COMPACTITY IN NARROW LIMIT TOWER SPACES

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ABSTRACT. We introduce a limit tower structure on the space of all bounded Radon measures on a completely regular space and we extend the Prohorov's theorem of narrow compactness. In the particular case of Polish spaces, we give a sequential version of this extension.

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1. Introduction. Let T be a completely regular space, $\mathcal{B}$ the borelians of T , and $\mathfrak{M}^b(T)$ the set of all bounded Radon measures on $(T, \mathcal{B})$ (i.e., the real bounded measures $\mu : \mathcal{B} \rightarrow \mathbb{R}$ such that $|\mu|(A) = \sup\{|\mu|(K) : K \text{ is compact, } K \subseteq A\}$, for all $A \in \mathcal{B}$, where $|\mu|$ is the variation of μ). Denote by $\mathcal{C}^b(T)$ the space of all bounded continuous real functions on T and let $\|f\| = \sup\{|f(t)| : t \in T\}$, for every $f \in \mathcal{C}^b(T)$. We recall that a filter $\mathfrak{f}$ on $\mathfrak{M}^b(T)$ is narrowly convergent to μ if and only if $V_{\varepsilon, f}(\mu) \in \mathfrak{f}$, for all $f \in \mathcal{C}^b(T)$, $\varepsilon > 0$, where $V_{\varepsilon, f}(\mu) = \{\nu : |\mu(f) - \nu(f)| < \varepsilon\}$.

We say that a set $H \subseteq \mathfrak{M}^b(T)$ is *relatively narrowly compact* if, for every filterbase $\mathfrak{B} \subseteq 2^H$ there exist a filter $\mathfrak{f}$ on $\mathfrak{M}^b(T)$ and $\mu \in \mathfrak{M}^b(T)$ such that $\mathfrak{f}$ converges to μ .

Prohorov's classical theorem states that *a bounded set $H \subseteq \mathfrak{M}^b(T)$ is relatively narrowly compact if the following condition is satisfied:*

$$\forall \varepsilon > 0, \quad \exists K_\varepsilon \text{ compact } \subseteq T : |\mu|(T \setminus K_\varepsilon) < \varepsilon, \quad \forall \mu \in H. \quad (1.1)$$

A set H as in (1.1) is called *tight*.

We remark that, if T is a Polish space (i.e., T is a separable, completely metrizable space) or T is a locally compact space, the converse is also true (relative narrow compactness implies tightness) (see [2, Section 5, Theorems 1 and 2]).

Limit tower spaces were first defined in 1997 by Kent and Brock [3] as an isomorphic gradated variant of convergence approach spaces of Löwen [8].

In this paper, we introduce on $\mathfrak{M}^b(T)$ a limit tower structure $\bar{p} = \{p_a : a \in [0, +\infty]\}$ (see [3]), where p_0 is the narrow convergence structure. Then, for every bounded set $H \subseteq \mathfrak{M}^b(T)$, there exists a number $t = t(H) \geq 0$ such that, for every filterbase $\mathfrak{B} \subseteq 2^H$ there exists a filter $\mathfrak{f}$ on $\mathfrak{M}^b(T)$, $\mathfrak{B} \subseteq \mathfrak{f}$, p_t -convergent in $\mathfrak{M}^b(T)$ (see Theorem 3.8); we say that H is p_t -relatively compact. The number $t(H)$ estimates the degree of tightness of H . If H is tight then $t(H) = 0$, so we obtain Prohorov's theorem.

If T is a locally compact space we extend also the converse of Prohorov's theorem (see Theorem 3.12).

We give some examples in the particular case of $T = \mathbb{N}$ when $\mathfrak{M}^b(\mathbb{N}) = \ell^1$.

In [Section 4](#), we obtain a sequential version of [Theorem 3.8](#) on the subset $\mathfrak{N}^1(T) \subseteq \mathfrak{N}^b(T)$ of all probabilities on the Polish space T . So, every sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathfrak{N}^1(T)$ contains a subsequence p_{2t} -convergent in $\mathfrak{N}^1(T)$, where $t = t(\{\mu_n : n \in \mathbb{N}\})$ (see [Theorem 4.9](#)). In particular, we prove that the limit tower structure $\bar{p}$ on the set of probabilities is induced by a probabilistic metric on this space.

2. Limit tower structures. Let X be a set, $\mathbb{B}(X)$ the set of all filterbases on X , and 2^X the power set of X ; for every $\mathfrak{f} \in \mathbb{B}(X)$, $\mathfrak{f}'$ is the filter generated by $\mathfrak{f}$. For $x \in X$, let $\dot{x}$ denote the fixed ultrafilter generated by $\{x\}$.

DEFINITION 2.1 (see [\[3, Definition 1\]](#)). A *limit structure* on X is a function $q : \mathbb{B}(X) \rightarrow 2^X$ satisfying

$$x \in q(\dot{x}), \quad x \in X, \quad (2.1)$$

$$q(\mathfrak{f}) = q(\mathfrak{f}'), \quad \forall \mathfrak{f} \in \mathbb{B}(X), \quad (2.2)$$

$$q(\mathfrak{f}' \cap \mathfrak{G}') = q(\mathfrak{f}) \cap q(\mathfrak{G}), \quad \forall \mathfrak{f}, \mathfrak{G} \in \mathbb{B}(X). \quad (2.3)$$

A pair (X, q) , where q is a limit structure on X is called a *limit space*.

REMARK 2.2. The statement “ $x \in q(\mathfrak{f})$ ” will be written $\mathfrak{f} \xrightarrow{q} x$ and we say that $\mathfrak{f}$ q -converges to x .

REMARK 2.3. In [\[3\]](#), a *limit structure* is a function $q : \mathbb{F}(X) \rightarrow 2^X$, where $\mathbb{F}(X)$ denotes the set of all filters on X , satisfying

$$\begin{aligned} x &\in q(\dot{x}), \quad x \in X, \\ \mathfrak{f} \subseteq \mathfrak{G} &\Rightarrow q(\mathfrak{f}) \subseteq q(\mathfrak{G}), \\ x \in q(\mathfrak{f}) &\Rightarrow x \in q(\mathfrak{f} \cap \dot{x}), \\ x \in q(\mathfrak{f}) \cap q(\mathfrak{G}) &\Rightarrow x \in q(\mathfrak{f} \cap \mathfrak{G}). \end{aligned} \quad (2.4)$$

If we extend such a function q to $\mathbb{B}(X)$ letting $q(\mathfrak{f}) = q(\mathfrak{f}')$, then [\(2.4\)](#) is equivalent to [\(2.1\)](#), [\(2.2\)](#), and [\(2.3\)](#).

REMARK 2.4. If τ is a topology on X and we define $\mathfrak{f} \xrightarrow{q_\tau} x$ if and only if $\mathcal{V}_\tau(x) \subseteq \mathfrak{f}'$, then q_τ is a limit structure on X (here $\mathcal{V}_\tau(x)$ denotes the neighborhood filter of x in (X, τ)). More exactly we have the following proposition.

PROPOSITION 2.5 (see [\[3, Proposition 2\]](#)). *Let q be a limit structure on X ; the necessary and sufficient condition for a topology τ to exist on X , such that $q = q_\tau$, is that q fulfills the following condition:*

(F) *Let $\{\mathfrak{f}_j : j \in J\}$ be a family of filterbases on X and $\{x_j : j \in J\} \subset X$ be such that $\mathfrak{f}_j \xrightarrow{q} x_j$ for all $j \in J$.*

If $\Phi \in \mathbb{B}(J)$ is such that $\mathfrak{f} \xrightarrow{q} x$, where $\mathfrak{f} = \{\{x_j : j \in \phi\} : \phi \in \Phi\}$, then

$$\bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{f}'_j \xrightarrow{q} x. \quad (2.5)$$

DEFINITION 2.6 (see [3, Definition 4]). A *limit tower* $\bar{p}$ on a set X is a family $\bar{p} = \{p_a : a \in [0, +\infty]\}$ of limit structures on X satisfying the following conditions:

$$p_a(\mathfrak{f}) \subseteq p_b(\mathfrak{f}), \quad \forall a \leq b, \quad \forall \mathfrak{f} \in \mathbb{B}(X), \quad (2.6)$$

$$p_\infty(\mathfrak{f}) = X, \quad \forall \mathfrak{f} \in \mathbb{B}(X), \quad (2.7)$$

$$p_a(\mathfrak{f}) = \bigcap_{b>a} p_b(\mathfrak{f}), \quad \forall a \in [0, +\infty), \quad \forall \mathfrak{f} \in \mathbb{B}(X). \quad (2.8)$$

If $x \in p_a(\mathfrak{f})$, we say that $\mathfrak{f}$ is p_a -convergent to x and we denote this by $\mathfrak{f} \xrightarrow{p_a} x$. If $\bar{p}$ is a limit tower on X , $(X, \bar{p})$ is called a *limit tower space*.

The axiom (F) defined in Proposition 2.5 has a natural extension to a limit tower space $(X, \bar{p})$:

(F) Let $a, b \in [0, +\infty]$, $\{\mathfrak{f}_j : j \in J\} \subseteq \mathbb{B}(X)$, and $\{x_j : j \in J\} \subseteq X$ such that $\mathfrak{f}_j \xrightarrow{p_a} x_j$, for all $j \in J$. If $\Phi \in \mathbb{B}(J)$ is such that $\mathfrak{f} \xrightarrow{p_b} x$, where $\mathfrak{f} = \{\{x_j : j \in \phi\} : \phi \in \Phi\}$, then

$$\bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{f}'_j \xrightarrow{p_{a+b}} x. \quad (2.9)$$

DEFINITION 2.7. A limit tower $\bar{p}$ on X which satisfies (F) is called a *topological limit tower*.

REMARK 2.8. From [3, Theorems 9, 13 and Proposition 12(b)] we know that a topological limit tower is an isomorphic form of a Löwen's approach structure (see [8]).

3. Narrow limit tower on $\mathfrak{M}^b(T)$. In this section, we introduce a topological limit tower $\bar{p} = \{p_a : a \in [0, +\infty]\}$ on the space of bounded Radon measures on a completely regular space such that p_0 -convergence is just the narrow convergence; then we extend the Prohorov's theorem of narrow compactness.

Let T be a completely regular space, let $\mathcal{B}$ be the σ -algebra of Borel subsets of T , and let $\mathfrak{M}^b(T)$ be the set of all bounded Radon measures on $(T, \mathcal{B})$. Denote by $C^b(T)$ the set of all bounded continuous real functions on T . For every $f \in C^b(T)$ and $\mu \in \mathfrak{M}^b(T)$, we denote $\mu(f) = \int_T f d\mu$.

Now, for every $a \in [0, +\infty]$, $\mu \in \mathfrak{M}^b(T)$, and $f \in C^b(T)$, we denote

$$V_{a,f}(\mu) = \left\{ \nu \in \mathfrak{M}^b(T) : |\mu(f) - \nu(f)| \leq a \|f\| \right\}. \quad (3.1)$$

Then, for every $a \in [0, +\infty)$, let $p_a : \mathbb{B}(\mathfrak{M}^b(T)) \rightarrow 2^{\mathfrak{M}^b(T)}$ defined by

$$p_a(\mathfrak{f}) = \left\{ \mu \in \mathfrak{M}^b(T) : \forall b > a, \forall f \in C^b(T), V_{b,f}(\mu) \in \mathfrak{f}' \right\}, \quad (3.2)$$

for all filterbases $\mathfrak{f}$ on $\mathfrak{M}^b(T)$; let p_∞ be the indiscrete convergence structure on $\mathfrak{M}^b(T)$ ($p_\infty(\mathfrak{f}) = \mathfrak{M}^b(T)$, for all $\mathfrak{f} \in \mathbb{B}(\mathfrak{M}^b(T))$).

We remark that $\mathfrak{f} \xrightarrow{p_a} \mu$ if and only if for all $b > a$, for all $f \in C^b(T)$, $V_{b,f}(\mu) \in \mathfrak{f}'$.

PROPOSITION 3.1. The limit tower $\bar{p} = \{p_a : a \in [0, +\infty]\}$ is a topological limit tower on $\mathfrak{M}^b(T)$.

PROOF. For every $a \in [0, +\infty)$, $\mu \in \mathfrak{M}^b(T)$, and $f \in C^b(T)$, we have $\mu \in V_{a,f}(\mu)$, so that we have (2.1).

Equations (2.2), (2.3), (2.6), and (2.7) are consequences of the definition of $\bar{p}$. From (2.6), $p_a(\mathfrak{f}) \subseteq \bigcap_{b>a} p_b(\mathfrak{f})$, for all $\mathfrak{f} \in \mathbb{B}(\mathfrak{N}^b(T))$. If $\mathfrak{f} \xrightarrow{p_c} \mu$, for all $c > a$, then for all $b > a$, there exists c such that $a < c < b$ hence $V_{b,f}(\mu) \in \mathfrak{f}$, for all $f \in C^b(T)$. Therefore $\mathfrak{f} \xrightarrow{p_a} \mu$ and so we have (2.8).

(F) Let $a, b \geq 0$, $\{\mathfrak{f}_j : j \in J\} \subseteq \mathbb{B}(\mathfrak{N}^b(T))$, and $\{\mu_j : j \in J\} \subseteq \mathfrak{N}^b(T)$ such that (1) $\mathfrak{f}_j \xrightarrow{p_a} \mu_j$, for all $j \in J$. Let Φ be a filterbase on J such that (2) $\mathfrak{f} \xrightarrow{b} \mu$, where $\mathfrak{f} = \{\{\mu_j\}_{j \in \phi} : \phi \in \Phi\}$. Then for all $u > a + b$, there exist $d > a$, $e > b$ such that $u = d + e$. Then for all $f \in C^b(T)$, from (2), $V_{e,f}(\mu) \in \mathfrak{f}'$ hence, there exists $\phi \in \Phi$ such that $\{\mu_j\}_{j \in \phi} \subseteq V_{e,f}(\mu)$. Then (3) $|\mu_j(f) - \mu(f)| \leq e\|f\|$, for all $j \in \phi$.

From (1), for all $j \in J$, $V_{a,f}(\mu_j) \in \mathfrak{f}'_j$. But, from (3), $V_{a,f}(\mu_j) \subseteq V_{u,f}(\mu)$, so that $V_{u,f} \in \mathfrak{f}'_j$, for all $j \in \phi$. Therefore,

$$V_{u,f}(\mu) \in \bigcap_{j \in \phi} \mathfrak{f}'_j \subseteq \bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{f}'_j. \quad (3.3)$$

It follows that $\mu \in p_{a+b}(\bigcup_{\phi \in \Phi} \bigcap_{j \in \phi} \mathfrak{f}'_j)$, so that $\bar{p} = \{p_a : a \in [0, +\infty]\}$ is a topological limit tower on $\mathfrak{N}^b(T)$. $\square$

DEFINITION 3.2. We say that $\bar{p} = \{p_a : a \in [0, +\infty]\}$ is the *narrow limit tower* on $\mathfrak{N}^b(T)$.

REMARK 3.3. Note that p_0 is the narrow convergence structure on $\mathfrak{N}^b(T)$. Indeed, $\mathfrak{f} \xrightarrow{p_0} \mu$ if and only if for all $\varepsilon > 0$, for all $f \in C^b(T)$, $V_{\varepsilon,f} \in \mathfrak{f}'$. But the sets $V_{\varepsilon,f}(\mu) = \{v : |\mu(f) - v(f)| \leq \varepsilon\|f\|\}$ form a subbase for the neighbourhood system of μ in the narrow topology on $\mathfrak{N}^b(T)$; so that $\mathfrak{f}$ is narrowly convergent to μ .

REMARK 3.4. If $\mathfrak{f} \xrightarrow{p_a} \mu$ then $\mathfrak{f} \xrightarrow{p_b} \mu$, for all $b \geq a$. Thus p_0 is the finest limit structure of $\bar{p}$.

REMARK 3.5. We may interpret $\inf\{a : \mathfrak{f} \xrightarrow{p_a} \mu\}$ as the degree of narrow convergence of filterbase $\mathfrak{f}$ to μ .

REMARK 3.6. For every net $(\mu_i)_{i \in I} \subseteq \mathfrak{N}^b(T)$ ($(I, \leq)$ is a directed set) let $\mathfrak{f} = \{\{x_j : j \geq i\} : i \in I\}$ be the filterbase generated by $(\mu_i)_{i \in I}$.

If $\bar{p} = \{p_a : a \in [0, +\infty]\}$ is the narrow limit tower on $\mathfrak{N}^b(T)$, then we say that $\mu_i \xrightarrow{a} \mu$ if $\mathfrak{f} \xrightarrow{p_a} \mu$. Therefore, $\mu_i \xrightarrow{a} \mu$ if and only if

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a \cdot \|f\|, \quad \forall f \in C^b(T). \quad (3.4)$$

DEFINITION 3.7. We say that a subset $H \subseteq \mathfrak{N}^b(T)$ is *a-relatively compact* if for every filterbase $\mathfrak{B} \subseteq 2^H$, there exist a filter $\mathfrak{f}$ on $\mathfrak{N}^b(T)$ and $\mu \in \mathfrak{N}^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{f}$ and $\mathfrak{f} \xrightarrow{p_a} \mu$.

We remark that H is 0-relatively compact if and only if H is relatively narrowly compact.

A subset $H \subseteq \mathfrak{N}^b(T)$ is bounded if $\sup\{|\mu|(T) : \mu \in H\} < +\infty$, where $|\mu|$ is the variation of μ . The mapping $\mu \mapsto |\mu|(T) = \|\mu\|$ is a norm on $\mathfrak{N}^b(T)$.

Let $\mathcal{K}(T)$ be the family of all compact sets on T ; for every bounded set $H \subseteq \mathfrak{N}^b(T)$

we denote

$$t(H) = \inf_{K \in \mathcal{H}(T)} \sup_{\mu \in H} |\mu|(T \setminus K). \quad (3.5)$$

We remark that $t(H) \in [0, +\infty)$ and $t(H) = 0$ if and only if H is tight. We say that $t(H)$ is the *degree of tightness* of H .

Now we give an extension of Prohorov's theorem.

THEOREM 3.8. *Every bounded set $H \subseteq \mathfrak{M}^b(T)$ is $t(H)$ -relatively compact.*

PROOF. Let X be the Stone-Čech compactification of T and $i: T \rightarrow X$ be the canonical injection of T in X . We remark that $C^b(X) = C(X)$ (X is compact); so $(\mathfrak{M}^b(X), \|\cdot\|)$ is the topological dual of the Banach space $(C(X), \|\cdot\|)$ and the narrow topology on $\mathfrak{M}^b(X)$ is the weak*-topology, w^* , of this dual space.

For every $\mu \in \mathfrak{M}^b(T)$ we define $\nu = I(\mu) \in \mathfrak{M}^b(X)$, where $I(\nu)(F) = \mu(F \circ i)$, for every $F \in C(X)$; $\|\nu\| = |\nu|(X) = |\mu|(T) = \|\mu\|$ so that $I: \mathfrak{M}^b(T) \rightarrow \mathfrak{M}^b(X)$, $\mu \mapsto I(\mu)$, is an isometric embedding.

Let H be a bounded subset of $\mathfrak{M}^b(T)$; then $I(H)$ is a bounded subset of $\mathfrak{M}^b(X)$. Therefore $I(H)$ is w^* -relatively compact. For every filterbase $\mathfrak{B} \subseteq 2^H$, $I(\mathfrak{B}) = \{I(B) : B \in \mathfrak{B}\}$ is a filterbase on $I(H)$. So that, there exists a filter $\mathfrak{G}$ on $\mathfrak{M}^b(X)$ w^* -convergent to a measure $\nu_0 \in \mathfrak{M}^b(X)$ such that $I(\mathfrak{B}) \subseteq \mathfrak{G}$. From the definition of $t(H)$, there exists a sequence $(K_n)_n \subseteq \mathcal{H}(T)$ such that (1) $|\mu|(T \setminus K_n) < t(H) + 1/n$, for all $n \in \mathbb{N}$, for all $\mu \in H$. We denote $T_0 = \bigcup_{n=1}^{\infty} K_n$ and (2) $X_0 = \bigcup_{n=1}^{\infty} i(K_n) = i(T_0)$.

For every $n \in \mathbb{N}$, $i(K_n) \in \mathcal{H}(X)$, so that X_0 is a Borel set of X . On the other hand, for every $n \in \mathbb{N}$, $X \setminus i(K_n)$ is an open subset of X so that the mapping $\lambda \mapsto |\lambda|(X \setminus i(K_n))$ is a w^* -lower semi-continuous mapping on $\mathfrak{M}^b(X)$ (see [2, Section 5, Proposition 6(a)]).

From $\mathfrak{G} \xrightarrow{w^*} \nu_0$, for all $n \in \mathbb{N}$, there exists $G_n \in \mathfrak{G}$ such that (3) $|\nu_0|(X \setminus i(K_n)) - 1/n < |\lambda|(X \setminus i(K_n))$, for all $\lambda \in G_n$.

The filterbase $\mathfrak{B}$ is a filterbase on H so that $\mathfrak{B} \neq \emptyset$. Let B_0 be a set in $\mathfrak{B}$; then $I(B_0) \in I(\mathfrak{B}) \subseteq \mathfrak{G}$.

For every $n \in \mathbb{N}$, there exists $\mu_n \in B_0$ such that $I(\mu_n) \in G_n$ ($I(B_0) \cap G_n \neq \emptyset$). Therefore, from (1) and (3), for every $n \in \mathbb{N}$,

$$\begin{aligned} |\nu_0|(X \setminus X_0) &\leq |\nu_0|(X \setminus i(K_n)) < |I(\mu_n)|(X \setminus i(K_n)) + \frac{1}{n} \\ &= |\mu_n|(X \setminus i(K_n)) + \frac{1}{n} = |\mu_n|(T \setminus K_n) + \frac{1}{n} < t(H) + \frac{2}{n}. \end{aligned} \quad (3.6)$$

Hence (4) $|\nu_0|(X \setminus X_0) \leq t(H)$.

Now, X being the Stone-Čech compactification of T , for every $f \in C^b(T)$ there exists $F \in C(X)$ such that $F \circ i = f$ and $\|F\| = \|f\|$ (see [10, Theorem 1.4.6, page 25]). Now we define $J: C^b(T) \rightarrow \mathbb{R}$ letting

$$J(f) = \nu_0(F \cdot \chi_{X_0}) = \int_{X_0} F d\nu_0. \quad (3.7)$$

Obviously, J is a continuous linear mapping on $C^b(T)$. For every $\varepsilon > 0$, from (2), there exists $K \in \mathcal{H}(T)$ such that $|\nu_0|(X_0 \setminus i(K)) < \varepsilon$ and $i(K) \subseteq X_0$. Then, for every $g \in C^b(T)$

with $|g| \leq 1$ and $g|_K = 0$, let $G \in C(X)$ such that $G \circ i = g$. Therefore, we have

$$\begin{aligned} |J(g)| &= \left| \nu_0(G \cdot \chi_{X_0}) \right| \leq \left| \nu_0(G \cdot \chi_{X_0 \setminus i(K)}) \right| + \left| \nu_0(G \cdot \chi_{i(K)}) \right| \\ &\leq |\nu_0|(X_0 \setminus i(K)) < \varepsilon. \end{aligned} \quad (3.8)$$

Hence, J is a linear mapping satisfying the condition (M) from [2, Section 5, Proposition 5] so that there exists exactly one measure $\mu_0 \in \mathfrak{M}^b(T)$ such that $\mu_0(f) = J(f)$, for every $f \in C^b(T)$. Then we have (5) $\mu_0(f) = \nu_0(F \cdot \chi_{X_0})$, for all $f \in C^b(T)$, where F is the continuous extension of f to X .

Now, for every $f_1, \dots, f_n \in C^b(T)$ with $\|f_k\| > 0$, for all $k = 1, \dots, n$, let $F_1, \dots, F_n \in C(X)$ such that $F_k \circ i = f_k$ and $\|F_k\| = \|f_k\|$, for every $k = 1, \dots, n$.

For all $b > t(H)$, let $\varepsilon = (b - t(H)) \cdot \min\{\|f_k\| : k = 1, \dots, n\} > 0$. The set

$$G = \bigcap_{k=1}^n \left\{ \lambda \in \mathfrak{M}^b(X) : \left| \lambda(F_k) - \nu_0(F_k) \right| < \varepsilon \right\} \quad (3.9)$$

is a w^* -neighborhood of ν_0 and so is a member of $\mathfrak{G}(\mathfrak{G} \xrightarrow{w^*} \nu_0)$. Therefore, for every $B \in \mathfrak{B}$, $G \cap I(B) \neq \emptyset$ ($I(\mathfrak{B}) \subseteq \mathfrak{G}$). Hence there exists $\mu \in B$ such that $I(\mu) \in G$. Then, for every $k = 1, \dots, n$, from (4) and (5), we have

$$\begin{aligned} \left| \mu(f_k) - \mu_0(f_k) \right| &= \left| \mu(F_k \circ i) - \mu_0(f_k) \right| = \left| I(\mu)(F_k) - \nu_0(F_k \cdot \chi_{X_0}) \right| \\ &\leq \left| I(\mu)(F_k) - \nu_0(F_k) \right| + \left| \nu_0(F_k \cdot \chi_{X \setminus X_0}) \right| < \varepsilon + \|F_k\| \cdot |\nu_0|(X \setminus X_0) \\ &< \varepsilon + \|f_k\| \cdot t(H) \leq (b - t(H)) \cdot \|f_k\| + \|f_k\| \cdot t(H) = b \cdot \|f_k\|. \end{aligned} \quad (3.10)$$

Therefore, $\mu \in \bigcap_{k=1}^n V_{b, f_k}(\mu_0)$. So, for every $b > t(H)$, $n \in \mathbb{N}$, $f_1, \dots, f_n \in C^b(T)$ and $B \in \mathfrak{B}$,

$$\bigcap_{k=1}^n V_{b, f_k}(\mu_0) \cap B \neq \emptyset. \quad (3.11)$$

Let $\mathfrak{f}$ be the filter generated by the filterbase

$$\left\{ \bigcap_{k=1}^n V_{b, f_k}(\mu_0) \cap B : b > t(H), f_1, \dots, f_n \in C^b(T), B \in \mathfrak{B} \right\}. \quad (3.12)$$

Then $\mathfrak{B} \subseteq \mathfrak{f}$ and $\mathfrak{f} \xrightarrow{p_{t(H)}} \mu_0$, so that H is a $t(H)$ -relatively compact set. $\square$

REMARK 3.9. If H is tight in $\mathfrak{M}^b(T)$ then $t(H) = 0$, so that H is a relatively narrowly compact set and we obtain Prohorov's theorem.

REMARK 3.10. Let $a \geq b \geq 0$; then, every b -relatively compact set is a -relatively compact set, also. Therefore, for every bounded set $H \subseteq \mathfrak{M}^b(T)$

$$[t(H), +\infty) \subseteq \{a \geq 0 : H \text{ is } a\text{-relatively compact}\}. \quad (3.13)$$

REMARK 3.11. We say that $H \subseteq \mathfrak{M}_+^b(T)$ is *a-relatively compact* in $\mathfrak{M}_+^b(T)$ if, for every filterbase $\mathfrak{B} \subseteq 2^H$, there exist a filter $\mathfrak{f}$ on $\mathfrak{M}_+^b(T)$ and $\mu \in \mathfrak{M}_+^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{f}$ and for all $b > a$, for all $f \in C^b(T)$, $V_{b, f}(\mu) \cap \mathfrak{M}_+^b(T) \in \mathfrak{f}$; we say in this case that $\mathfrak{f} \xrightarrow{p_a} \mu$ in $\mathfrak{M}_+^b(T)$.

The subset of all positive measures, $\mathfrak{M}_+^b(X)$, is closed in the narrow topology of $\mathfrak{M}^b(X)$ (see [2, Section 5, Remark 2]) so that, if $H \subseteq \mathfrak{M}_+^b(T)$ is a bounded subset, then $I(H)$ is w^* -relatively compact in $\mathfrak{M}_+^b(X)$. Then we follow the proof of [Theorem 3.8](#) and we obtain that every bounded subset $H \subseteq \mathfrak{M}_+^b(T)$ is $t(H)$ -relatively compact in $\mathfrak{M}_+^b(T)$. Also, we have

$$[t(H), +\infty) \subseteq \{a \geq 0 : H \text{ is } a\text{-relatively compact in } \mathfrak{M}_+^b(T)\}. \quad (3.14)$$

In the particular case where T is locally compact, we have the converse of [Theorem 3.8](#) in the subspace $\mathfrak{M}_+^b(T)$.

THEOREM 3.12. *Let T be a locally compact space and H an a -relatively compact set in $\mathfrak{M}_+^b(T)$; then $t(H) \leq a$.*

PROOF. We suppose that H is an a -relatively compact subset of $\mathfrak{M}_+^b(T)$ and $t(H) = \inf_{K \in \mathcal{K}(T)} \sup_{\mu \in H} \mu(T \setminus K) > a$. Then, for every $\varepsilon > 0$ and $K \in \mathcal{K}(T)$, there exists $\mu_K \in H$ such that (1) $\mu_K(T \setminus K) > a + \varepsilon$.

For every $K \in \mathcal{K}(T)$ we denote $B_K = \{\mu_L : L \in \mathcal{K}(T), K \subseteq L\}$. Then $\mathfrak{B} = \{B_K : K \in \mathcal{K}(T)\}$ is a filterbase on H so that there exist a filter $\mathfrak{F}$ on $\mathfrak{M}_+^b(T)$ and $\mu \in \mathfrak{M}_+^b(T)$ such that $\mathfrak{B} \subseteq \mathfrak{F}$ and (2) $\mathfrak{F} \xrightarrow{p_a} \mu$, in $\mathfrak{M}_+^b(T)$ (see [Remark 3.11](#)). Since μ is a Radon measure, there exists $K_0 \in \mathcal{K}(T)$ such that (3) $\mu(T \setminus K_0) < \varepsilon/2$.

Let U be a relatively compact neighborhood of K and $f : T \rightarrow [0, 1]$ a continuous function such that (4) $f|_{K_0} = 0$ and $f|_{T \setminus U} = 1$.

We remark that $f \in C^b(T)$ and $\|f\| = 1$. Now let $b = a + \varepsilon/2 > a$ and $f \in C^b(T)$; from (2), $V_{b,f}(\mu) \in \mathfrak{F} \supseteq \mathfrak{B}$ so that (5) $V_{b,f}(\mu) \cap B_{\bar{U}} \neq \emptyset$.

Hence, there exists $K \in \mathcal{K}(T)$, $K \supseteq \bar{U}$ such that (6) $|\mu_K(f) - \mu(f)| \leq b \cdot \|f\| = b$.

From (1), (3), (4), and (6) we obtain the following contradiction:

$$\begin{aligned} a + \varepsilon &< \mu_K(T \setminus K) \leq \mu_K(T \setminus \bar{U}) \leq \mu_K(f) \leq \mu(f) + a + \frac{\varepsilon}{2} \\ &\leq \mu(T \setminus K_0) + a + \frac{\varepsilon}{2} < a + \varepsilon. \end{aligned} \quad (3.15)$$

□

REMARK 3.13. If H is a relatively narrowly compact subset of $\mathfrak{M}_+^b(T)$ (i.e., 0-relatively compact set), then $t(H) = 0$ so that H is tight. Therefore, we obtain the converse of Prohorov's theorem; so [Theorem 3.12](#) is an extension of [2, Section 5, Theorem 2].

REMARK 3.14. From [Remark 3.11](#) and [Theorem 3.12](#), we obtain (in the case of locally compact spaces)

$$[t(H), +\infty) = \{a \geq 0 : H \text{ is } a\text{-relatively compact in } \mathfrak{M}_+^b(T)\}. \quad (3.16)$$

EXAMPLE 3.15. Let $T = \mathbb{N}$ be the set of natural numbers and $\mathfrak{B} = \mathcal{P}(\mathbb{N})$. Then $\mathfrak{M}^b(\mathbb{N}) = \ell^1$ (the space of all sequences of real numbers $(x_n)_{n \in \mathbb{N}}$ such that $\sum_{n=1}^{\infty} |x_n| < +\infty$) and $C^b(T) = \ell^\infty$ (the space of all bounded sequences of real numbers). Indeed,

$$\begin{aligned} \forall x = (x_n)_n \in \ell^1, \quad x : \mathfrak{B} &\longrightarrow \mathbb{R}, \quad x(A) = \sum_{n \in A} x_n, \\ x(y) &= \sum_n x_n y_n, \quad \forall y = (y_n)_n \in \ell^\infty. \end{aligned} \quad (3.17)$$

Let $(x^p)_{p \in \mathbb{N}} \subseteq \mathfrak{W}^b(\mathbb{N})$ and $x \in \mathfrak{W}^b(\mathbb{N})$, where $x^p = (x_n^p)_n$, for every $p \in \mathbb{N}$ and $x = (x_n)_n$. Then $x^p \xrightarrow{a} x$ if and only if (1) $\limsup_p |\sum_{n \in \mathbb{N}} (x_n^p - x_n) \cdot y_n| \leq a \cdot \sup_n |y_n|$, for all $(y_n)_n \in \ell^\infty$ (see Remark 3.6).

For every bounded set $H = \{x^p : p \in \mathbb{N}\} \subseteq \mathfrak{W}^b(\mathbb{N})$ (2) $t(H) = \inf_m \sup_p \sum_{n=m}^\infty |x_n^p|$.

Let $(x_p)_{p \in \mathbb{N}} \subseteq [0, 1]$ be a sequence; we define

$$x_n^p = \begin{cases} 1 - x_p, & n = 0, \\ x_p, & n = p, \\ 0, & \text{otherwise.} \end{cases} \quad (3.18)$$

Then $x^p = (x_n^p)_{n \in \mathbb{N}} \in \mathfrak{W}^b(\mathbb{N})$ and, from (2), we obtain

$$t(\{x^p : p \in \mathbb{N}\}) = \limsup_n x_n = t. \quad (3.19)$$

We easily remark that $x^p \xrightarrow{t} x$, where $x = (x_n)_n$ and

$$x_n = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \quad (3.20)$$

From Remark 3.14, $\inf\{a \geq 0 : x^p \xrightarrow{a} x\} = \limsup_n x_n$. If $x_n \rightarrow 0$, then $(x^p)_p$ is narrowly convergent to x . In the particular case where $x_n = 1$, for every $n \in \mathbb{N}$, x^p is the Dirac measure δ_p and $\delta_p \xrightarrow{1} \delta_0$.

We remark that

$$\inf\{a \geq 0 : \delta_p \xrightarrow{a} \delta_0\} = 1. \quad (3.21)$$

4. Probabilistic metric on $\mathfrak{W}^1(T)$. Let (T, d) be a Polish space and let $\mathfrak{W}^1(T) \subseteq \mathfrak{W}_+^b(T)$ be the subset of all probabilities on T . We say that a net $(\mu_i)_{i \in I} \subseteq \mathfrak{W}^1(T)$ is p_a -convergent to $\mu \in \mathfrak{W}^1(T)$ ($a \geq 0$) if

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a \cdot \|f\|, \quad \forall f \in C^b(T). \quad (4.1)$$

We denote this by $\mu_i \xrightarrow{a} \mu$. So, $\bar{p} = \{p_a : a \in [0, +\infty]\}$ is the narrow limit tower induced on $\mathfrak{W}^1(T)$ (see Remark 3.6). If X is the Stone-Ćech compactification of T , the subset $\mathfrak{W}^1(X)$ is a compact set of $\mathfrak{W}^b(X)$ (see [2, Section 5, Proposition 11]). So, with a similar argument to that of Remark 3.11, we deduce that every subset $H \subseteq \mathfrak{W}^1(T)$ is $t(H)$ -relatively compact in $\mathfrak{W}^1(T)$ (i.e., every net $(\mu_i)_{i \in I}$ has a subnet p_a -convergent).

Theorem 4.1 has a similar proof to that of Portmanteau's theorem (see [1, Theorem 2.1, Appendix III, Theorem 3]) which we omit.

THEOREM 4.1. *Let $(\mu_i)_{i \in I}$ be a net in $\mathfrak{W}^1(T)$, $\mu \in \mathfrak{W}^1(T)$ and $a \geq 0$; the following statements are equivalent:*

$$\mu_i \xrightarrow{a} \mu, \quad (4.2)$$

$$\limsup_i |\mu_i(f) - \mu(f)| \leq a, \quad \forall f \in C^b(T) \text{ with } \|f\| \leq 1, \quad (4.3)$$

$$\limsup_i \mu_i(F) \leq \mu(F), \quad \forall F = \bar{F} \subseteq T, \quad (4.4)$$

$$\liminf_i \mu_i(D) \geq \mu(D), \quad \forall D = D^\circ \subseteq T, \quad (4.5)$$

$$\limsup_i |\mu_i(A) - \mu(A)| \leq a, \quad \forall A \in \mathcal{B} \text{ with } \mu(\bar{A} - A^\circ) = 0. \quad (4.6)$$

In [Theorem 4.1](#), $\bar{A}$ and A° denote the closure and the interior of A in the topological space (T, τ_d) , respectively.

REMARK 4.2. In [Theorem 4.1](#), we can suppose that $a \in [0, 1]$.

REMARK 4.3. R. Löwen gave a similar result in [\[7, Theorem 6\]](#).

DEFINITION 4.4. For every $F = \bar{F} \subseteq T$ and $\varepsilon > 0$ we denote $F^\varepsilon = \{t \in T : d(t, F) < \varepsilon\}$. For every $a \in [0, 1]$ we define $L_a : \mathfrak{M}^1(T) \times \mathfrak{M}^1(T) \rightarrow \mathbb{R}_+$ letting

$$L_a(\mu, \nu) = \inf \left\{ \varepsilon > 0 : \mu(F) < \nu(F^\varepsilon) + a + \varepsilon, \nu(F) < \mu(F^\varepsilon) + a + \varepsilon, \forall F = \bar{F} \subseteq T \right\}. \quad (4.7)$$

REMARK 4.5. L_0 is the metric of Lévy-Prohorov on $\mathfrak{M}^1(T)$. Therefore, L_0 induces the narrow topology on $\mathfrak{M}^1(T)$ and $(\mathfrak{M}^1(T), L_0)$ is a Polish space [\[2, Section 5, Examples 8 and 9\]](#).

REMARK 4.6. The family $\mathcal{L} = \{L_a : a \in [0, 1]\}$ has the following properties:

$$\begin{aligned} L_a(\mu, \nu) &= 0, \quad \forall a \geq 0 \iff \mu = \nu, \\ L_a(\mu, \nu) &= L_a(\nu, \mu), \quad \forall \mu, \nu \in \mathfrak{M}^1(T), \quad \forall a \in [0, 1], \\ L_{a+b}(\mu, \nu) &\leq L_a(\mu, \lambda) + L_b(\lambda, \nu), \quad \forall \mu, \nu, \lambda \in \mathfrak{M}^1(T), \quad \forall a, b \in [0, 1], \\ L_a(\mu, \nu) &= \sup_{b>a} L_b(\mu, \nu), \quad \forall \mu, \nu \in \mathfrak{M}^1(T), \quad \forall a \in [0, 1]. \end{aligned} \quad (4.8)$$

In [\[4, Theorem 1\]](#) we proved that such a family $\mathcal{L}$ is an equivalent gradated form of a probabilistic metric (F, T_m) , where, for every $\mu, \nu \in \mathfrak{M}^1(T)$ and $a > 0$,

$$F(\mu, \nu)(a) = \sup_{\varepsilon>0} \inf_{F=\bar{F}} \left\{ \min \left[\mu(F^{a-\varepsilon}) - \nu(F), \nu(F^{a-\varepsilon}) - \mu(F) \right] + 1 + a \right\} \wedge 1 \quad (4.9)$$

and $T_m(a, b) = \max\{a + b - 1, 0\}$. For the space of distribution functions, equivalent probabilistic metrics are introduced in [\[5, 6, 9\]](#).

In [Theorem 4.7](#) we compare the narrow limit tower with the convergence structures induced by the family of semi-pseudometrics $\mathcal{L} = \{L_a : a \in [0, 1]\}$. So, this theorem is an important step to obtain a sequential version of [Theorem 3.8](#).

THEOREM 4.7. Let $(\mu_i)_{i \in I}$ be a net in $\mathfrak{M}^1(T)$, $\mu \in \mathfrak{M}^1(T)$ and $a \in [0, 1]$.

$$\text{If } L_a(\mu_i, \mu) \rightarrow 0, \quad \text{then } \mu_i \xrightarrow{a} \mu, \quad (4.10)$$

$$\text{If } \mu_i \xrightarrow{a} \mu, \quad \text{then } L_{2a}(\mu_i, \mu) \rightarrow 0. \quad (4.11)$$

PROOF. (i) We suppose that $L_a(\mu_i, \mu) \rightarrow 0$; then, for every $n \in \mathbb{N}^*$, there exists $i_n \in I$ such that, for every $i \geq i_n$, $L_a(\mu_i, \mu) < 1/n$. Therefore,

$$\mu_i(F) < \mu(F^{1/n}) + a + \frac{1}{n}, \quad \forall F = \bar{F}, \quad (4.12)$$

so that, for every $F = \bar{F} \subseteq T$,

$$\limsup_i \mu_i(F) \leq \sup_{i \geq i_n} \mu_i(F) \leq \mu(F^{1/n}) + a + \frac{1}{n}. \quad (4.13)$$

But $\mu(F^{1/n}) \rightarrow \mu(F)$, so that $\limsup_i \mu_i(F) \leq \mu(F) + a$, for all $F = \bar{F}$.

From (4.4) this is equivalent to $\mu_i \xrightarrow{a} \mu$.

(ii) Let now $\mu_i \xrightarrow{a} \mu$ and let $\varepsilon > 0$. For every $r > 0$ and $t \in T$, let $S_r(t) = \{s \in T : d(s, t) < r\}$. Then $\overline{S_r(t)} \setminus S_r^\circ(t) \subseteq \{s \in T : d(s, t) = r\} = C_r$. But $C_{r_1} \cap C_{r_2} = \emptyset$, for all $r_1 \neq r_2$ and $\mu(\cup_{r>0} C_r) \leq 1$. It follows that there exists a countable set $N \subseteq (0, +\infty)$ such that $\mu(C_r) = 0$, for all $r \in (0, +\infty) \setminus N$. Therefore, T being separable, there exists a countable family $\{S_{r_n}(t_n) : n \in \mathbb{N}\}$ such that (1) $T = \cup_{n=1}^\infty S_{r_n}(t_n)$, $\mu(\overline{S_{r_n}(t_n)} \setminus S_{r_n}^\circ(t_n)) = 0$ and $r_n < \varepsilon/6$, for all $n \in \mathbb{N}$.

We denote for all $n \in \mathbb{N}$, $S_n = S_{r_n}(t_n)$. Let $K \subseteq T$ be a compact set such that $\mu(T \setminus K) < \varepsilon/3$ and let $p \in \mathbb{N}$ such that $K \subseteq \cup_{n=1}^p S_n = A_0$; then (2) $\mu(T \setminus A_0) < \varepsilon/3$.

We denote $\mathcal{A} = \{\cup_{i=1}^q S_{k_i} : q \in \mathbb{N}, k_1, \dots, k_n \leq p\}$; obviously, $A_0 \in \mathcal{A}$. For every $A \in \mathcal{A}$, $\mu(\bar{A} \setminus A^\circ) = 0$ so that, from (4.6),

$$\limsup_i |\mu_i(A) - \mu(A)| \leq a. \quad (4.14)$$

Therefore there exists $i_0 \in I$ such that, for every $i \geq i_0$ and $A \in \mathcal{A}$, (3) $|\mu_i(A) - \mu(A)| < a + \varepsilon/3$.

Now, for every $F = \bar{F} \subseteq T$, let

$$A_F = \bigcup \left\{ S_n : n \leq p, S_n \cap F \neq \emptyset \right\} \in \mathcal{A}. \quad (4.15)$$

Then (4) $F \subseteq A_F \cup (T \setminus A_0)$, $A_F \subseteq F^{\varepsilon/3}$.

Indeed, $F = (F \cap A_0) \cup (F \setminus A_0) \subseteq A_F \cup (T \setminus A_0)$. For every $t \in A_F$ there exists S_n such that $t \in S_n$ and $S_n \cap F \neq \emptyset$. Then, from (1), $d(t, F) \leq 2 \cdot r_n < \varepsilon/3$, so that $t \in F^{\varepsilon/3}$. Then, from (2), (3), and (4), we have

$$\begin{aligned} \mu_i(F) &< \mu_i(A_F) + \mu_i(T \setminus A_0) < \mu(F) + a + \frac{\varepsilon}{3} + 1 - \mu_i(A_0) \\ &< \mu(A_F) + a + \frac{\varepsilon}{3} + 1 - \mu(A_0) + a + \frac{\varepsilon}{3} = \mu(A_F) + \mu(T \setminus A_0) + 2 \cdot a + \frac{2\varepsilon}{3} \\ &< \mu(F^{\varepsilon/3}) + 2 \cdot a + \varepsilon \leq \mu(F^\varepsilon) + 2 \cdot a + \varepsilon, \\ \mu(F) &\leq \mu(A_F) + \mu(T \setminus A_0) < \mu_i(A_F) + a + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \mu_i(F^{\varepsilon/3}) + a + \frac{2\varepsilon}{3} < \mu_i(F^\varepsilon) + 2 \cdot a + \varepsilon, \end{aligned} \quad (4.16)$$

for every $F = \bar{F} \subseteq T$. Then $L_{2a}(\mu_i, \mu) \leq \varepsilon$, for every $i \geq i_0$. Therefore $L_{2a}(\mu_i, \mu) \rightarrow 0$. $\square$

COROLLARY 4.8. *Let H be an a -relatively compact subset in $\mathfrak{N}^1(T)$; then, for every sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq H$, there exist a subsequence $(\mu_{k_n})_{n \in \mathbb{N}}$ and $\mu \in \mathfrak{N}^1(T)$ such that $\mu_{k_n} \xrightarrow{2 \cdot a} \mu$.*

PROOF. For every sequence $(\mu_n)_{n \in \mathbb{N}} \subseteq H$, there exist a subnet $(\mu_{n_i})_{i \in I}$ and $\mu \in \mathfrak{N}^1(T)$ such that $\mu_{n_i} \xrightarrow{a} \mu$. From (4.11), $L_{2a}(\mu_{n_i}, \mu) \rightarrow 0$. So, for every $p \in \mathbb{N}$, there exists $i_p \in I$ such that $n_{i_p} \geq p$ and $L_{2a}(\mu_{n_{i_p}}, \mu) < 1/p$.

Therefore, we can choose a subsequence $(\mu'_n)_{n \in \mathbb{N}}$ of $(\mu_n)_{n \in \mathbb{N}}$ such that $L_{2a}(\mu'_n, \mu) \rightarrow 0$. From (4.10) it follows that $\mu'_n \xrightarrow{2 \cdot a} \mu$. $\square$

Now we are able to give the sequential version of [Theorem 3.8](#).

THEOREM 4.9. *Let $(\mu_n)_{n \in \mathbb{N}} \subseteq \mathfrak{M}^1(T)$ and $t = t(\{\mu_n : n \in \mathbb{N}\})$ be the degree of tightness of $(\mu_n)_{n \in \mathbb{N}}$. Then there exist a subsequence $(\mu_{k_n})_{n \in \mathbb{N}}$ and $\mu \in \mathfrak{M}^1(T)$ such that*

$$\mu_{k_n} \xrightarrow{2 \cdot t} \mu. \quad (4.17)$$

REFERENCES

- [1] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, 1968. [MR 38 # 1718](#). [Zbl 172.21201](#).
- [2] N. Bourbaki, *Éléments de Mathématique. Fasc. XXXV. Livre VI: Intégration. Chapitre IX: Intégration sur les Espaces Topologiques Séparés*, Actualités Scientifiques et Industrielles, no. 1343, Hermann, Paris, 1969 (French). [MR 43#2183](#). [Zbl 189.14201](#).
- [3] P. Brock and D. C. Kent, *Approach spaces, limit tower spaces, and probabilistic convergence spaces*, Appl. Categ. Structures 5 (1997), no. 2, 99–110. [MR 98b:54017](#).
- [4] L. C. Florescu, *Probabilistic pseudometrics*, Mathematica (Cluj) 24(47) (1982), no. 1-2, 21–29. [MR 85c:54047](#).
- [5] ———, *A p -complete Menger structure for distribution functions space*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat. 34 (1988), no. 1, 1–6. [MR 89m:54044](#). [Zbl 656.60019](#).
- [6] U. Höhle, *Probabilistische Metriken auf der Menge der nicht negativen Verteilungsfunktionen*, Aequationes Math. 18 (1978), no. 3, 345–356 (German). [MR 80k:54057](#).
- [7] R. Lowen, *Convergence in a localizable fuzzy topology on spaces of probability measures*, Convergence Structures 1984 (Bechyně, 1984), Math. Res., 24, Akademie-Verlag, Berlin, 1985, pp. 217–226. [MR 87g:54017](#).
- [8] ———, *Approach spaces: a common supercategory of TOP and MET*, Math. Nachr. 141 (1989), 183–226. [MR 90i:54025](#).
- [9] V. Radu, *On obtaining Lévy's metric by probabilistic metrics*, An. Univ. Timișoara Ser. Ştiinţ. Mat. 24 (1986), no. 1-2, 60–65. [MR 88j:54046](#).
- [10] R. C. Walker, *The Stone-Čech Compactification*, Springer-Verlag, New York, 1974, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 83. [MR 52 # 1595](#).

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