\( \Box \)

**Definition 3.12.** Let \( \sigma \) be a fuzzy subset of \( X \). A fuzzy relation \( \mu \) on \( X \) is called a **fuzzy \( \sigma \)-product relation** if \( \mu(x, y) \geq \sigma(x) \cdot \sigma(y) \) for all \( x, y \in X \).

**Definition 3.13.** Let \( \sigma \) be a fuzzy subset of \( X \). A fuzzy relation \( \mu \) on \( X \) is called a **left fuzzy relation on \( \sigma \)** if \( \mu(x, y) = \sigma(x) \) for all \( x, y \in X \).

Similarly, we can define a right fuzzy relation on \( \sigma \). Note that a left (resp., right) fuzzy relation on \( \sigma \) is a fuzzy \( \sigma \)-product relation.

**Theorem 3.14.** Let \( \mu \) be a left fuzzy relation on a fuzzy subset \( \sigma \) of \( X \). If \( \mu \) is a fuzzy dot subalgebra of \( X \times X \), then \( \sigma \) is a fuzzy dot subalgebra of \( X \).

**Proof.** Assume that a left fuzzy relation \( \mu \) on \( \sigma \) is a fuzzy dot subalgebra of \( X \times X \). Then

\[
\sigma(x_1 \ast x_2) = \mu(x_1 \ast x_2, y_1 \ast y_2) = \mu((x_1, y_1) \ast (x_2, y_2))
\]

\[
\geq \mu(x_1, y_1) \cdot \mu(x_2, y_2) = \sigma(x_1) \cdot \sigma(x_2)
\]

(3.19)

for all \( x_1, x_2, y_1, y_2 \in X \). Hence \( \sigma \) is a fuzzy dot subalgebra of \( X \).

\( \Box \)
**THEOREM 3.15.** Let \( \mu \) be a fuzzy relation on \( X \) satisfying the inequality \( \mu(x, y) \leq \mu(x, 0) \) for all \( x, y \in X \). Given \( z \in X \), let \( \sigma_z \) be a fuzzy subset of \( X \) defined by \( \sigma_z(x) = \mu(x, z) \) for all \( x \in X \). If \( \mu \) is a fuzzy dot subalgebra of \( X \times X \), then \( \sigma_z \) is a fuzzy dot subalgebra of \( X \) for all \( z \in X \).

**Proof.** Let \( z, x, y \in X \), then

\[
\sigma_z(x \ast y) = \mu(x \ast y, z) = \mu((x, z) \ast (y, 0)) \geq \mu(x, z) \cdot \mu(y, 0) \geq \mu(x, z) \cdot \mu(y, z) = \sigma_z(x) \cdot \sigma_z(y),
\]

completing the proof. \( \Box \)

**THEOREM 3.16.** Let \( \mu \) be a fuzzy relation on \( X \) and let \( \sigma_\mu \) be a fuzzy subset of \( X \) given by \( \sigma_\mu(x) = \inf_{y \in X} \mu(x, y) \cdot \mu(y, x) \) for all \( x \in X \). If \( \mu \) is a fuzzy dot subalgebra of \( X \times X \) satisfying the equality \( \mu(x, 0) = 1 = \mu(0, x) \) for all \( x \in X \), then \( \sigma_\mu \) is a fuzzy dot subalgebra of \( X \).

**Proof.** For any \( x, y, z \in X \), we have

\[
\mu(x \ast y, z) = \mu((x, z) \ast (y, 0)) \geq \mu(x, z) \cdot \mu(y, 0) = \mu(x, z),
\]

\[
\mu(z, x \ast y) = \mu((z, x) \ast (0, y)) \geq \mu(z, x) \cdot \mu(0, y) = \mu(z, x).
\]

It follows that

\[
\mu(x \ast y, z) \cdot \mu(z, x \ast y) \geq \mu(x, z) \cdot \mu(z, x)
\]

\[
\geq (\mu(x, z) \cdot \mu(z, x)) \cdot (\mu(y, z) \cdot \mu(z, y)) \quad (3.22)
\]

so that

\[
\sigma_\mu(x \ast y) = \inf_{z \in X} \mu(x \ast y, z) \cdot \mu(z, x \ast y)
\]

\[
\geq \left( \inf_{z \in X} \mu(x, z) \cdot \mu(z, x) \right) \cdot \left( \inf_{z \in X} \mu(y, z) \cdot \mu(z, y) \right) \quad (3.23)
\]

\[
= \sigma_\mu(x) \cdot \sigma_\mu(y).
\]

This completes the proof. \( \Box \)

**Definition 3.17** (see Choudhury et al. [3]). A **fuzzy map** \( f \) from a set \( X \) to a set \( Y \) is an ordinary map from \( X \) to the set of all fuzzy subsets of \( Y \) satisfying the following conditions:

(C1) for all \( x \in X \), there exists \( y_x \in X \) such that \( (f(x))(y_x) = 1 \),

(C2) for all \( x \in X \), \( f(x)(y_1) = f(x)(y_2) \) implies \( y_1 = y_2 \).

One observes that a fuzzy map \( f \) from \( X \) to \( Y \) gives rise to a unique ordinary map \( \mu_f : X \times X \rightarrow I \), given by \( \mu_f(x, y) = f(x)(y) \). One also notes that a fuzzy map from \( X \) to \( Y \) gives a unique ordinary map \( f_1 : X \rightarrow Y \) defined as \( f_1(x) = y_x \).
A fuzzy map \( f \) from a BCH-algebra \( X \) to a BCH-algebra \( Y \) is called a fuzzy homomorphism if
\[
\mu_f(x_1 \ast x_2, y_1 \ast y_2) = \sup_{y = y_1 \ast y_2} \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2)
\]  
(3.24)
for all \( x_1, x_2 \in X \) and \( y \in Y \).

One notes that if \( f \) is an ordinary map, then the above definition reduces to an ordinary homomorphism. One also observes that if a fuzzy map \( f \) is a fuzzy homomorphism, then the induced ordinary map \( f_1 \) is an ordinary homomorphism.

**Proposition 3.19.** Let \( f : X \to Y \) be a fuzzy homomorphism of BCH-algebras. Then
\begin{enumerate}
\item \( \mu_f(x_1 \ast x_2, y_1 \ast y_2) \leq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2) \) for all \( x_1, x_2 \in X \) and \( y_1, y_2 \in Y \).
\item \( \mu_f(0, 0) = 1 \).
\item \( \mu_f(0 \ast x, 0 \ast y) \geq \mu_f(x, y) \) for all \( x \in X \) and \( y \in Y \).
\item If \( Y \) is medial and \( \mu_f(x, y) = t \neq 0 \), then \( \mu_f(0, y_x \ast y) = t \) for all \( x \in X \) and \( y \in Y \), where \( y_x \in Y \) with \( \mu_f(x, y_x) = 1 \).
\end{enumerate}

**Proof.**
\begin{enumerate}
\item For every \( x_1, x_2 \in X \) and \( y_1, y_2 \in Y \), we have
\[
\mu_f(x_1 \ast x_2, y_1 \ast y_2) = \sup_{y_1 = y_1 \ast y_2} \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2)
\]  
(3.25)
\[
\geq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2).
\]

\item Let \( x \in X \) and \( y_x \in Y \) be such that \( \mu_f(x, y_x) = 1 \). Using (i) and (i), we get
\[
\mu_f(0, 0) = \mu_f(x \ast x, y_x \ast y_x) \geq \mu_f(x, y_x) \cdot \mu_f(x, y_x) = 1
\]  
(3.26)
and so \( \mu_f(0, 0) = 1 \).

\item The proof follows from (i) and (ii).

\item Assume that \( Y \) is medial and \( \mu_f(x, y) = t \neq 0 \) for all \( x \in X \) and \( y \in Y \), and let \( y_x \in Y \) be such that \( \mu_f(x, y_x) = 1 \). Then
\[
\mu_f(0, y_x \ast y) = \mu_f(x \ast x, y_x \ast y) \geq \mu_f(x, y_x) \cdot \mu_f(x, y)
\]
\[
= t = \mu_f(x, y_x) = \mu_f(x \ast 0, y_x \ast (y_x \ast y))
\]  
(3.27)
\[
\geq \mu_f(x, y_x) \cdot \mu_f(0, y_x \ast y) = \mu_f(0, y_x \ast y),
\]
and hence \( \mu_f(0, y_x \ast y) = t \). This completes the proof.
\end{enumerate}

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**References**
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