VALUE DISTRIBUTION OF CERTAIN DIFFERENTIAL POLYNOMIALS

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Abstract. We prove a result on the value distribution of differential polynomials which improves some earlier results.

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1. Introduction and definitions. Let \( f \) be a transcendental meromorphic function in the open complex plane. The problem of possible Picard values of derivatives of \( f \) reduces to the problem of whether certain polynomials in a meromorphic function and its derivatives necessarily have zeros. We do not explain the standard definitions and notations of value distribution theory as those are available in [6].

Definition 1.1. A meromorphic function \( \alpha \) is said to be a small function of \( f \) if \( T(r, \alpha ) = S(r, f) \).

Definition 1.2 (see [1, 4, 10]). Let \( n_{0j}, n_{1j}, \ldots, n_{kj} \) be nonnegative integers. The expression \( M_{j}[f] = (f)^{n_{0j}} (f^{(1)})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}} \) is called a differential monomial generated by \( f \) of degree \( \gamma_{M_{j}} = \sum_{i=0}^{k} n_{ij} \) and weight \( \Gamma_{M_{j}} = \sum_{i=0}^{k} (i+1) n_{ij} \).

The sum \( P[f] = \sum_{j=1}^{l} b_{j} M_{j}[f] \) is called a differential polynomial generated by \( f \) of degree \( \gamma_{p} = \max \{ \gamma_{M_{j}} : 1 \leq j \leq l \} \) and weight \( \Gamma_{p} = \max \{ \Gamma_{M_{j}} : 1 \leq j \leq l \} \), where \( T(r, b_{j}) = S(r, f) \) for \( j = 1, 2, \ldots, l \).

The numbers \( \gamma_{p} = \min \{ \gamma_{M_{j}} : 1 \leq j \leq l \} \) and \( k \) (the highest order of the derivative of \( f \) in \( P[f] \)) are called, respectively, the lower degree and order of \( P[f] \).

\( P[f] \) is said to be homogeneous if \( \gamma_{p} = \gamma_{p} \).

Also \( P[F] \) is called a quasi differential polynomial generated by \( f \) if, instead of assuming \( T(r, b_{j}) = S(r, f) \), we just assume that \( m(r, b_{j}) = S(r, f) \) for the coefficients \( b_{j}(j = 1, 2, \ldots, l) \).

Definition 1.3. Let \( m \) be a positive integer. We denote by \( N(r, a; f \leq m) \) \((N(r, a; f \geq m))\) the counting function of those \( a \)-points of \( f \) whose multiplicities are not greater (less) than \( m \), where each \( a \)-point is counted according to its multiplicity.

In a similar manner, we define \( N(r, a; f < m) \) and \( N(r, a; f > m) \).

Also \( \overline{N}(r, a; f \leq m), \overline{N}(r, a; f \geq m), \overline{N}(r, a; f < m), \) and \( \overline{N}(r, a; f > m) \) are defined similarly, where in counting the \( a \)-points of \( f \) we ignore the multiplicities.

Finally, we agree to take \( \overline{N}(r, a; f \leq \infty) \equiv \overline{N}(r, a; f) \) and \( N(r, a; f \leq \infty) \equiv N(r, a; f) \).

Definition 1.4. For two meromorphic functions \( f, g \) and positive integer \( m \), we denote by \( N(r, a; f \mid g = b, > m) \) the counting function of those \( a \)-points of \( f \), counted...
with proper multiplicities, which are the b-points of g with multiplicities greater than m.

**Definition 1.5** (see [2]). Let m be a positive integer. We denote by \( N_m(r,a;f) \) the counting function of a-points of f, where an a-point of multiplicity \( \mu \) is counted \( \mu \) times if \( \mu \leq m \) and m times if \( \mu > m \).

As the standard convention, we mean by \( N(r,f) \) and \( N(r,\infty;f) \) the counting functions \( N(r,\infty;f) \) and \( N(r,\infty;f) \), respectively.

Hayman [5] proved the following theorems.

**Theorem 1.6.** If \( f \) is a transcendental meromorphic function and \( n (\geq 5) \) is a positive integer, then \( \psi = f' - af^n \) assumes all finite values infinitely often.

**Theorem 1.7.** If \( f \) is a transcendental meromorphic function and \( n (\geq 3) \) is a positive integer, then \( \psi = f'f^n \) assumes all finite values, except possibly zero, infinitely often.

When \( f \) is transcendental, entire conclusions of Theorems 1.6 and 1.7 hold, respectively for \( n \geq 3 \) (cf. [5]) and \( n \geq 1 \) (cf. [3]).

To study the value distribution of differential polynomials Yang [7] proved the following results.

**Theorem 1.8.** Let \( f \) be a transcendental meromorphic function with \( N(r,f) = S(r,f) \), and let \( \psi = f^n + P[f] \), where \( n (\geq 2) \) is an integer and \( P[f] \) is a differential polynomial generated by \( f \) with \( \gamma_P \leq n-2 \). Then \( \delta(a;\psi) < 1 \) for \( a \neq 0, \infty \).

**Theorem 1.9.** Let \( f \) be a transcendental meromorphic function with \( N(r,f) = S(r,f) \), and let \( \psi = f^nP[f] \), where \( n (\geq 2) \) is an integer and \( P[f] \) is a differential polynomial generated by \( f \). Then \( \delta(a;\psi) < 1 \) for \( a \neq 0, \infty \).

Improving all the above results, Yi [9] proved the following theorem.

**Theorem 1.10.** Let \( f \) be a transcendental meromorphic function and \( Q_1[f], Q_2[f] \) be two differential polynomials generated by \( f \) such that \( Q_1[f] \neq 0, Q_2[f] \neq 0 \), and \( P[f] = \sum_{j=0}^{n} a_jf^j \) \( (a_n \neq 0) \), where \( a_1, a_2, \ldots, a_n \) are small functions of \( f \). If \( F = P[f]Q_1[f] + Q_2[f] \), then

\[
(n - \gamma_{Q_2}) T(r,f) \leq \mathcal{N}(r,0;F) + \mathcal{N}(r,0;P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) \mathcal{N}(r,f) + S(r,f). \tag{1.1}
\]

In Theorem 1.10 we see that the influence of \( Q_1[f] \) on the value distribution of \( F \) is ignored. In this paper, we show that Theorem 1.10 can further be improved if the influence of \( Q_1[f] \) is taken into consideration. Throughout, we ignore zeros and poles of any small function of \( f \) because the corresponding counting function is absorbed in \( S(r,f) \).

2. **Lemmas.** In this section, we present some lemmas which will be needed in the sequel.

**Lemma 2.1** (see [4]). Let \( f \) be a nonconstant meromorphic function and \( Q^*[f], Q[f] \) denote differential polynomials generated by \( f \) with arbitrary meromorphic coefficients
Let $q_1^*, q_2^*, \ldots, q_s^*$ and $q_1, q_2, \ldots, q_t$, respectively. Further let $P[f] = \sum_{j=0}^{n} a_j f^j$ ($a_n \neq 0$) and $y_Q \leq n$. If $P[f]Q^*[f] = Q[f]$, then

$$m(r, Q^*[f]) \leq \sum_{j=1}^{s} m(r, q_j^*) + \sum_{j=1}^{t} m(r, q_j) + S(r, f).$$ (2.1)

**Lemma 2.2.** Let $Q[f] = \sum_{j=1}^{l} b_j M_j[f]$ be a differential polynomial generated by $f$ of order and lower degree $k$ and $\gamma_Q$ of $Q[f]$. Then $Q[f]$ be a transcendental meromorphic function in the open complex plane, and $Q[f]$ with multiplicity at least $k$ and $\gamma_Q$, respectively. If $z_0$ is a zero of $f$ with multiplicity $\mu$ ($>k$) and $z_0$ is not a pole of any of the coefficients $b_j$ ($j = 1, 2, \ldots, l$), then $z_0$ is a zero of $Q[f]$ with multiplicity at least $(\mu - k)\gamma_Q$.

**Proof.** Clearly $z_0$ is a zero of $M_j[f]$ with multiplicity $\mu n_{0j} + (\mu - 1) n_{1j} + \cdots + (\mu - k) n_{kj}$

$$= \mu \gamma_{M_j} - (\Gamma_{M_j} - \gamma_{M_j}) = (\mu - k) \gamma_{M_j} + (k + 1) \gamma_{M_j} - \Gamma_{M_j} \tag{2.2}$$

$$\geq (\mu - k) \gamma_{M_j} \geq (\mu - k) \gamma_Q.$$ Since $z_0$ is assumed not to be a pole of the coefficients $b_j$ ($j = 1, 2, \ldots, l$) we see that $z_0$ is a zero of $Q[f]$ with multiplicity at least $(\mu - k)\gamma_Q$. This proves the lemma. \hfill \Box

**Lemma 2.3** (see [1]). The following inequality holds:

$$m(r, P[f]) \leq y_P N(r, f) + (\Gamma_P - y_P) N(r, f) + S(r, f).$$ (2.3)

**Lemma 2.4** (see [7]). Let $P[f] = \sum_{i=0}^{n} a_i f^i$, where $a_n (\neq 0), a_{n-1}, \ldots, a_1, a_0$ are small functions of $f$. Then $m(r, P[f]) = nm(r, f) + S(r, f)$.

**Lemma 2.5** (see [4]). If $Q[f]$ is a differential polynomial generated by $f$ with arbitrary meromorphic coefficients $q_j$ ($1 \leq j \leq n$), then

$$m(r, Q[f]) \leq y_Q m(r, f) + \sum_{j=1}^{n} m(r, q_j) + S(r, f).$$ (2.4)

**Lemma 2.6** (see [8]). If $P[f]$ is as in Lemma 2.4, then $T(r, P[f]) = nT(r, f) + S(r, f)$.

3. The main result. In this section, we present the main result of the paper.

**Theorem 3.1.** Let $f$ be a transcendental meromorphic function in the open complex plane, and $Q_1[f]$ ($\neq 0$), $Q_2[f]$ ($\neq 0$) be two differential polynomials generated by $f$ such that $k$ and $\gamma_{Q_1}$ be the order and lower degree of $Q_1[f]$, respectively and $P[f] = \sum_{i=0}^{n} a_i f^i$, where $a_n (\neq 0), a_{n-1}, \ldots, a_0$ are small functions of $f$. If

$$F = P[f]Q_1[f] + Q_2[f],$$ (3.1)

then

$$\begin{align*}
(n - y_{Q_2}) T(r, f) &\leq N(r, 0; F) + N(r, 0; P[f]) + (\Gamma_{Q_2} - y_{Q_2} + 1) N(r, f) \\
- y \{ N(r, 0; F) - N_{k+1} (r, 0; f) \} + S(r, f) \tag{3.2}
\end{align*}$$

where $y = \gamma_{Q_1}$ if $n \geq y_{Q_2}$ and $y = 0$ if $n < y_{Q_2}$.
**Proof.** If $n < y_{Q_2}$, the theorem is obvious. So we suppose that $n \geq y_{Q_2}$. Differentiating (3.1) we get

$$F' = P' [f]Q_1 [f] + P [f] Q_1' [f] + Q_2' [f],$$

(3.3)

where $P' [f] = (d/dz)P [f]$ and $Q_i' [f] = (d/dz)Q_i [f]$ for $i = 1, 2$.

Multiplying (3.1) by $(F' / F)$, and substituting in (3.3) we get

$$P [f] Q^* [f] = Q [f],$$

(3.4)

where

$$Q^* [f] = \left( \frac{F'}{F} - \frac{P' [f]}{P [f]} \right) Q_1 [f] - Q_1' [f],$$

(3.5)

$$Q [f] = Q_2' [f] - \left( \frac{F'}{F} \right) Q_2 [f].$$

(3.6)

First we suppose that $Q^* [f] \neq 0$. By Lemma 2.1, it follows from (3.4) that $m(r, Q^* [f]) = S(r, f)$ because $y_Q = y_{Q_2} \leq n$.

Since $P [f] = Q [f] / Q^* [f]$, we get by Lemma 2.5 and the first fundamental theorem

$$m(r, P [f]) \leq m(r, Q [f]) + m(r, 0; Q^* [f])$$

$$\leq y_{Q_2} m(r, f) + m(r, Q^* [f]) + N(r, Q^* [f]) - N(r, 0; Q^* [f]) + S(r, f)$$

$$= y_{Q_2} m(r, f) + N(r, Q^* [f]) - N(r, 0; Q^* [f]) + S(r, f).$$

(3.7)

So by Lemma 2.4

$$(n - y_{Q_2}) m(r, f) \leq N(r, Q^* [f]) - N(r, 0; Q^* [f]) + S(r, f).$$

(3.8)

From (3.5) we see that possible poles of $Q^* [f]$ occur at the poles of $f$ and zeros of $F$ and $P [f]$. Also we note that the zeros of $F$ and $P [f]$ are at most simple poles of $Q^* [f]$. Let $z_0$ be a pole of $f$ with multiplicity $\mu$. Then $z_0$ is a pole of $Q [f]$ with multiplicity not exceeding $(\mu - 1) y_{Q_2} + \Gamma_{Q_2} + 1 = \mu y_{Q_2} + \Gamma_{Q_2} - y_{Q_2} + 1$ and $z_0$ is a pole of $P [f]$ with multiplicity $n \mu$. Hence, from (3.4) it follows that $z_0$ is a pole of $Q^* [f]$ with multiplicity not exceeding $\mu y_{Q_2} + \Gamma_{Q_2} - y_{Q_2} + 1 - n \mu = \Gamma_{Q_2} - y_{Q_2} + 1 - (n - y_{Q_2}) \mu$.

Therefore

$$N(r, Q^* [f]) \leq N(r, 0; F) + N(r, 0; P [f]) + (\Gamma_{Q_2} - y_{Q_2} + 1) N(r, f)$$

$$- (n - y_{Q_2}) N(r, f) + S(r, f).$$

(3.9)

Now we note that the order of the differential polynomial $Q_1' [f]$ is $k + 1$. Let $z_0$ be a zero of $f$ with multiplicity $\mu > k + 1$. Let $y_{Q_1} \geq 1$. Then by Lemma 2.2, we see that $z_0$ is a zero of $Q_1' [f]$ with multiplicity at least $(\mu - 1) y_{Q_1}$. Also $z_0$ may be a pole of $(F' / F) - P' [f] / P [f]$ with multiplicity not exceeding 1. So $z_0$ is a zero of $((F' / F) - P' [f] / P [f]) Q_1 [f]$ with multiplicity at least $(\mu - k) y_{Q_1} - 1$. 
Since the lower degree of $Q_1[f]$ is $\gamma_{Q_1}$, it follows from Lemma 2.2 that $z_0$ is a zero of $Q_1[f]$ with multiplicity at least $(\mu - k - 1)\gamma_{Q_1}$.

Therefore $z_0$ is a zero of $Q^*[f]$ with multiplicity at least $(\mu - k - 1)\gamma_{Q_1}$. Hence

\[
N(r, 0; Q^*[f]) 
\geq N(r, 0; Q^*[f])|f = 0, > k + 1)
\]

\[
\geq \gamma_{Q_1}N(r, 0; f) > k + 1 - \gamma_{Q_1} (k + 1)N(r, 0; f > k + 1) + S(r, f)
\]

\[
= \gamma_{Q_1}N(r, 0; f) - \gamma_{Q_1} \{N(r, 0; f) \leq k + 1) + (k + 1)N(r, 0; f) > k + 1)\} + S(r, f).
\]

(3.10)

So

\[
N(r, 0; Q^*[f]) \geq \gamma_{Q_1} \{N(r, 0; f) - N_{k+1}(r, 0; f)\} + S(r, f).
\]

(3.11)

If $\gamma_{Q_1} = 0$, inequality (3.11) obviously holds. Now from (3.8), (3.9), and (3.11) we get

\[
(n - \gamma_{Q_2})T(r, f) \leq N(r, 0; F) + N(r, 0; P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f)
\]

\[
- \gamma_{Q_1} \{N(r, 0; f) - N_{k+1}(r, 0; f)\} + S(r, f).
\]

(3.12)

Next we suppose that $Q^*[f] \equiv 0$. Then from (3.4) it follows that $Q[f] \equiv 0$, and so using (3.1) we get $P[f]Q_1[f] = cQ_2[f]$, where $c$ is a nonzero constant. Then in a similar line of calculation for inequalities (3.8), (3.9), and (3.11) we get

\[
(n - \gamma_{Q_2})m(r, f) \leq N(r, Q_1[f]) - N(r, 0; Q_1[f]) + S(r, f),
\]

\[
N(r, Q_1[f]) \leq (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f) - (n - \gamma_{Q_2})N(r, f) + S(r, f),
\]

(3.13)

\[
N(r, 0; Q_1[f]) \geq \gamma_{Q_1} \{N(r, 0; f) - N_{k+1}(r, 0; f)\} + S(r, f).
\]

Now from (3.13) we get

\[
(n - \gamma_{Q_2})T(r, f) \leq N(r, 0; F) + N(r, 0; P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f)
\]

\[
- \gamma_{Q_1} \{N(r, 0; f) - N_{k+1}(r, 0; f)\} + S(r, f).
\]

(3.14)

This proves the theorem.

\[\square\]

**Remark 3.2.** The following example shows that Theorem 3.1 is sharp.

**Example 3.3.** Let $f = e^2 - 2, P[f] = f + 2, Q_1[f] = f$, and $Q_2[f] = 1$. Then $F = P[f]Q_1[f] + Q_2[f] = (e^2 - 1)^2$ and $k = 0, \gamma_{Q_1} = 1, \gamma_{Q_2} = 0, n = 1$. Also we see that

\[
(n - \gamma_{Q_2})T(r, f) = N(r, 0; F) + N(r, 0; P[f]) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f)
\]

\[
- \gamma_{Q_1} \{N(r, 0; f) - N_{k+1}(r, 0; f)\} + S(r, f).
\]

(3.15)

**4. Applications.** As applications of Theorem 1.10, Yi [9] proved the following theorems which improve Theorems 1.8 and 1.9.
**Theorem 4.1.** Let $f$ be a transcendental meromorphic function and $Q_1[f] \neq 0$, $Q_2[f] \neq 0$ be two differential polynomials generated by $f$. Let $F = f^n Q_1[f] + Q_2[f]$ and

$$\limsup_{r \to \infty} \frac{N(r, 0; f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) N(r, f)}{T(r, f)} < n - \gamma_{Q_2}. \quad (4.1)$$

Then $\Theta(a; F) < 1$ for any small function $a \not\equiv 0, Q_2[f]$ of $f$.

**Theorem 4.2.** Let $F = f^n Q[f]$, where $Q[f]$ is a differential polynomial generated by $f$ and $Q[f] \not\equiv 0$. If

$$\limsup_{r \to \infty} \frac{N(r, 0; f) + N(r, f)}{T(r, f)} < n, \quad (4.2)$$

then $\Theta(a; F) < 1$, where $a \not\equiv 0, \infty$ is a small function of $f$.

Considering the following examples, Yi [9] claimed that Theorems 4.1 and 4.2 are sharp.

**Example 4.3.** Let $f = (e^{4z} + 1)/(e^{4z} - 1)$, $Q_1[f] = 1$, $Q_2[f] = f - 1$, and $F = f^4 Q_1[f] + Q_2[f]$. Then $n = 4$, $\gamma_{Q_2} = 1$, $\Gamma_{Q_2} = 2$, and

$$\limsup_{r \to \infty} \frac{N(r, 0; f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) N(r, f)}{T(r, f)} = n - \gamma_{Q_2}. \quad (4.3)$$

Also we see that $\Theta(0; F) = 1$.

**Example 4.4.** Let $f = (e^z - 1)/(e^z + 1)$, $Q_1[f] = 1$, $F = f^n Q_1[f]$, where $n = 2$. It is easy to verify that

$$\limsup_{r \to \infty} \frac{N(r, 0; f) + N(r, f)}{T(r, f)} = n \quad (4.4)$$

and $\Theta(1; F) = 1$.

The following examples suggest that some improvements of Theorems 4.1 and 4.2 are possible.

**Example 4.5.** Let $f = ((e^z - 1)/(e^z + 1))^2$, $Q_1[f] = f$, $Q_2[f] = 1$, and $F = f Q_1[f] + Q_2[f]$. Then $n = 1$, $\gamma_{Q_1} = 1$, $\gamma_{Q_2} = 0$, $\Gamma_{Q_2} = 0$, and the order of the differential polynomial $Q_1[f]$ is zero. Clearly

$$\limsup_{r \to \infty} \frac{N(r, 0; f) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) N(r, f)}{T(r, f)} = n - \gamma_{Q_2}. \quad (4.5)$$

Also we see that $\Theta(1; F) = \Theta(\infty; F) = 3/4$, $\Theta(2; F) = 1/2$ and so, by Nevanlinna’s three small functions theorem (cf. [6, page 47]), $\Theta(a; F) \leq 2 - 3/4 - 1/2 = 3/4$ for any small function $a \not\equiv 1, 2, \infty$. However, we note that

$$\limsup_{r \to \infty} \frac{N(r, 0; f) \leq 1 + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) N(r, f)}{T(r, f)} = \frac{1}{2} < n - \gamma_{Q_2}. \quad (4.6)$$
**Example 4.6.** Let \( f = ((e^z - 1)/(e^z + 1))^2 \), \( Q[f] = f \), and \( F = fQ[f] \). Then \( n = 1 \), \( \gamma_Q = 1 \), and the order of the differential polynomial \( Q[f] \) is zero. Clearly

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) + N(r, f)}{T(r, f)} = n
\]

and \( \Theta(a; F) < 1 \) for any small function \( a \) of \( f \). We note that

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) \leq 1 + N(r, f)}{T(r, f)} = \frac{1}{2} < n.
\]

The following two theorems improve Theorems 4.1 and 4.2.

**Theorem 4.7.** Let \( f \) be a transcendental meromorphic function and \( Q_1[f], Q_2[f] \) be two differential polynomials generated by \( f \) which are not identically zero. Let \( F = f^nQ_1[f] + Q_2[f] \). If

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) \leq \chi_{Q_1} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f)}{T(r, f)} < n - \gamma_{Q_2},
\]

then \( \Theta(a; F) < 1 \) for any small function \( a \) \( (\neq \infty, Q_2[f]) \) of \( f \), where

\[
\chi_{Q_1} = \begin{cases} 
1 + k & \text{if } \gamma_{Q_1} \geq 1, \\
\infty & \text{if } \gamma_{Q_1} = 0,
\end{cases}
\]

and \( k \) is the order of the differential polynomial \( Q_1[f] \).

**Theorem 4.8.** Let \( f \) be a transcendental meromorphic function and \( Q[f] \) \( (\neq 0) \) be a differential polynomial generated by \( f \). If \( F = f^nQ[f] \) and

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) \leq \chi_{Q} + N(r, f)}{T(r, f)} < n,
\]

then \( \Theta(a; F) < 1 \) for every small function \( a \) \( (\neq 0, \infty) \) of \( f \), where

\[
\chi_{Q} = \begin{cases} 
1 + k & \text{if } \gamma_{Q} \geq 1, \\
\infty & \text{if } \gamma_{Q} = 0,
\end{cases}
\]

and \( k \) is the order of the differential polynomial \( Q[f] \).

**Remark 4.9.** Theorem 4.7 improves Theorems 1.8 and 4.1, and Theorem 4.8 improves Theorems 1.9 and 4.2.

**Remark 4.10.** The following examples show that Theorems 4.7 and 4.8 are sharp.

**Example 4.11.** Let \( f = e^z - 1 \), \( Q_1[f] = f' - f \), \( Q_2[f] = 2f' \), and \( F = f^2Q_1[f] + Q_2[f] \). Then \( n = 2 \), \( k = 1 \), \( \Gamma_{Q_2} = 2 \), \( \gamma_{Q_2} = 1 \), and

\[
\limsup_{r \to \infty} \frac{N(r, 0; f) \leq 2 + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)N(r, f)}{T(r, f)} = n - \gamma_{Q_2}.
\]

Also we see that \( \Theta(1; F) = 1 \).
Example 4.12. Let $f = e^z + 1$, $Q[f] = f - f'$, and $F = f Q[f]$. Then $\chi_Q = 1$, $k = 1$, $n = 1$, and
\[
\limsup_{r \to \infty} \frac{N(r, 0; f \leq 2) + N(r, f)}{T(r, f)} = n.
\] (4.14)

Also we see that $\Theta(1; F) = 1$.

As other applications of Theorem 3.1, we obtain the following results which improve Theorems 1.6 and 1.7.

Theorem 4.13. Let $f$ be a transcendental meromorphic function, and $F = f' - a f^n$, where $a (\neq 0)$ is a small function of $f$. If $n \geq 5$ is an integer, then $\Theta(b; F) \leq 4/n$ for any small function $b (\neq \infty)$ of $f$.

Theorem 4.14. Let $f$ be a transcendental meromorphic function. If $F = f^n f'$ and $n \geq 3$ is an integer, then $\Theta(a; F) \leq 4/(n + 2)$ for any small function $a (\neq 0, \infty)$ of $f$.

We prove Theorems 4.8 and 4.14 only.

Proof of Theorem 4.8. First we treat the case $\chi_Q \geq 1$. Then by Theorem 3.1 we get
\[
n T(r, f) \leq N(r, a; F) + N(r, 0; P[f]) + N(r, f) - \chi_Q [N(r, 0; f) - N(k+1, 0; f)] + S(r, f)
\]
\[
\leq N(r, a; F) + N(r, 0; f) - N(r, 0; f) + N(k+1, 0; f) + N(r, f) + S(r, f),
\]
(4.15)

that is,
\[
n T(r, f) \leq N(r, a; F) + N(r, 0; f) + N(r, f) + S(r, f).
\] (4.16)

Now we treat the case $\chi_Q = 0$. Then from Theorem 3.1 we get
\[
n T(r, f) \leq N(r, a; F) + N(r, 0; f) + N(r, f) + S(r, f).
\] (4.17)

Combining (4.16) and (4.17), we obtain
\[
n T(r, f) \leq N(r, a; F) + N(r, 0; f) \leq \chi_Q + N(r, f) + S(r, f)
\] (4.18)

from which the theorem follows.

Proof of Theorem 4.14. Proceeding in the line of the proof of Theorem 4.8 we get
\[
n T(r, f) \leq N(r, a; F) + N(r, 0; f) \leq k + 1) + N(r, f) + S(r, f),
\] (4.19)

that is,
\[
(n - 2) T(r, f) \leq N(r, a; F) + S(r, f).
\] (4.20)
Now by Lemmas 2.3 and 2.5 we see that

\[ T(r,F) \leq (n+2)T(r,f) + S(r,f). \]  
(4.21)

If possible let \( \Theta(a;F) > 4/(n+2) \). Then there exits an \( \varepsilon(>0) \) such that for all large values of \( r \)

\[ N(r,a;F) < \left( \frac{n-2}{n+2} - \varepsilon \right) T(r,F). \]  
(4.22)

From (4.20), (4.21), and (4.22) we get

\[ \varepsilon(n+2)T(r,f) \leq S(r,f), \]  
(4.23)

which is a contradiction. This proves the theorem.

\[ \square \]

REFERENCES


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