ABOUT INTERPOLATION OF SUBSPACES OF REARRANGEMENT INVARIANT SPACES GENERATED BY RADEMACHER SYSTEM

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Abstract. The Rademacher series in rearrangement invariant function spaces “close” to the space \( L_\infty \) are considered. In terms of interpolation theory of operators, a correspondence between such spaces and spaces of coefficients generated by them is stated. It is proved that this correspondence is one-to-one. Some examples and applications are presented.

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1. Introduction. Let

\[ r_k(t) = \text{sign} \sin 2^{k-1} \pi t \quad (k = 1, 2, \ldots) \quad (1.1) \]

be the Rademacher functions on the segment \([0, 1]\). Define the linear operator

\[ Ta(t) = \sum_{k=1}^{\infty} a_k r_k(t) \quad \text{for } a = (a_k)_{k=1}^{\infty} \in l_2. \quad (1.2) \]

It is well known (cf. [23, pages 340–342]) that \( Ta \) is an almost everywhere finite function on \([0, 1]\). Moreover, from Khintchine’s inequality it follows that

\[ \| Ta \|_{L_p} \asymp \| a \|_2 \quad \text{for } 1 \leq p < \infty, \quad (1.3) \]

where \( \| a \|_p = (\sum_{k=1}^{\infty} |a_k|^p)^{1/p} \). The symbol \( \asymp \) means the existence of two-sided estimates with constants depending only on \( p \). Also, it can easily be checked that

\[ \| Ta \|_{L_\infty} = \| a \|_1. \quad (1.4) \]

A more detailed information on the behaviour of Rademacher series can be obtained by treating them in the framework of general rearrangement invariant spaces.

Recall that a Banach space \( X \) of measurable functions \( x = x(t) \) on \([0, 1]\) is said to be a rearrangement invariant space (r.i.s.) if the inequality \( x^*(t) \leq y^*(t) \), for \( t \in [0, 1] \) and \( y \in X \), implies \( x \in X \) and \( \| x \| \leq \| y \| \). Here and in what follows \( z^*(t) \) is the nonincreasing rearrangement of a function \( |z(t)| \) with respect to the Lebesgue measure denoted by \( \text{meas} \) [10, page 83].

Important examples of r.i.s.’s are Marcinkiewicz and Orlicz spaces. Let \( \mathcal{P} \) denote the cone of nonnegative increasing concave functions on the semiaxis \((0, \infty)\).

If \( \varphi \in \mathcal{P} \), then the Marcinkiewicz space \( M(\varphi) \) consists of all measurable functions \( x = x(t) \) such that

\[ \| x \|_{M(\varphi)} = \sup \left\{ \frac{1}{\varphi(t)} \int_0^t x^*(s) \, ds : 0 < t \leq 1 \right\} < \infty. \quad (1.5) \]
If $S(t)$ is a nonnegative convex continuous function on $[0, \infty)$, $S(0) = 0$, then the Orlicz space $L_S$ consists of all measurable functions $x = x(t)$ such that
\[
\|x\|_S = \inf \left\{ u > 0 : \int_0^1 S\left( \frac{|x(t)|}{u} \right) \, dt \leq 1 \right\} < \infty.
\]

In particular, if $S(t) = t^p$ ($1 \leq p < \infty$), then $L_S = L^p$.

For any r.i.s. $X$ on $[0, 1]$ we have $L_\infty \subset X \subset L_1$ [10, page 124]. Let $X^0$ denote the closure of $L_\infty$ in an r.i.s. $X$.

In problems discussed below, a special role is played by the Orlicz space $L^N$, where $N(t) = \exp(t^2) - 1$ or, more precisely, by the space $G = L^0$. In [19], V. A. Rodin and E. M. Semenov proved a theorem about the equivalence of Rademacher system to the standard basis in the space $l^2$.

**Theorem 1.1.** Suppose that $X$ is an r.i.s. Then
\[
\|Ta\|_X = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X \asymp \|a\|_2
\]
if and only if $X \supset G$.

By Theorem 1.1, the space $G$ is the minimal space among r.i.s.'s $X$ such that the Rademacher system is equivalent in $X$ to the standard basis of $l^2$.

In this paper, we consider problems related to the behaviour of Rademacher series in r.i.s.'s intermediate between $L_\infty$ and $G$. Here a major role is played by concepts and methods of interpolation theory of operators.

For a Banach couple $(X_0, X_1)$, $x \in X_0 + X_1$ and $t > 0$, we introduce the Peetre $\mathcal{K}$-functional
\[
\mathcal{K}(t; x_0, x_1) = \inf \left\{ \|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \right\}.
\]

Let $Y_0$ be a subspace of $X_0$ and $Y_1$ a subspace of $X_1$. A couple $(Y_0, Y_1)$ is called a $\mathcal{K}$-subcouple of a couple $(X_0, X_1)$ if
\[
\mathcal{K}(t, y; Y_0, Y_1) \asymp \mathcal{K}(t, y; X_0, X_1),
\]
with constants independent of $y \in Y_0 + Y_1$ and $t > 0$.

In particular, if $Y_i = P(X_i)$, where $P$ is a linear projector bounded from $X_i$ into itself for $i = 0, 1$, then $(Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $(X_0, X_1)$ (see [3] or [21, page 136]). At the same time, there are many examples of subcouples that are not $\mathcal{K}$-subcouples (see [21, page 589], [22], and Remark 3.2 of this paper).

Let $T(l_1)$ (respectively $T(l_2)$) denote the subspace of $L_\infty$ (of $G$) consisting of all functions of the form $x = Ta$, where $T$ is given by (1.2) and $a \in l_1 (\in l_2)$. From (1.4) and Theorem 1.1 it follows that
\[
\mathcal{K}(t, Ta; T(l_1), T(l_2)) \asymp \mathcal{K}(t, a; l_1, l_2).
\]

In spite of the fact that $T(l_1)$ is uncomplemented in $L_\infty$ (see [17] or [11, page 134]) the following assertion holds.
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**Theorem 1.2.** The couple \((T(l_1),T(l_2))\) is a \(\mathcal{K}\)-subcouple of the couple \((L_\infty,G)\). In other words (see (1.10)),

\[
\mathcal{K}(t,Ta;L_\infty,G) \cong \mathcal{K}(t,a;l_1,l_2),
\]

with constants independent of \(a = (a_k)_{k=1}^\infty \in l_2\) and \(t > 0\).

We will use in the proof of Theorem 1.2 an assertion about the distribution of Rademacher sums. It was proved by S. Montgomery-Smith [13].

**Theorem 1.3.** There exists a constant \(A \geq 1\) such that for all \(a = (a_k)_{k=1}^\infty \in l_2\) and \(t > 0\)

\[
\begin{align*}
\text{meas} \left\{ s \in [0,1] : \sum_{k=1}^\infty a_k r_k(s) > \varphi_a(t) \right\} &\leq \exp \left(-\frac{t^2}{2}\right), \\
\text{meas} \left\{ s \in [0,1] : \sum_{k=1}^\infty a_k r_k(s) > A^{-1} \varphi_a(t) \right\} &\geq A^{-1} \exp (-A t^2),
\end{align*}
\]

where \(\varphi_a(t) = \mathcal{K}(t,a;L_1,l_1,l_2)\).

Now we need some definitions from interpolation theory of operators. We say that a linear operator \(U\) is bounded from a Banach couple \(X = (X_0,X_1)\) into a Banach couple \(Y = (Y_0,Y_1)\) (in short, \(U : X \rightarrow Y\)) if \(U\) is defined on \(X_0 + X_1\) and acts as bounded operator from \(X_i\) into \(Y_i\) for \(i = 0,1\).

Let \(X = (X_0,X_1)\) be a Banach couple. A space \(X\) such that \(X_0 \cap X_1 \subset X \subset X_0 + X_1\) is called an interpolation space between \(X_0\) and \(X_1\) if each linear operator \(U : X \rightarrow X\) is bounded from \(X\) into itself.

To every r.i.s. \(X\) assign the sequence space \(FX\) of Rademacher coefficients of functions of the form (1.2) from \(X\):

\[
\|(a_k)\|_{FX} = \left\| \sum_{k=1}^\infty a_k r_k \right\|_X.
\]

Well-known properties of Rademacher functions imply that \(FX\) is an r.i. sequence space [19]. Furthermore, Theorem 1.3 and properties of the \(\mathcal{K}\)-functional show that \(FX\) is an interpolation space between \(l_1\) and \(l_2\) (see the proof of Theorem 1.2 later). For interpolation r.i.s. between \(L_\infty\) and \(G\) the correspondence \(X \rightarrow FX\) can be defined by using the real interpolation method.

For every \(p \in [1,\infty]\), we denote by \(l_p(u_k), u_k \geq 0 (k = 0,1,\ldots)\) the space of all two-sided sequences of real numbers \(a = (a_k)_{k=-\infty}^\infty\) such that the norm \(\|a\|_{l_p(u_k)} = \|(a_k u_k)\|_p\) is finite. Let \(E\) be a Banach lattice of two-sided sequences, \((\min(1,2^k))_{k=-\infty}^\infty \in E\). If \((X_0,X_1)\) is a Banach couple, then the space of the real \(\mathcal{K}\)-method of interpolation \((X_0,X_1)_E\) consists of all \(x \in X_0 + X_1\) such that

\[
\|x\| = \|(\mathcal{K}(2^k,x;X_0,X_1))_k\|_E < \infty.
\]

It is readily checked that the space \((X_0,X_1)_E\) is an interpolation space between \(X_0\) and \(X_1\) (cf. [15, page 422]). In the special case \(E = l_p(2^{-k\theta}) (0 < \theta < 1, 1 \leq p \leq \infty)\) we obtain the spaces \((X_0,X_1)_{\theta,p}\) (for the detailed exposition of their properties see [4]).
A couple \( \overline{X} = (X_0, X_1) \) is said to be a \( \mathcal{K} \)-monotone couple if for every \( x \in X_0 + X_1 \) and \( y \in X_0 + X_1 \) there exists a linear operator \( U : \overline{X} \to \overline{X} \) such that \( y = Ux \) whenever
\[
\mathcal{K}(t, y; X_0, X_1) \leq \mathcal{K}(t, x; X_0, X_1) \quad \forall t > 0.
\] (1.15)

As it is well known (cf. [15, page 482]), any interpolation space \( X \) with respect to a \( \mathcal{K} \)-monotone couple \( (X_0, X_1) \) is described by the real \( \mathcal{K} \)-method. It means that for some \( E \)
\[
X = (X_0, X_1)^{\mathcal{K}}_E.
\] (1.16)

In particular, by the Sparr theorem [20] the couple \( (l_1, l_2) \) is a \( \mathcal{K} \)-monotone couple. Therefore, if \( F \) is an interpolation space between \( l_1 \) and \( l_2 \), then there exists \( E \) such that
\[
F = (l_1, l_2)^{\mathcal{K}}_E.
\] (1.17)

Hence Theorem 1.2 allows to find an r.i.s. that contains Rademacher series with coefficients belonging to an arbitrary interpolation space between \( l_1 \) and \( l_2 \). In [19], the similar result was obtained for sequence spaces satisfying more restrictive conditions (see Remark 3.3).

**Theorem 1.4.** Let \( F \) be an interpolation sequence space between \( l_1 \) and \( l_2 \) and \( F = (l_1, l_2)^{\mathcal{K}}_E \). Then for the r.i.s. \( X = (L_\infty, G)^{\mathcal{K}}_E \) we have
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \asymp \|a\|_F
\] (1.18)
with constants independent of \( a = (a_k)_{k=1}^{\infty} \).

Combining Theorem 1.4 with the above remarks, we get the following assertion. If \( F \) is a sequence space, then
\[
\| (a_k) \|_F \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_X \quad \text{for some r.i.s. } X
\] (1.19)
if and only if \( F \) is an interpolation space between \( l_1 \) and \( l_2 \).

The last result shows that the restriction of the correspondence (1.13) to interpolation r.i.s. between \( L_\infty \) and \( G \) is bijective.

**Theorem 1.5.** Let r.i.s.’s \( X_0 \) and \( X_1 \) be two interpolation spaces between \( L_\infty \) and \( G \). If
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_0} \asymp \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{X_1},
\] (1.20)
then \( X_0 = X_1 \) and the norms of \( X_0 \) and \( X_1 \) are equivalent.

In [16, 19], the similar results were obtained by additional conditions with respect to spaces \( X_0 \) and \( X_1 \).
2. Proofs

**Proof of Theorem 1.2.** It is known [10, page 164] that the \( \mathcal{K} \)-functional of a couple of Marcinkiewicz spaces is given by the formula

\[
\mathcal{K}(t,x;M(\varphi_0),M(\varphi_1)) = \sup_{0 < u \leq 1} \frac{\int_0^u x^*(s) \, ds}{\max(\varphi_0(u), \varphi_1(u)/t)}.
\]

(2.1)

If \( N(t) = \exp(t^2) - 1 \), then the Orlicz space \( L_N \) coincides with the Marcinkiewicz space \( M(\varphi_1) \), where \( \varphi_1(u) = u \log_2^{1/2}(2/u) \) [12]. In addition, \( L_\infty = M(\varphi_0) \), where \( \varphi_0(u) = u \).

Therefore,

\[
\mathcal{K}(t,x;L_\infty,G) = \sup_{0 < u \leq 1} \left( \frac{1}{u} \int_0^u x^*(s) \, ds \min \left( 1, t \log_2^{-1/2} \left( \frac{u}{t} \right) \right) \right) \quad \text{for } x \in G.
\]

(2.2)

Since \( x^*(u) \leq 1/u \int_0^u x^*(s) \, ds \), then from (2.2) it follows that

\[
\mathcal{K}(t,x;L_\infty,G) \geq \sup_{k=0,1,\ldots} \{ x^*(2^{-k}) \min \left( 1, t(k+1)^{-1/2} \right) \}.
\]

(2.3)

Hence,

\[
\mathcal{K}(t,x;L_\infty,G) \geq x^*(2^{-kt}) \quad \text{for } t \geq 1,
\]

(2.4)

where \( k_t = [t^2] - 1 \) ([z] is the integral part of a number z).

Now let \( a = (a_k)_{k=1}^\infty \in l_2 \) and \( x(t) = Ta(t) = \sum_{k=1}^\infty a_k r_k(t) \). By the Holmstedt formula [7],

\[
\varphi_a(t) \leq \sum_{k=1}^{[t^2]} a_k^2 + t \left\{ \sum_{k=[t^2]+1}^\infty (a_k^*)^2 \right\}^{1/2} \leq B \varphi_a(t),
\]

(2.5)

where \( \varphi_a(t) = \mathcal{K}(t,a;L_1,l_2) \), \( (a_k^*)_{k=1}^\infty \) is a nonincreasing rearrangement of the sequence \( (|a_k|)_{k=1}^\infty \), and \( B > 0 \) is a constant independent of \( a = (a_k)_{k=1}^\infty \) and \( t > 0 \).

Assume, at first, that \( a \notin l_1 \). Then inequality (2.5) shows that

\[
\lim_{t \to 0^+} \varphi_a(t) = 0, \quad \lim_{t \to \infty} \varphi_a(t) = \infty.
\]

(2.6)

The function \( \varphi_a \) belongs to the class \( \mathcal{P} \) [4, page 55]. Therefore it maps the semiaxis \( (0,\infty) \) onto \( (0,\infty) \) one-to-one, and there exists the inverse function \( \varphi_a^{-1} \). By Theorem 1.3, we have

\[
n|s| \left( \tau \right) = \max \{ s \in [0,1]: |x(s)| > \tau \} \geq \psi(\tau) \quad \text{for } \tau > 0,
\]

(2.7)

where \( \psi(\tau) = A^{-1} \exp\{ -A[\varphi_a^{-1}(\tau A)]^2 \} \). Passing to rearrangements we obtain

\[
x^*(s) \geq \psi^{-1}(s) \quad \text{for } 0 < s < A^{-1}.
\]

(2.8)

Obviously, by condition \( t \geq C_1 = C_1(A) = \sqrt{2 \log_2(2A)} \), it holds

\[
2^{-k_t/2} < A^{-1} \quad \text{for } k_t = [t^2] - 1.
\]

(2.9)

Hence (2.4) and (2.8) imply

\[
\mathcal{K}(t,x;L_\infty,G) \geq \psi^{-1}(2^{-k_t}).
\]

(2.10)
Combining the definition of the function $\psi$ with (2.9), we obtain
\[
\psi^{-1}(2^{-k_1}) = A^{-1} \varphi_a(A^{-1/2} \ln^{1/2} (A^{-1} 2^{k_1})) \geq A^{-1} \varphi_a\left(\sqrt{\frac{k_1 \ln 2}{2A}}\right)
\]
\[
\geq A^{-3/2} \sqrt{\frac{\ln 2}{2}} \varphi_a\left(\sqrt{k_1}\right) \geq A^{-3/2} t^{-1/2} \sqrt{k_1} \varphi_a(t).
\]
(2.11)

From the inequality $t \geq C_1 \geq \sqrt{2}$ it follows that
\[
\frac{\sqrt{k_1}}{t} \geq \frac{\sqrt{[t^2] - 1}}{\sqrt{[t^2] + 1}} \geq 3^{-1/2}.
\]
(2.12)

Therefore, by (2.10), we have
\[
\mathcal{K}(t,x;L_\infty,G) \geq C_2 \varphi_a(t) \quad \text{for} \quad t \geq C_1,
\]
(2.13)

where $C_2 = C_2(A) = \sqrt{\ln 2/6A^{-3/2}}$.

If now $t \geq 1$, then the concavity of the $\mathcal{K}$-functional and the previous inequality yield
\[
\mathcal{K}(t,x;L_\infty,G) \geq C_1^{-1}\mathcal{K}(tC_1,x;L_\infty,G) \geq \frac{C_2}{C_1} \varphi_a(C_1t) \geq \frac{C_2}{C_1} \varphi_a(t).
\]
(2.14)

Using the inequalities $\|a\|_2 \leq \|a\|_1$ ($a \in l_1$) and $\|x\|_G \leq \|x\|_\infty$ ($x \in L_\infty$), the definition of the $\mathcal{K}$-functional, and Theorem 1.1, we obtain
\[
\mathcal{K}(t,x;L_\infty,G) = t\|x\|_G \geq C_3 t\|a\|_2 = C_3 \varphi_a(t) \quad \text{for} \quad 0 < t \leq 1.
\]
(2.15)

Thus,
\[
\mathcal{K}(t,a;l_1,l_2) \leq C\mathcal{K}(t,Ta;L_\infty,G),
\]
(2.16)

if $C = \max(C_3^{-1},C_1/C_2)$.

Suppose now $a \in l_1$. By (2.5), without loss of generality, we can assume that the function $\varphi_a$ maps the semiaxis $(0, \infty)$ injectively onto the interval $(0, \|a\|_1)$. Hence we can define the mappings $\varphi_a^{-1} : (0,\|a\|_1) \rightarrow (0,\infty)$, $\psi : (0, A^{-1}\|a\|_1) \rightarrow (0, A^{-1})$, and $\psi^{-1} : (0, A^{-1}) \rightarrow (0, A^{-1}\|a\|_1)$. Arguing as above, we get inequality (2.16).

The opposite inequality follows from Theorem 1.1 and relation (1.4). Indeed,
\[
\mathcal{K}(t,Ta;L_\infty,G) \leq \inf \{ \|Ta^{a_0}\|_\infty + t\|Ta^1\|_G : a = a_0 + a^1, a_0 \in l_1, a^1 \in l_2 \}
\leq D\mathcal{K}(t,a;l_1,l_2).
\]
(2.17)

**Proof of Theorem 1.4.** It is sufficient to use Theorem 1.2 and the definition of the real $\mathcal{K}$-method of interpolation. 

For the proof of Theorem 1.5 we need some definitions and auxiliary assertions. These results are also of some independent interest.

Let $f(t)$ be a function defined on the interval $(0,l)$, where $l = 1$ or $l = \infty$. Then the dilation function of $f$ is defined as follows:
\[
\mathcal{M}_f(t) = \sup \left\{ \frac{f(st)}{f(s)} : s, st \in (0,l) \right\}, \quad \text{if} \quad t \in (0,l).
\]
(2.18)
Since this function is semimultiplicative, then there exist numbers

\[ \gamma_f = \lim_{t \to 0^+} \frac{\ln M_f(t)}{\ln t}, \quad \delta_f = \lim_{t \to \infty} \frac{\ln M_f(t)}{\ln t}. \]  

(2.19)

A Banach couple \( \vec{X} = (X_0, X_1) \) is called a partial retract of a couple \( \vec{Y} = (Y_0, Y_1) \) if each element \( x \in X_0 + X_1 \) is orbitally equivalent to some element \( y \in Y_0 + Y_1 \). The last means that there exist linear operators \( U : \vec{X} \to \vec{Y} \) and \( V : \vec{Y} \to \vec{X} \) such that \( Ux = y \) and \( Vy = x \).

**Proposition 2.1.** Suppose that \( M(\varphi) \) is a Marcinkiewicz space on \([0,1]\). If \( \gamma_\varphi > 0 \), then \( \vec{X} = (L_\infty, M(\varphi)) \) is a \( \mathcal{H}\)-monotone couple.

**Proof.** It is sufficient to show that the couple \( \vec{X} \) is a partial retract of the couple \( \vec{Y} = (L_\infty, L_\infty(\tilde{\varphi})) \), where

\[ \|x\|_{L_\infty(\tilde{\varphi})} = \sup_{0 < t \leq 1} \tilde{\varphi}(t) |x(t)|, \quad \tilde{\varphi}(t) = \frac{t}{\varphi}. \]  

(2.20)

Indeed, a partial retract of a \( \mathcal{H}\)-monotone couple is a \( \mathcal{H}\)-monotone couple \([15,\text{page} 420]\), and by the Sparr theorem \([20]\) \( \vec{Y} \) is a \( \mathcal{H}\)-monotone couple.

First note that the inclusion \( L_\infty \subset M(\varphi) \) implies \( L_\infty + M(\varphi) = M(\varphi) \). So, let \( x \in M(\varphi) \). Without loss of generality \([10, \text{page} 87]\), assume that \( x(t) = x^*(t) \). Define the operator

\[ U_1 y(t) = \sum_{k=1}^{\infty} 2^k \int_{2^{-k}}^{2^{-k+1}} y(s) ds x_{(2^{-k},2^{-k+1})}(t) \quad \text{for} \quad y \in M(\varphi). \]  

(2.21)

Clearly, \( U_1 \) maps \( L_\infty \) into itself. In addition, the concavity of the function \( \varphi \) and properties of the nonincreasing rearrangement imply

\[ \|U_1 y\|_{L_\infty(\tilde{\varphi})} \leq 2 \sup_{k=1,2,\ldots} (\varphi(2^{-k+1}))^{-1} \int_{0}^{2^{-k}} y^*(s) ds \leq 2 \|y\|_{M(\varphi)}. \]  

(2.22)

Hence \( U_1 : \vec{X} \to \vec{Y} \). Since \( x(t) \) is nonincreasing, then \( U_1 x(t) \geq x(t) \). Therefore the linear operator

\[ U y(t) = \frac{x(t)}{U_1 x(t)} U_1 y(t) \]  

(2.23)

is bounded from the couple \( \vec{X} \) into the couple \( \vec{Y} \). In addition, \( Ux(t) = x(t) \).

Take for \( V \) the identity mapping, that is, \( V y(t) = y(t) \). Since \( \gamma_f > 0 \), then, by \([10, \text{page} 156]\), we have

\[ \|Vy\|_{M(\varphi)} \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) y^*(t) \leq C \sup_{0 < t \leq 1} \tilde{\varphi}(t) |y(t)| = C \|y\|_{L_\infty(\tilde{\varphi})}. \]  

(2.24)

Therefore \( V : \vec{Y} \to \vec{X} \) and \( VX = x \).

Thus an arbitrary element \( x \in M(\varphi) \) is orbitally equivalent to itself as to element of the space \( L_\infty + L_\infty(\tilde{\varphi}) \). This completes the proof. \( \Box \)
**Corollary 2.2.** If $\gamma_{\phi} > 0$, then $(L_{\infty}, M(\phi))$ is a $\mathcal{K}$-monotone couple.

**Proof.** Assume that $x$ and $y$ belong to the space $M(\phi)$ and
\[
\mathcal{K}(t, y; L_{\infty}, M(\phi)) \leq \mathcal{K}(t, x; L_{\infty}, M(\phi)) \quad \text{for } t > 0. \tag{2.25}
\]

If $z \in M(\phi)$, then
\[
\mathcal{K}(t, z; L_{\infty}, M(\phi)) = \mathcal{K}(t, z; L_{\infty}, M(\phi)). \tag{2.26}
\]

Therefore,
\[
\mathcal{K}(t, y; L_{\infty}, M(\phi)) \leq \mathcal{K}(t, x; L_{\infty}, M(\phi)) \quad \text{for } t > 0. \tag{2.27}
\]

Hence, by Proposition 2.1, there exists an operator $T : (L_{\infty}, M(\phi)) \to (L_{\infty}, M(\phi))$ such that $y = Tx$. It is readily seen that $M(\phi)$ is an interpolation space of the couple $(L_{\infty}, M(\phi))$. Therefore $T : (L_{\infty}, M(\phi)) \to (L_{\infty}, M(\phi))$.

We define now two subcones of the cone $\mathcal{P}$. Denote by $\mathcal{P}_0$ the set of all functions $f \in \mathcal{P}$ such that $\lim_{t \to 0} f(t) = \lim_{t \to \infty} f(t)/t = 0$. If $f \in \mathcal{P}$, then $0 \leq \gamma_f \leq \delta_f \leq 1$ [10, page 76]. Let $\mathcal{P}^{+-}$ be the set of all $f \in \mathcal{P}$ such that $0 < \gamma_f \leq \delta_f < 1$. It is obvious that $\mathcal{P}^{+-} \subset \mathcal{P}_0$.

A couple $(X_0, X_1)$ is called a $\mathcal{K}_0$-complete couple if for any function $f \in \mathcal{P}_0$ there exists an element $x \in X_0 + X_1$ such that
\[
\mathcal{K}(t, x; X_0, X_1) = f(t). \tag{2.28}
\]

In other words, the set $\mathcal{K}(X_0 + X_1)$ of all $\mathcal{K}$-functionals of a $\mathcal{K}_0$-complete couple $(X_0, X_1)$ contains, up to equivalence, the whole of the subcone $\mathcal{P}_0$.

**Proposition 2.3.** The Banach couple $(L_1(0, \infty), L_2(0, \infty))$ is a $\mathcal{K}_0$-complete couple.

**Proof.** By the Holmstedt formula for functional spaces [7],
\[
\mathcal{K}(t, x; L_1, L_2) \equiv \max \left\{ \int_0^t x^*(s) ds, t \left[ \int_t^\infty (x^*(s))^2 ds \right]^{1/2} \right\}. \tag{2.29}
\]

If $f \in \mathcal{P}_0$, then $g(t) = f(t^{1/2})$ belongs to $\mathcal{P}_0$. We denote $x(t) = g'(t)$. Then $x(t) = x^*(t)$ and
\[
\int_0^t x(s) ds = g(t). \tag{2.30}
\]

Assume that $f \in \mathcal{P}^{+-}$. If $\delta_f < 1$, then there exists $\varepsilon > 0$ such that for some $C > 0$
\[
G(s) = f(s^{1/2}) \leq C \left( \sqrt{\frac{s}{t}} \right)^{1-\varepsilon} f(t^{1/2}), \quad \text{if } s \geq t. \tag{2.31}
\]

Since $g \in \mathcal{P}_0$, then $g'(t) \leq g(t)/t$. Therefore for $t > 0$
\[
\int_t^\infty (x(s))^2 ds \leq \int_t^\infty \frac{g^2(s)}{s^2} ds \leq C^2 t^{\varepsilon-1} (f(t^{1/2}))^2 \int_t^\infty s^{-1-\varepsilon} ds = C^2 \varepsilon t^{-1} (g(t))^2. \tag{2.32}
\]

Combining this with (2.29) and (2.30), we obtain
\[
\mathcal{K}(t, x; L_1, L_2) \approx g(t^2) = f(t). \tag{2.33}
\]
Thus $\mathcal{H}(L_1 + L_2) \supset \mathcal{P}^{++}$. Hence, in particular, the intersection $\mathcal{H}(X_0 + X_1) \cap \mathcal{P}^{++}$ is not empty. Therefore, by [6, Theorem 4.5.7], $(L_1, L_2)$ is a $\mathcal{H}_0$-complete Banach couple. This completes the proof.

Let $\mathcal{H}(l_1 + l_2)$ be the set of all $\mathcal{H}$-functionals corresponding to the couple $(l_1, l_2)$. By $F$ we denote the set of all functions $f \in \mathcal{P}$ such that

$$f(t) = f(1) t \quad \text{for } 0 < t \leq 1, \quad \lim_{t \to \infty} \frac{f(t)}{t} = 0. \quad (2.34)$$

**Corollary 2.4.** Up to equivalence,

$$\mathcal{H}(l_1 + l_2) \supset F. \quad (2.35)$$

**Proof.** It is well known (cf. [4, page 142]) that for $x \in L_1(0, \infty) + L_\infty(0, \infty)$ and $u > 0$

$$\mathcal{H}(u, x; L_1, L_\infty) = \int_0^u x^*(s) \, ds. \quad (2.36)$$

In addition,

$$L_1 = (L_1, L_\infty)_{l_1}^\mathcal{X}, \quad L_2 = (L_1, L_\infty)_{l_2}^{(2^{-k/2})}. \quad (2.37)$$

The spaces $l_\infty$ and $l_2(2^{-k/2})$ are interpolation spaces with respect to the couple $(l_\infty, l_\infty(2^{-k}))$ [4]. Therefore, by the reiteration theorem (see [5] or [14]),

$$\mathcal{H}(t, x; L_1, L_2) \simeq \mathcal{H}(t, x; l_\infty, l_\infty(2^{-k/2})); \quad \forall x \in L_1 + L_2. \quad (2.38)$$

Introduce the average operator:

$$Q x(t) = \sum_{k=1}^\infty \int_{k-1}^k x(s) \, ds \chi(k-1, k](t), \quad \text{if } t > 0. \quad (2.39)$$

From (2.36) it follows that

$$\mathcal{H}(t, Q x^*; L_1, L_\infty) = \mathcal{H}(t, x^*; L_1, L_\infty) \quad (2.40)$$

for all positive integers $t$. Both functions in (2.40) are concave. Therefore,

$$\mathcal{H}(t, Q x^*; L_1, L_\infty) \simeq \mathcal{H}(t, x; L_1 \cdot L_\infty) \quad \forall t \geq 1. \quad (2.41)$$

Hence (2.38) yields

$$\mathcal{H}(t, Q x^*; L_1, L_2) \simeq \mathcal{H}(t, x; L_1, L_2), \quad \text{if } t \geq 1. \quad (2.42)$$

Now let $f \in F$. Since $F \subset P_0$, then, by Proposition 2.3, there exists a function $x \in L_1(0, \infty) + L_2(0, \infty)$ such that

$$\mathcal{H}(t, x; L_1, L_2) \simeq f(t). \quad (2.43)$$

Clearly, the operator $Q$ is a projector in the spaces $L_1$ and $L_2$ with norm 1. Moreover, $Q(L_1) = l_1$ and $Q(L_2) = l_2$. Hence, by the theorem about complemented subcouples
mentioned in Section 1 (see [3] or [21, page 136]),
\[ \mathcal{H}(t, Qx^*; L_1, L_2) \sim \mathcal{H}(t, a; l_1, l_2) \quad \text{for } t > 0, \]  
(2.44)

where \( a = \left( \int_{k-1}^k x^*(s) \, ds \right)_{k=1}^\infty. \)

Thus (2.42) and (2.43) imply
\[ \mathcal{H}(t, a; l_1, l_2) \sim f(t) \quad \text{for } t \geq 1. \]  
(2.45)
The last relation also holds if \( 0 < t \leq 1. \) Indeed, in this case
\[ \mathcal{H}(t, a; l_1, l_2) = t \langle a \rangle_2 = t \mathcal{H}(1, a; l_1, l_2) \times t f(1) = f(t). \]  
(2.46)

This completes the proof.

**Proof of Theorem 1.5.** As it was already mentioned in the proof of Theorem 1.2, the Orlicz space \( L_N, N(t) = \exp(t^2) - 1, \) coincides with the Marcinkiewicz space \( M(\varphi_1), \) for \( \varphi_1(u) = u \log_2^1(2/u). \) Since \( \gamma_{\varphi_1} = 1, \) then Corollary 2.2 implies that the couple \( (L_\infty, G) \) is a \( \mathcal{H} \)-monotone couple. Hence,
\[ X_0 = (L_\infty, G)^\mathcal{H}_{E_0}, \quad X_1 = (L_\infty, G)^\mathcal{H}_{E_1}, \]  
(2.47)

for some parameters of the real \( \mathcal{H} \)-method of interpolation \( E_0 \) and \( E_1. \) By Theorem 1.4,
\[ \left\| \sum_{k=1}^\infty a_k r_k \right\|_{X_i}\sim \| (a_k) \|_{F_i}, \]  
(2.48)

where \( F_i = (l_1, l_2)^\mathcal{H}_{E_i} \) (\( i = 0, 1 \)). So
\[ (l_1, l_2)^\mathcal{H}_{E_0} = (l_1, l_2)^\mathcal{H}_{E_1}. \]  
(2.49)

Equation (2.49) means that the norms of spaces \( E_0 \) and \( E_1 \) are equivalent on the set \( \mathcal{H}(l_1 + l_2). \) It is readily to check that this set coincides, up to the equivalence, with the set \( \mathcal{H}(L_\infty + G) \) of all \( \mathcal{H} \)-functionals corresponding to the couple \( (L_\infty, G). \) More precisely,
\[ \mathcal{H}(l_1 + l_2) = \mathcal{H}(L_\infty + G) = \mathcal{F}. \]  
(2.50)

In fact, by Theorem 1.2 and Corollary 2.2, \( \mathcal{F} \subset \mathcal{H}(l_1 + l_2) \subset \mathcal{H}(L_\infty + G). \) On the other hand, since \( L_\infty \subset G \) with the constant 1 and \( L_\infty \) is dense in \( G, \) then \( \mathcal{H}(L_\infty + G) \subset \mathcal{F} [15, \text{ page 386}]. \)

Now let \( x \in X_0. \) By (2.47), we have \((\mathcal{H}(2^k, x; L_\infty, G))_k \in X_0. \) Using (2.50), we can find \( a \in l_2 \) such that
\[ \mathcal{H}(2^k, a; l_1, l_2) \sim \mathcal{H}(2^k, x; L_\infty, G) \]  
(2.51)

for all positive integers \( k. \) Since a parameter of \( \mathcal{H} \)-method is a Banach lattice, then this implies \((\mathcal{H}(2^k, a; l_1, l_2))_k \in E_0. \) Therefore, by (2.49), \((\mathcal{H}(2^k, a; l_1, l_2))_k \in E_1, \) that is, \((\mathcal{H}(2^k, x; L_\infty, G))_k \in E_1 \) or \( x \in X_1. \) Thus \( X_0 \subset X_1. \) Arguing as above, we obtain the converse inclusion, and \( X_0 = X_1 \) as sets. Since \( X_0 \) and \( X_1 \) are Banach lattices, then their norms are equivalent. This completes the proof. \( \square \)
3. Final remarks and examples

**Remark 3.1.** Combining Theorems 1.2, 1.4, and 1.5 with results obtained in [8], we can prove similar assertions for lacunary trigonometric series. Moreover, taking into account the main result of [1], we can extend Theorems 1.2, 1.4, and 1.5 to Sidon systems of characters of a compact abelian group.

**Remark 3.2.** In Theorem 1.2, we cannot replace the space $G$ by $L_q$ with some $q < \infty$. Indeed, suppose that the couple $(T(l_1), T(l_2))$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, L_q)$, that is,
\[
\mathcal{K}(t, a; l_1, l_2) \cong \mathcal{K}(t, Ta; L_\infty, L_q).
\]
Let $E = l_p(2^{-\theta k})$, where $0 < \theta < 1$ and $p = q/\theta$. Applying the $\mathcal{K}$-method of interpolation $(\cdot, \cdot)^E_{\mathcal{K}}$ to the couples $(l_1, l_2)$ and $(L_\infty, L_q)$, we obtain
\[
\|Ta\|_p \cong \|a\|_{r,p} = \left\{ \sum_{k=1}^{\infty} (a_k^*)^{p/r} k^{p/r-1} \right\}^{1/p}.
\]
(3.2)
Since $r = 2/(2-\theta) < 2$ [4, page 142], then this contradicts with (1.3).

**Remark 3.3.** Clearly, a partial retract of a couple $\overrightarrow{Y} = (Y_0, Y_1)$ is a $\mathcal{K}$-subcouple of $\overrightarrow{Y}$. The opposite assertion is not true, in general (nevertheless, some interesting examples of $\mathcal{K}$-subcouples and partial retracts simultaneously are given in [9]). Indeed, by Theorem 1.2, the subcouple $(l_1, l_2)$ is a $\mathcal{K}$-subcouple of the couple $(L_\infty, G)$. Assume that $(l_1, l_2)$ is a partial retract of this couple. Then (see the proof of Proposition 2.1), $(l_1, l_2)$ is a partial retract of the couple $(L_\infty, L_\infty(\log_2^{-1/2}(2/t)))$, as well. Therefore, by Lemma 1 from [2] and [4, page 142] it follows that
\[
[l_1, l_2]_\theta = (l_1, l_2)_{\theta, \infty} = l_p, \infty,
\]
(3.3)
where $[l_1, l_2]_\theta$ is the space of the complex method of interpolation [4], $0 < \theta < 1$, and $p = 2/(2-\theta)$. On the other hand, it is well known [4, page 139] that
\[
[l_1, l_2]_\theta = l_p \quad \text{for} \quad p = \frac{2}{2-\theta}.
\]
(3.4)
This contradiction shows that the couple $(l_1, l_2)$ is not a partial retract of the couple $(L_\infty, G)$.

Using Theorem 1.4, we can find coordinate sequence spaces of coefficients of Rademacher series belonging to certain r.i.s.’s.

**Example 3.4.** Let $X$ be the Marcinkiewicz space $M(\varphi)$, where $\varphi(t) = t \log_2 \log_2(16/t)$, $0 < t \leq 1$. Show that
\[
\left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{M(\varphi)} \cong \|a\|_{l_1(\log)},
\]
(3.5)
where $l_1(\log)$ is the space of all sequences $a = (a_k)_{k=1}^{\infty}$ such that the norm
\[
\|a\|_{l_1(\log)} = \sup_{k=1, 2, \ldots} \log_2^{-1} (2k) \sum_{i=1}^{k} a_i
\]
(3.6)
is finite. Taking into account Theorem 1.4, it is sufficient to check that

\[
(l_1, l_2)^\mathcal{K} = l_1 (\log),
\]

\[
(l_\infty, G)^\mathcal{K} = M(q),
\]

for some parameter \( F \) of the \( \mathcal{K} \)-method of interpolation. More precisely, we will prove that (3.7) and (3.8) are true for \( F = l_\infty (u_k) \), where \( u_k = 1 / (k + 1) \) \((k \geq 0)\) and \( u_k = 1 \) \((k < 0)\).

By the Holmstedt formula (2.5),

\[
\varphi_a(2^k) \leq \sum_{i=1}^{2k} a_i^* + 2^k \left[ \sum_{i=1}^\infty \left( a_i^* \right)^2 \right]^{1/2} \leq B\varphi_a(2^k) \quad \text{for } k = 0, 1, 2, \ldots, \quad (3.9)
\]

where, as before, \( \varphi_a(t) = \mathcal{K}(t; a_l l_1, l_2) \). Without loss of generality, assume that \( a_i = a_i^* \). If \( \|a\|_{l_1(\log)} = R < \infty \), then by (3.6),

\[
\sum_{i=1}^{2k} a_i^* \leq 2R(k + 1). \quad (3.10)
\]

In particular, this implies \( a_{2^k} \leq 2^{-2k+1}R(k + 1) \), for nonnegative integer \( k \). Using (3.10), we obtain

\[
\sum_{i=1}^{2k} a_i^2 = \sum_{j=k}^{2j+1} a_i^2 \leq 3 \sum_{j=k}^{2j} a_{2^j}^2 \leq 12R^2 \sum_{j=k}^{\infty} 2^{-2j} (j + 1)^2 \leq 192R^2 \int_{k+1}^{\infty} x^2 2^{-2x} dx \leq 144R^2 (k + 1)^2 2^{-2k}. \quad (3.11)
\]

Hence the second term in (3.9) does not exceed \( 12R(k + 1) \). Therefore, if \( E = (l_1, l_2)^\mathcal{K}_F \), then (3.10) implies

\[
\|a\|_E = \sup_{k=0,1,\ldots} \frac{\varphi_a(2^k)}{k + 1} \leq 14\|a\|_{l_1(\log)}. \quad (3.12)
\]

Conversely, if \( 2^j + 1 \leq k \leq 2^{j+1} \) for some \( j = 0, 1, 2, \ldots \), then from (3.9) it follows that

\[
\sum_{i=1}^{k} a_i \leq B\varphi_a(2^{j+1}) \leq \sum_{i=1}^{2^{j+1}} a_i \leq B\|a\|_E (j + 2) \leq 2B\log_2(2k)\|a\|_E. \quad (3.13)
\]

Therefore, \( \|a\|_{l_1(\log)} \leq 2B\|a\|_E \) and (3.7) is proved.

We pass now to function spaces. At first, we introduce one more interpolation method which is, actually, a special case of the real method of interpolation. For a function \( \varphi \in \mathcal{P} \) and an arbitrary Banach couple \((X_0, X_1)\) define generalized Marcinkiewicz space as follows:

\[
M_{\varphi}(X_0, X_1) = \left\{ x \in X_0 + X_1 : \sup_{t>0} \frac{\mathcal{K}(t, x; X_0, X_1)}{\varphi(t)} < \infty \right\}. \quad (3.14)
\]
Let \( q_0(t) = \min(1,t) \), \( q_1(t) = \min(1,t \log_2^{1/2} [\max(2,2/t)]) \), and \( N(t) = \exp(t^2) - 1 \), as before. By equation (2.36), we have

\[
L_\infty = M_{q_0}(L_1,L_\infty), \quad L_N = M_{q_1}(L_1,L_\infty),
\]

(3.15)

(here \( L_\infty \) and \( L_N \) are functional spaces on the segment \([0,1]\)). In addition, using similar notation, it is easy to check that

\[
(X_0,X_1)^{\infty}_F = M_\rho(X_0,X_1),
\]

(3.16)

for an arbitrary Banach couple \((X_0,X_1)\) and \( \rho(t) = \log_2(4 + t) \). Hence, by the reiteration theorem for generalized Marcinkiewicz spaces [15, page 428], we obtain

\[
(L_\infty,L_N)^{\infty}_F = M_\rho(M_{q_0}(L_1,L_\infty),M_{q_1}(L_1,L_\infty)) = M_{\rho(t)}(L_1,L_\infty) = M(\rho(t)),
\]

(3.17)

where \( \rho(t) = q_0(t) \rho(q_1(t)/q_0(t)) \). A simple calculation gives \( \rho(t) \approx \rho(t) \), if \( t > 0 \). Thus,

\[
(L_\infty,L_N)^{\infty}_F = M(\rho).
\]

(3.18)

It is readily seen that \( \mathcal{K}(t,x;L_\infty,G) = \mathcal{K}(t,x;L_\infty,L_N) \), for all \( x \in G \). Therefore, for such \( x \) the norm \( \|x\|_{M(\rho)} \) is equal to the norm \( \|x\|_Y \), where \( Y = (L_\infty,G)^{\infty}_F \). On the other hand, for \( x \in M(\rho) \)

\[
\frac{1}{t \log_2^{1/2}(2/t)} \int_0^t x^*(s) \, ds \leq \|x\|_{M(\rho)} \frac{\log_2 \log_2(16/t)}{\log_2^{1/2}(2/t)} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0 + .
\]

(3.19)

This implies that \( M(\rho) \subset G \) [10, page 156]. Thus \( Y = M(\rho) \), and (3.8) is proved. Equivalence (3.5) follows now, as already stated, from (3.7) and (3.8).

**Remark 3.5.** Theorems 1.4 and 1.5 strengthen results of [18, 19], where similar assertions are obtained for sequence spaces \( F \) satisfying more restrictive conditions. For instance, we can readily show that the norm of the dilation operator

\[
\sigma_n a = \left( a_1, \ldots, a_n^{1/n}, a_2, \ldots \right)
\]

(3.20)
in the space \( l_1(\ln) \) (see Example 3.6) is equal to \( n \). Therefore, condition (11) from [19] fails for this space and the theorems obtained in [18, 19] cannot be applied to it. Similarly, the Marcinkiewicz space \( M(\rho) \) from Example 3.4 does not satisfy the conditions of Theorem 8 of [19].

Using Theorems 1.4 and 1.5, we can derive certain interpolation relations.

**Example 3.6.** Let \( \varphi \in \mathcal{P} \) and \( 1 \leq p < \infty \). Recall that the Lorentz space \( \Lambda_p(\varphi) \) consists of all measurable functions \( x = x(s) \) such that

\[
\|x\|_{\varphi,p} = \left\{ \int_0^1 (x^*(s))^p \, d\varphi(s) \right\}^{1/p} < \infty.
\]

(3.21)
In [19], V. A. Rodin and E. M. Semenov proved that
\[ \left\| \sum_{k=1}^{\infty} a_k r_k \right\|_{\varphi,p} \approx \left\| (a_k) \right\|_{\varphi,p}, \] (3.22)
where \( \varphi(s) = \log^{1-p}(2/s) \) and \( 1 < p < 2 \). Moreover, the space \( \Lambda_p(\varphi) \) is the unique r.i.s. having this property. Note that \( l_p = (l_1,l_2)_{\theta,p} \), where \( \theta = 2(p-1)/p \) [4, page 142]. Therefore, by Theorem 1.4, we obtain
\[ (L_\infty,G)_{\theta,p} = \Lambda_p(\varphi) \] (3.23)
for the same \( p \) and \( \theta \).

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**References**


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