SOME APPLICATIONS OF MINIMAL OPEN SETS

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ABSTRACT. We characterize minimal open sets in topological spaces. We show that any nonempty subset of a minimal open set is pre-open. As an application of a theory of minimal open sets, we obtain a sufficient condition for a locally finite space to be a pre-Hausdorff space.

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1. Introduction. Let $X$ be a topological space. We call a nonempty open set $U$ of $X$ a minimal open set when the only open subsets of $U$ are $U$ and $\emptyset$.

In this paper, we study fundamental properties of minimal open sets and apply them to obtain some results on pre-open sets (cf. [2]) and pre-Hausdorff spaces.

In Section 2, we characterize minimal open sets, that is, we show that a nonempty open set $U$ is a minimal open set if and only if $\text{Cl}(U) = \text{Cl}(S)$ for any nonempty subset $S$ of $U$. This result implies that any nonempty subset $S$ of a minimal open set $U$ is a pre-open set.

In Section 3, we study minimal open sets in locally finite spaces. The results of this section are closely related to the work of James [1], and these results will be used in the next section.

In Section 4, we apply the theory of minimal open sets to study pre-open sets. Our first main result of this section is a property of the set of all minimal open sets in any nonempty finite open set which is not a minimal open set. This result enables us to prove a generalization of Theorem 2.5, when $U$ is a nonempty finite open set, in Theorem 4.4. Theorem 4.5 shows that our theory of minimal open set is useful to study pre-open sets.

Finally, we show that some conditions on minimal open sets implies pre-Hausdorffness of a space, that is, if any minimal open set of a locally finite space $X$ has two elements at least, then $X$ is a pre-Hausdorff space.

2. Minimal open sets. Let $(X, \tau)$ be a topological space.

**Definition 2.1.** A nonempty open set $U$ of $X$ is said to be a minimal open set if and only if any open set which is contained in $U$ is $\emptyset$ or $U$.

**Lemma 2.2.** (1) Let $U$ be a minimal open set and $W$ an open set. Then $U \cap W = \emptyset$ or $U \subseteq W$.

(2) Let $U$ and $V$ be minimal open sets. Then $U \cap V = \emptyset$ or $U = V$. 
Proof. (1) Let $W$ be an open set such that $U \cap W \neq \emptyset$. Since $U$ is a minimal open set and $U \cap W \subset U$, we have $U \cap W = U$. Therefore $U \subset W$.

(2) If $U \cap V \neq \emptyset$, then we see that $U \subset V$ and $V \subset U$ by (1). Therefore $U = V$.

Proposition 2.3. Let $U$ be a minimal open set. If $x$ is an element of $U$, then $U \subset W$ for any open neighborhood $W$ of $x$.

Proof. Let $W$ be an open neighborhood of $x$ such that $U \not\subset W$. Then $U \cap W$ is an open set such that $U \cap W \subset U$ and $U \cap W \neq \emptyset$. This contradicts our assumption that $U$ is a minimal open set.

Proposition 2.4. Let $U$ be a minimal open set. Then

$$U = \cap \{ W \mid W \text{ is an open neighborhood of } x \}$$

(2.1)

for any element $x$ of $U$.

Proof. By Proposition 2.3 and the fact that $U$ is an open neighborhood of $x$, we have $U \subset \cap \{ W \mid W \text{ is an open neighborhood of } x \} \subset U$. Therefore we have the result.

Theorem 2.5. Let $U$ be a nonempty open set. Then the following three conditions are equivalent:

(1) $U$ is a minimal open set.

(2) $U \subset \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

(3) $\text{Cl}(U) = \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

Proof. (1)$\Rightarrow$(2). Let $S$ be any nonempty subset of $U$. By Proposition 2.3, for any element $x$ of $U$ and any open neighborhood $W$ of $x$, we have

$$S = U \cap S \subset W \cap S.$$  

(2)$\Rightarrow$(3). For any nonempty subset $S$ of $U$, we have $\text{Cl}(S) \subset \text{Cl}(U)$. On the other hand, by (2), we see $\text{Cl}(U) \subset \text{Cl}(\text{Cl}(S)) = \text{Cl}(S)$. Therefore we have $\text{Cl}(U) = \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

(3)$\Rightarrow$(1). Suppose that $U$ is not a minimal open set. Then there exists a nonempty open set $V$ such that $V \subset U$ and hence there exists an element $a \in U$ such that $a \notin V$. Then we have $\text{Cl}\{a\} \subset V^c$, the complement of $V$. It follows that $\text{Cl}\{a\} \neq \text{Cl}(U)$.

A subset $M$ of a space $(X, \tau)$ is called a pre-open set if $M \subset \text{Int}\text{Cl}(M)$. The family of all pre-open sets in $(X, \tau)$ will be denoted by $\text{PO}(X, \tau)$, (cf. [2]).

A space $(X, \tau)$ is called pre-Hausdorff if for each $x, y \in X$, $x \neq y$ there exist subsets $U, V \in \text{PO}(X, \tau)$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Theorem 2.6. Let $U$ be a minimal open set. Then any nonempty subset $S$ of $U$ is a pre-open set.

Proof. By Theorem 2.5(2), we have $\text{Int}U \subset \text{Int}\text{Cl}(S)$. Since $U$ is an open set, we have $S \subset U = \text{Int}(U) \subset \text{Int}\text{Cl}(S)$. 


Theorem 2.7. Let $U$ be a minimal open set and $M$ a nonempty subset of $X$. If there exists an open neighborhood $W$ of $M$ such that $W \subset \text{Cl}(M \cup U)$, then $M \cup S$ is a pre-open set for any nonempty subset $S$ of $U$.

Proof. By Theorem 2.5(3), we have $\text{Cl}(M \cup S) = \text{Cl}(M) \cup \text{Cl}(S) = \text{Cl}(M \cup U)$. Since $W \subset \text{Cl}(M \cup U) = \text{Cl}(M \cup S)$ by assumption, we have $\text{Int}(W) \subset \text{Int}\text{Cl}(M \cup S)$. Since $W$ is an open neighborhood of $M$, namely $W$ is an open set such that $M \subset W$, we have $M \subset W = \text{Int}(W) \subset \text{Int}\text{Cl}(M \cup S)$. Moreover we have $\text{Int}(U) \subset \text{Int}\text{Cl}(M \cup U)$, for $\text{Int}(U) = U \subset \text{Cl}(U) \subset \text{Cl}(M) \cup \text{Cl}(U) = \text{Cl}(M \cup U)$. Since $U$ is an open set, we have $S \subset U = \text{Int}U \subset \text{Int}\text{Cl}(M \cup U) = \text{Int}\text{Cl}(M \cup S)$. Therefore $M \cup S \subset \text{Int}\text{Cl}(M \cup S)$.

Corollary 2.8. Let $U$ be a minimal open set and $M$ a nonempty subset of $X$. If there exists an open neighborhood $W$ of $M$ such that $W \subset \text{Cl}(U)$, then $M \cup S$ is a pre-open set for any nonempty subset $S$ of $U$.

Proof. By assumption, we have $W \subset \text{Cl}(M) \cup \text{Cl}(U) = \text{Cl}(M \cup U)$. So by Theorem 2.7, we see that $M \cup S$ is a pre-open set.

The condition of Theorem 2.7, namely $W \subset \text{Cl}(M \cup S)$, does not necessarily imply the condition of Corollary 2.8, namely $W \subset \text{Cl}(S)$. We have the following example.

Example 2.9. Let $X = \{a, b, c, d\}$ with topology $\emptyset = \{\emptyset, \{d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$, $X\}$, $U = \{a, b\}$ and $M = W = \{d\}$. Then $W = \{d\} \subset \text{Cl}(\{a, b\} \cup \{d\}) = \text{Cl}(M \cup U)$ and $W = \{d\} \not\subset \text{Cl}(\{a, b\}) = \text{Cl}(U)$.

Theorem 2.10. Let $U$ be a minimal open set and $x$ an element of $X \setminus U$. Then $W \cap U = \emptyset$ or $U \subset W$ for any open neighborhood $W$ of $x$.

Proof. Since $W$ is an open set, we have the result by Lemma 2.2.

Corollary 2.11. Let $U$ be a minimal open set and $x$ an element of $X \setminus U$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then $U_x \cap U = \emptyset$ or $U \subset U_x$.

Proof. If $U \subset W$ for any open neighborhood $W$ of $x$, then $U \cap \cap\{W \mid W$ is an open neighborhood of $x\}$. Therefore $U \subset U_x$. Otherwise there exists an open neighborhood $W$ of $x$ such that $W \cap U = \emptyset$. Then we have $U \cap U_x = \emptyset$.

3. Finite open sets. In this section, we study some properties of minimal open sets in finite open sets and locally finite spaces.

Theorem 3.1. Let $V$ be a nonempty finite open set. Then there exists at least one (finite) minimal open set $U$ such that $U \subset V$.

Proof. If $V$ is a minimal open set, we may set $U = V$. If $V$ is not a minimal open set, then there exists an (finite) open set $V_1$ such that $\emptyset \neq V_1 \subset V$. If $V_1$ is a minimal open set, we may set $U = V_1$. If $V_1$ is not a minimal open set, then there exists an (finite) open set $V_2$ such that $\emptyset \neq V_2 \subset V_1 \subset V$. Continuing this process, we have a sequence of open sets

$$V \ni V_1 \ni V_2 \cdots \ni V_k \ni \cdots$$

(3.1)
Since $V$ is a finite set, this process repeats only finitely. Then, finally we get a minimal open set $U = V_n$ for some positive integer $n$.

A topological space is said to be a \textit{locally finite space} if each of its elements is contained in a finite open set.

\textbf{Corollary 3.2.} Let $X$ be a locally finite space and $V$ a nonempty open set. Then there exists at least one (finite) minimal open set $U$ such that $U \subset V$.

\textbf{Proof.} Since $V$ is a nonempty set, there exists an element $x$ of $V$. Since $X$ is a locally finite space, we have a finite open set $V_x$ such that $x \in V_x$. Since $V \cap V_x$ is a finite open set, we get a minimal open set $U$ such that $U \subset V \cap V_x \subset V$ by Theorem 3.1.

\textbf{Theorem 3.3.} Let $V_\lambda$ be an open set for any $\lambda \in \Lambda$ and $W$ a nonempty finite open set. Then $W \cap (\cap_{\lambda \in \Lambda} V_\lambda)$ is a finite open set.

\textbf{Proof.} We see that there exists an integer $n$ such that $W \cap (\cap_{\lambda \in \Lambda} V_\lambda) = W \cap (\cap_{\lambda = 1}^n V_\lambda)$ and hence we have the result.

\textbf{Theorem 3.4.} Let $V_\lambda$ be an open set for any $\lambda \in \Lambda$ and $W_\mu$ a nonempty finite open set for any $\mu \in M$. Let $S = \cup_{\mu \in M} W_\mu$. Then $S \cap (\cap_{\lambda \in \Lambda} V_\lambda)$ is an open set.

\textbf{Proof.} Since $W_\mu$ is a finite open set, by Theorem 3.3, we have $W_\mu \cap (\cap_{\lambda \in \Lambda} V_\lambda)$ is a finite open set for any $\mu \in M$. Since

\begin{equation}
S \cap (\cap_{\lambda \in \Lambda} V_\lambda) = (\cup_{\mu \in M} W_\mu) \cap (\cap_{\lambda \in \Lambda} V_\lambda) = \cup_{\mu \in M} (W_\mu \cap (\cap_{\lambda \in \Lambda} V_\lambda)),
\end{equation}

we have the result.

\textbf{Corollary 3.5} (see [1]). Any locally finite space is an Alexandroff space.

\section{Applications.} Let $U$ be a nonempty finite open set. We see, by Lemma 2.2 and Corollary 3.2, that there exists a positive integer $k$ such that $\{U_1, U_2, \ldots, U_k\}$ is the set of all minimal open sets in $U$. Then it satisfies the following two conditions:

(a) $U_i \cap U_j = \emptyset$ for any $i, j$ with $1 \leq i, j \leq k$, and $i \neq j$.

(b) If $U'$ is a minimal open set in $U$, then there exists $i$ with $1 \leq i \leq k$ such that $U' = U_i$.

\textbf{Theorem 4.1.} Let $U$ be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \ldots, U_n\}$ be the set of all minimal open sets in $U$ and $x$ an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Define $U_x \equiv \cap\{W \mid W$ is an open neighborhood of $x\}$. Then there exists a positive integer $i$ of $\{1, \ldots, n\}$ such that $U_i \subset U_x$.

\textbf{Proof.} Assume that $U_i \notin U_x$ for any positive integer $i$ of $\{1, \ldots, n\}$. Then we have $U_i \cap U_x = \emptyset$ for any minimal open set $U_i$ in $U$ by Corollary 2.11. Since $U_x$ is a nonempty finite open set by Theorem 3.3, there exists a minimal open set $U'$ such that $U' \subset U_x$ by Theorem 3.1. Since $U' \subset U_x \subset U$, we have $U'$ is a minimal open set in $U$. By assumption, we have $U_i \cap U' \subset U_i \cap U_x = \emptyset$ for any minimal open set $U_i$. Therefore $U' \neq U_i$ for any positive integer $i$ of $\{1, 2, \ldots, n\}$. This contradicts our assumption.
**Proposition 4.2.** Let $U$ be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \ldots, U_n\}$ be the set of all minimal open sets in $U$ and $x$ an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer $i$ of $\{1, \ldots, n\}$ such that $U_i \subset W_x$ for any open neighborhood $W_x$ of $x$.

**Proof.** Since $W_x \supset \cap \{W \mid W$ is an open neighborhood of $x\}$, we have the result by Theorem 4.1.

**Theorem 4.3.** Let $U$ be a nonempty finite open set which is not a minimal open set. Let $\{U_1, U_2, \ldots, U_n\}$ be the set of all minimal open sets in $U$ and $x$ an element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$. Then there exists a positive integer $i$ of $\{1, \ldots, n\}$ such that $x$ is an element of $\text{Cl}(U_i)$.

**Proof.** By Proposition 4.2, there exists a positive integer $i$ of $\{1, \ldots, n\}$ such that $U_i \subset W$ for any open neighborhood $W$ of $x$. Therefore $U_i \cap W \supset U_i \cap U_i \neq \emptyset$ for any open neighborhood $W$ of $x$. Therefore we have the result.

The following result is a generalization of Theorem 2.5, when $U$ is a nonempty finite open set.

**Theorem 4.4.** Let $U$ be a nonempty finite open set and $U_i$ a minimal open set in $U$ for each $i \in \{1, 2, \ldots, n\}$. Then the following three conditions are equivalent:

1. $\{U_1, U_2, \ldots, U_n\}$ is the set of all minimal open sets in $U$.
2. $U \subset \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets $S_i$ of $U_i$ for $i \in \{1, 2, \ldots, n\}$.
3. $\text{Cl}(U) = \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subsets $S_i$ of $U_i$ for $i \in \{1, 2, \ldots, n\}$.

**Proof.** (1) $\Rightarrow$ (2). If $U$ is a minimal open set, then this is the result of Theorem 2.5(2). Otherwise $U$ is not a minimal open set. If $x$ is any element of $U - (U_1 \cup U_2 \cup \cdots \cup U_n)$, we have $x \in \text{Cl}(U_1) \cup \text{Cl}(U_2) \cup \cdots \cup \text{Cl}(U_n)$ by Theorem 4.3. Therefore

$$U \subset \text{Cl}(U_1) \cup \text{Cl}(U_2) \cup \cdots \cup \text{Cl}(U_n) = \text{Cl}(S_1) \cup \text{Cl}(S_2) \cup \cdots \cup \text{Cl}(S_n) = \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$$

(4.1)

by Theorem 2.5(3).

(2) $\Rightarrow$ (3). For any nonempty subset $S_i$ of $U_i$ with $i \in \{1, 2, \ldots, n\}$, we have $\text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n) \subset \text{Cl}(U)$. On the other hand, by (2), we see

$$\text{Cl}(U) \subset \text{Cl}(\text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)) = \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n).$$

(4.2)

Therefore we have $\text{Cl}(U) = \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$ for any nonempty subset $S_i$ of $U_i$ with $i \in \{1, 2, \ldots, n\}$.

(3) $\Rightarrow$ (1). Suppose that $V$ is a minimal open set in $U$ and $V \neq U_i$ for $i \in \{1, 2, \ldots, n\}$. Then we have $V \cap \text{Cl}(U_i) = \emptyset$ for each $i \in \{1, 2, \ldots, n\}$. It follows that any element of $V$ is not contained in $\text{Cl}(U_1 \cup U_2 \cup \cdots \cup U_n)$. This contradicts the condition (3) because $V \subset U \subset \text{Cl}(U) = \text{Cl}(S_1 \cup S_2 \cup \cdots \cup S_n)$.

Let $U$ be a nonempty finite open set, $\{U_1, U_2, \ldots, U_n\}$ the set of all minimal open sets in $U$ and $x_i$ an element of $U_i$ for each $i \in \{1, 2, \ldots, n\}$. Then we see that the set $\{x_1, x_2, \ldots, x_n\}$ is a pre-open set by Theorem 4.4. Moreover, we have the following result.
**Theorem 4.5.** Let \( U \) be a nonempty finite open set and \( \{U_1, U_2, \ldots, U_n\} \) the set of all minimal open sets in \( U \). Let \( S \) be any subset of \( U - (U_1 \cup U_2 \cup \cdots \cup U_n) \) and \( S_i \) be any nonempty subset of \( U_i \) for each \( i \in \{1, 2, \ldots, n\} \). Then \( S \cup S_1 \cup S_2 \cdots \cup S_n \) is a pre-open set.

**Proof.** By Theorem 4.4(2), we have
\[
U \subset \overline{(S_1 \cup S_2 \cdots \cup S_n)} \subset \overline{(S \cup S_1 \cup S_2 \cdots \cup S_n)}.
\]
Since \( U \) is an open set, then we have
\[
S \cup S_1 \cup S_2 \cdots \cup S_n \subset U = \text{Int}(U) \subset \text{Int} \overline{(S \cup S_1 \cup S_2 \cdots \cup S_n)}.
\]
Then we have the result.

**Theorem 4.6.** Let \( X \) be a locally finite space. If any minimal open set of \( X \) has two elements at least, then \( X \) is a pre-Hausdorff space.

**Proof.** Let \( x, y \) be elements of \( X \) such that \( x \neq y \). Since \( X \) is a locally finite space, there exists finite open sets \( U \) and \( V \) such that \( x \in U \) and \( y \in V \). By Theorem 3.1, there exists the set \( \{U_1, U_2, \ldots, U_n\} \) of all minimal open sets in \( U \) and the set \( \{V_1, V_2, \ldots, V_m\} \) of all minimal open sets in \( V \).

**Case 1.** If there exists \( i \) of \( \{1, 2, \ldots, n\} \) and \( j \) of \( \{1, 2, \ldots, m\} \) such that \( x \in U_i \) and \( y \in V_j \), then, by Theorem 2.6, \( \{x\} \) and \( \{y\} \) are disjoint pre-open sets which contains \( x \) and \( y \), respectively.

**Case 2.** If there exists \( i \) of \( \{1, 2, \ldots, n\} \) such that \( x \in U_i \) and \( y \notin V_j \) for any \( j \) of \( \{1, 2, \ldots, m\} \), then we find an element \( y_j \) of \( V_j \) for each \( j \) such that \( \{x\} \cap \{y, y_1, y_2, \ldots, y_m\} = \emptyset \) by Theorems 2.6, 4.5 and the assumption.

**Case 3.** If \( x \notin U_i \) for any \( i \) of \( \{1, 2, \ldots, n\} \) and \( y \notin V_j \) for any \( j \) of \( \{1, 2, \ldots, m\} \), then we find elements \( x_j \) of \( U_i \) and \( y_j \) of \( V_j \) for each \( i, j \) such that \( \{x, x_1, x_2, \ldots, x_n\} \) and \( \{y, y_1, y_2, \ldots, y_m\} \) are pre-open sets and \( \{x, x_1, x_2, \ldots, x_n\} \cap \{y, y_1, y_2, \ldots, y_m\} = \emptyset \) by Theorem 4.5 and the assumption. We remark that we use the assumption that any minimal open set of \( X \) has at least two elements for the case \( U_i = V_j \) for some \( i \) and \( j \) in the argument of cases (2) and (3).

Therefore \( X \) is a pre-Hausdorff space.

**References**


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