ON A NONRESONANCE CONDITION BETWEEN THE FIRST 
AND THE SECOND EIGENVALUES FOR THE $p$-LAPLACIAN

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Abstract. We are concerned with the existence of solution for the Dirichlet problem

$$-\Delta_p u = f(x,u) + h(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,$$

when $f(x,u)$ lies in some sense between the first and the second eigenvalues of the $p$-Laplacian $\Delta_p$. Extensions to more general operators which are $(p-1)$-homogeneous at infinity are also considered.

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1. Introduction. In this paper, we are concerned with the existence of solution to the following quasilinear elliptic problem:

$$-\Delta_p u = f(x,u) + h(x) \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$  \hfill (1.1)

Here $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$, $\Delta_p$ denotes the $p$-Laplacian $\Delta_p u = \text{div}(\nabla |\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$, $h$ belongs to $W^{-1, p'}(\Omega)$ with $p'$ the Hölder conjugate of $p$ and $f$ is a Carathéodory function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that

$$\lambda_1 \leq \liminf_{s \rightarrow \pm \infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s \rightarrow \pm \infty} \frac{f(x,s)}{|s|^{p-2}s} < \lambda_2 \quad \text{a.e. in} \quad \Omega,$$  \hfill (1.2)

where $\lambda_1$ (resp., $\lambda_2$) is the first (resp., the second) eigenvalue of the problem

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega.$$  \hfill (1.3)

Problems of this sort have been extensively studied in the 70s and 80s in the semilinear case $p = 2$. In the quasilinear case $p \neq 2$, (1.1) was investigated for $N = 1$ in [6] and for $N \geq 1$ in [3]. In this latter work nonresonance is studied at the left of $\lambda_1$.

One of the difficulties to deal with the partial differential equation case $N \geq 1$ is the lack of knowledge of the spectrum of the $p$-Laplacian in that case. The basic properties of $\lambda_1$ were established in [2], while a variational characterization of $\lambda_2$ was derived recently in [4]. This variational characterization of $\lambda_2$ allows the study of its (strict) monotonicity dependence with respect to a weight. This is the property which is used in our approach to (1.1). The asymmetry in our assumption (1.2) between $\lambda_1$ and $\lambda_2$ also comes from that property. In fact it remains an open question whether the last strict inequality in (1.2) can be replaced by $\leq$. 

In Section 3 we extend our existence result to more general operators. We consider
\[ A(u) = f(x,u) + h(x) \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]
where \( A = -\sum_{i=1}^{N} (\partial / \partial x_i) A_i(x,u, \nabla u) \) verifies a \((p-1)\)-homogeneity condition at infinity. Such operators were studied by Anane [1] in the variational case. Here we use degree theory for mappings of type \((S)_+\) as developed by Browder [7] and Berkowitz and Mustonen [5]. No variational structure is consequently needed.

2. A result for the \(p\)-Laplacian. We seek a weak solution of (1.1), that is,
\[ \text{find } u \in W^{1,p}_0(\Omega) \text{ such that } \forall v \in W^{1,p}_0(\Omega) : \]
\[ \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x,u) v \, dx + \langle h, v \rangle, \]
where \( \langle \cdot, \cdot \rangle \) denotes the duality product between \( W^{-1,p'}(\Omega) \) and \( W^{1,p}_0(\Omega) \). We assume that \( f \) satisfies
\[ \max_{|s| \leq R} |f(x,s)| \in L^{p'}(\Omega), \quad \forall R > 0, \]
\[ \lambda_1 \leq l(x) \leq k(x) < \lambda_2 \text{ a.e. in } \Omega, \]
where
\[ l(x) = \liminf_{s \to \pm \infty} \frac{f(x,s)}{|s|^{p-2}}, \quad k(x) = \limsup_{s \to \pm \infty} \frac{f(x,s)}{|s|^{p-2}}. \]
The first inequality in (2.3) must be understood as “less or equal almost everywhere together with strict inequality on a set of positive measure.” We also assume that some uniformity holds in the inequalities in (2.3):
\[ \forall \epsilon > 0, \exists \eta(\epsilon) > 0 : \lambda_1 - \epsilon \leq \frac{f(x,s)}{|s|^{p-2}}, \quad \forall |s| \geq \eta(\epsilon), \text{ a.e. in } \Omega, \]
\[ \forall \epsilon > 0, \exists \eta(\epsilon) > 0 : \lambda_2 + \epsilon \leq \frac{f(x,s)}{|s|^{p-2}}, \quad \forall |s| \geq \eta(\epsilon), \text{ a.e. in } \Omega. \]

**Remark 2.1.** It is clear that (2.2) and (2.5) imply the growth condition
\[ |f(x,s)| \leq a |s|^{p-1} + b(x) \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega, \]
where \( a > 0 \) and \( b(\cdot) \in L^{p'}(\Omega) \).

**Remark 2.2.** Equations (2.2) and (2.5) also imply
\[ \forall \epsilon > 0, \exists b_\epsilon \in L^{p'}(\Omega) \text{ such that } \]
\[ |s|^{p} (\lambda_1 - \epsilon) - b_\epsilon(x) \leq sf(x,s) \leq |s|^{p} (\lambda_2 + \epsilon) + b_\epsilon(x), \quad \forall s \in \mathbb{R}, \text{ a.e. in } \Omega. \]
Theorem 2.3. Suppose that $f$ satisfies (2.2), (2.3), and (2.5). Then for any $h \in W^{-1,p'}(\Omega)$, problem (2.1) admits a solution $u$ in $W^{1,p}_0(\Omega)$.

Proof. We denote by $(T_t)_{t \in [0,1]}$ the family of operators from $W^{1,p}_0(\Omega)$ to $W^{1,p}_0(\Omega)$ defined by

$$T_t(u) = (-\Delta_p)^{-1}[(1-t)\alpha|u|^{p-2}u + tf(\cdot,u) + th(\cdot)]$$

(2.8)

where $\alpha$ is some fixed number with $\lambda_1 < \alpha < \lambda_2$.

To prove Theorem 2.3, we first establish the following estimate:

$$\exists R > 0 \text{ such that } \forall t \in [0,1], \forall u \in \partial B(O,R) \text{ such that } [I - T_t](u) \neq 0,$$

(2.9)

where $B(O,R)$ denotes the ball of center $O$ and radius $R$ in $W^{1,p}_0(\Omega)$.

To prove (2.9) we assume by contradiction that

$$\forall n > 0, \exists t_n \in [0,1], \exists u_n \in W^{1,p}_0(\Omega) \text{ with } \|u_n\|_{1,p} = n \text{ such that } T_{t_n}(u_n) = u_n,$$

(2.10)

where $\| \cdot \|_{1,p}$ denotes the norm in $W^{1,p}_0(\Omega)$.

Let $w_n = u_n/n$. We can extract from $(w_n)$ a subsequence, still denoted by $(w_n)$, which converges weakly in $W^{1,p}_0(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in $\Omega$ to $w \in W^{1,p}_0(\Omega)$. We can also suppose that $t_n$ converges to $t \in [0,1]$. To reach a contradiction, we use the following lemmas which give various information on $w_n$ and $w$.

Lemma 2.4. The sequence $g_n$ defined by

$$g_n = \frac{f(x,nw_n)}{n^{p-1}}$$

(2.11)

is bounded in $L^{p'}(\Omega)$, and consequently, for a subsequence, $g_n$ converges weakly to some $g$ in $L^{p'}(\Omega)$.

Proof. This is an immediate consequence of (2.6).

Lemma 2.5. $w \neq 0$.

Proof. Since $w_n$ verifies,

$$\int_\Omega |\nabla w_n|^p dx = (1-t_n)\alpha \int_\Omega |w_n|^p dx + t_n \left[ \int_\Omega g_n(x)w_n(x) dx + \frac{1}{n^{p-1}} \langle h, w_n \rangle \right]$$

(2.12)

we deduce from Lemma 2.4 that

$$1 = (1-t)\alpha \int_\Omega |w|^p dx + t \int_\Omega g(x)w(x) dx,$$

(2.13)

which clearly implies the conclusion of Lemma 2.5.

Lemma 2.6. $g = 0$ a.e. in $\Omega \setminus A$, where $A = \{x \in \Omega : w(x) \neq 0\}$.
**Proof.** By (2.6), we have
\[ |g_n(x)| \leq a \left| w_n \right|^{p-1} + \frac{b(x)}{n^{p-1}} \text{ a.e. in } \Omega, \tag{2.14} \]
and so
\[ \left\| g_n \right\|_{L^p(\Omega \setminus A)} \leq a \left\| w_n \right\|_{L^p(\Omega \setminus A)}^{p/p'} + \frac{1}{np-1} \left\| b \right\|_{L^p(\Omega \setminus A)}, \tag{2.15} \]
which implies
\[ \lim_{n \to +\infty} \left\| g_n \right\|_{L^p(\Omega \setminus A)} = 0. \tag{2.16} \]
Set \( D = \{ x \in \Omega \setminus A : g(x) \neq 0 \}. \) By Lemma 2.4 we have, for \( \phi(x) = \text{sign}[g(x)] \chi_D(x) \in L^p(D) \)
\[ \lim_{n \to +\infty} \int_D g_n(x) \phi(x) \, dx = \int_D |g(x)| \, dx, \tag{2.17} \]
and consequently by (2.16),
\[ \int_D |g(x)| \, dx = 0, \tag{2.18} \]
which implies \( \text{meas}(D) = 0, \) that is, the conclusion of Lemma 2.6.

**Lemma 2.7.** Set
\[ \tilde{g}(x) = \begin{cases} g(x) & \text{on } A, \\ \frac{g(x)}{|w(x)|^{p-2} w(x)} & \text{on } \Omega \setminus A, \end{cases} \tag{2.19} \]
where \( \beta \) is a fixed number with \( \lambda_1 < \beta < \lambda_2. \) We have
\[ \lambda_1 \leq \tilde{g}(x) < \lambda_2 \text{ a.e. in } \Omega. \tag{2.20} \]

**Proof.** Set
\[ B_l = \{ x \in A : w(x)g(x) < l(x) |w(x)|^p \}, \]
\[ B_k = \{ x \in A : w(x)g(x) > k(x) |w(x)|^p \}. \tag{2.21} \]
We first prove that \( \text{meas}(B_l) = 0 \) and \( \text{meas}(B_k) = 0. \)
By (2.7), we have that \( \forall \varepsilon \geq 0, \exists b_\varepsilon \in L^{p'}(\Omega) \) such that
\[ -\frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p [l(x) - \varepsilon] \leq w_n(x) g_n(x) \leq \frac{b_\varepsilon(x)}{n^p} + |w_n(x)|^p [k(x) + \varepsilon] \text{ a.e. in } \Omega. \tag{2.22} \]
The first inequality gives
\[ -\frac{1}{n^p} \int_{B_l} b_\varepsilon(x) \, dx + \int_{B_l} |w_n(x)|^p [l(x) - \varepsilon] \, dx \leq \int_{B_l} w_n(x) g_n(x) \, dx. \tag{2.23} \]
Letting first \( x \to \infty, \) then \( \varepsilon \to 0, \) we deduce
\[ \int_{B_l} \left[ w(x)g(x) - |w(x)|^p l(x) \right] \, dx \geq 0, \tag{2.24} \]
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which implies \( \text{meas}(B_1) = 0 \). Similarly one gets \( \text{meas}(B_k) = 0 \). We thus have

\[
l(x) \leq \tilde{g}(x) \leq k(x) \quad \text{a.e. in } A.
\]  

(2.25)

Since

\[
\lambda_1 < \tilde{g}(x) = \beta < \lambda_2 \quad \text{a.e. in } \Omega \setminus A,
\]  

(2.26)

we obtain the conclusion of the lemma.

\[ \square \]

**Lemma 2.8.** \( w \) is a solution of

\[
-\Delta_p w = m|w|^{p-2}w \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega,
\]  

(2.27)

where \( m(x) = (1 - t)\alpha + t\tilde{g}(x) \).

**Proof.** We first prove that \( w \) is a solution of

\[
-\Delta_p w = (1 - t)\alpha|w|^{p-2}w + tg \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega.
\]  

(2.28)

We recall that \( w_n \) satisfies

\[
-\Delta_p w_n = (1 - t_n)\alpha|w_n|^{p-2}w_n + t_n \left[ g_n + \frac{1}{n^{p-1}} h \right] \quad \text{in } \Omega,
\]

\[
w_n = 0 \quad \text{on } \partial \Omega.
\]  

(2.29)

Since \( (-\Delta_p)(w_n) \) is bounded in \( W^{-1,p'}(\Omega) \), there exists a subsequence, still denoted by \( (w_n) \), and a distribution \( T \in W^{-1,p'}(\Omega) \), such that \( (-\Delta_p)(w_n) \) converges weakly to \( T \) in \( W^{-1,p'}(\Omega) \); in particular

\[
\lim_{n \to +\infty} \langle -\Delta_p w_n, w \rangle = \langle T, w \rangle.
\]  

(2.30)

We also have

\[
\langle -\Delta_p w_n, w_n - w \rangle = (1 - t_n)\alpha \int_{\Omega} |w_n|^{p-2}w_n(w_n - w) \, dx
\]

\[
+ t_n \left[ \int_{\Omega} g_n(x)(w_n - w) \, dx + \frac{1}{n^{p-1}} \langle h, w_n - w \rangle \right],
\]  

(2.31)

which implies

\[
\lim_{n \to +\infty} \langle -\Delta_p w_n, w_n - w \rangle = 0,
\]  

(2.32)

and therefore

\[
\lim_{n \to +\infty} \langle -\Delta_p w_n, w_n \rangle = \langle T, w \rangle.
\]  

(2.33)

Since \( (-\Delta_p) \) is an operator of type \( (M) \), we deduce

\[
T = -\Delta_p w.
\]  

(2.34)
Going to the limit in (2.29) then yields (2.28). But by Lemma 2.6, we have
\[(1 - t)\alpha |w|^{p-2}w + tg = m|w|^{p-2}w \quad \text{a.e. in } \Omega.\] (2.35)
So \(w\) is a solution of (2.27).

We denote by \(\lambda_1(\Omega, r(x))\) (resp., \(\lambda_2(\Omega, r(x))\)) the first (resp., the second) eigenvalue in the problem with weight
\[-\Delta_p u = \lambda r(x)|u|^{p-2}u \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega.\] (2.36)
By Lemma 2.7 and the fact that \(\lambda_1 < \alpha < \lambda_2\), we have
\[\lambda_1 \leq m(x) < \lambda_2 \quad \text{a.e. in } \Omega.\] (2.37)
It follows, by the strict monotonicity property of the second eigenvalue with respect to the weight (cf. [4]), that
\[1 = \lambda_2(\Omega, \lambda_2) < \lambda_2(\Omega, m).\] (2.38)
It also follows by the strict monotonicity of the first eigenvalue with respect to the weight (cf. [8]), that
\[\lambda_1(\Omega, m) < \lambda_1(\Omega, \lambda_1) = 1.\] (2.39)
Consequently,
\[\lambda_1(\Omega, m) < 1 < \lambda_2(\Omega, m).\] (2.40)
But by Lemmas 2.5 and 2.8, 1 is an eigenvalue of \((-\Delta_p)\) for the weight \(m\). This contradicts the definition of the second eigenvalue \(\lambda_2(\Omega, m)\). We have thus proved that the estimate (2.9) holds.

We can now conclude by a standard degree argument. Indeed \(T_t\) is clearly completely continuous, since \((\Delta_p)^{-1}\) is continuous from \(W^{-1,p'}(\Omega)\) to \(W_0^{1,p}(\Omega)\). Therefore,
\[\deg (I - T_0, B(O, R), O) = \deg (I - T_1, B(O, R), O).\] (2.41)
Since \(T_0\) is odd, we have, by Borsuk theorem, that \(\deg(I - T_0, B(O, R), O)\) is an odd integer and so nonzero. It then follows that there exists \(u \in B(O, R)\) such that \(T_1(u) = u\), which proves Theorem 2.3.

3. Generalization. Theorem 2.3 will now be extended to the case of nonhomogeneous operators. We consider the problem
\[A(u) = f(x, u) + h(x) \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega,\] (3.1)
where
\[A(u) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} A_i(x, u(x), \nabla u(x)).\] (3.2)
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The method used in Section 2 for \((-\Delta_p)\) can be adapted under suitable assumptions on \(A\). We basically assume that \(A\) is a Leray-Lions operator which is \((p-1)\)-homogeneous at infinity. Our precise assumptions are the following:

Each \(A_i(x,s,\xi)\) is a Carathéodory function,

\[
\sum_{i=1}^{N} \left[ A_i(x,s,\xi) - A_i(x,s,\xi') \right] (\xi_i - \xi_i') > 0, \quad \text{for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ all } \xi \neq \xi' \in \mathbb{R}^N, \tag{3.3}
\]

\(\exists K \in L^{p'}(\Omega), \exists c(t)\) a function defined on \(\mathbb{R}^+\) with \(\lim_{t \to +\infty} c(t) = 0\) such that

\[
|A_i(x,ts,t\xi) - t^{p-1}|\xi|^p|\xi'|^{p-2}\xi_i| \leq t^{p-1}c(t)\left[|\xi|^p + |s|^{p-1} + K(x)\right], \tag{3.5}
\]

for a.e. \(x \in \Omega, \text{ all } s \in \mathbb{R}, \text{ all } \xi \in \mathbb{R}^N, \text{ all } t \in \mathbb{R}^+\).

We will be able to solve (3.1) when \(f(x,s)\) lies at infinity between the first and the second eigenvalues of the \(p\)-Laplacian \((-\Delta_p), \text{ in the sense of (1.2).}\)

**Remark 3.1.** Equation (3.5) is a hypothesis which means that \(A\) is asymptotically homogeneous to \((-\Delta_p)\). An example of an operator which verifies (3.3), (3.4), and (3.5) is the following regularized version of the \(p\)-Laplacian:

\[
A = -\Delta_{p,\epsilon} = -\text{div}\left[(\epsilon + |\nabla u|^2)^{(p-2)/2}\nabla u\right] \tag{3.6}
\]

with \(\epsilon > 0\).

**Remark 3.2.** Equations (3.3), (3.4), and (3.5) imply the following usual growth and coercivity conditions:

\(\exists c_4 > 0, \exists K_4 \in L^{p'}(\Omega)\) such that \(|A_i(x,s,\xi)| \leq c_4\left(|\xi|^p + |s|^{p-1} + K_4(x)\right), \tag{3.7}\)

a.e. \(x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N, \text{ for } i = 1, \ldots, N,\)

\(\exists c_5 > 0, c_5' > 0, K_5 \in L^1(\Omega)\) such that \(\sum_{i=1}^{N} A_i(x,s,\xi)\xi_i \geq c_5|\xi|^p - c_5'|s|^p - K_5(x), \tag{3.8}\)

a.e. \(x \in \Omega, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N.\)

Indeed (3.7) follows immediately from (3.5). To verify (3.8), one observes that by (3.5) one has, for each \(t > 0,\)

\[
A_i(x,ts,t\xi)\xi_i - t^{p-1}|\xi|^p|\xi'|^{p-2}\xi_i^2 \geq -t^{p-1}c(t)|\xi_i|\left[|\xi|^p + |s|^{p-1} + K(x)\right], \tag{3.9}
\]

and so

\[
\sum_{i=1}^{N} A_i(x,ts,t\xi)\xi_i \geq t^{p-1}|\xi|^p\left[1 - Nc(t)\left(1 + \frac{2}{p}\right)\right] - \frac{1}{p'}t^{p-1}c(t)N(\left(|s|^p + |K(x)|^p\right). \tag{3.10}
\]

Choosing \(t\) sufficiently large yields (3.8).
Remark 3.3. Equations (3.3) and (3.5) imply that $A$ is well defined, continuous, and bounded from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Equations (3.3), (3.4), and (3.5) also imply that $A$ is of type $(S)_+$. This latter fact can be proved along similar lines as in the argument given by Berkovits and Mustonen in [5].

We are now ready to state the following theorem.

**Theorem 3.4.** Assume (2.2), (2.3), (2.5), (3.3), (3.4), and (3.5). Then for any $h \in W^{-1,p'}(\Omega)$, there exists a weak solution $u \in W_0^{1,p}(\Omega)$ of (3.1), that is,

$$
\int_\Omega \sum_{i=1}^N A_i(x,u(x),\nabla u(x)) \frac{\partial v}{\partial x_i} \, dx = \int_\Omega f(x,u) v \, dx + \langle h,v \rangle, \quad \forall v \in W_0^{1,p}(\Omega).
$$

(3.11)

**Proof.** The proof is rather similar to that of Theorem 2.3, and we will only detail below those points which really involve the operator $A$.

Let $(S_t)^t \in [0,1]$ be the family of operators from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$ defined by

$$
S_t(u) = tA(u) - (1-t)(\Delta_p u) - t[f(x,u) + h(x)] - (1-t)\alpha|u|^{p-2}u,
$$

(3.12)

for some fixed number $\alpha$ with $\lambda_1 < \alpha < \lambda_2$. Since the operator $A$ is of type $(S)_+$, $S_t$ is also of type $(S)_+$. By the degree theory for mappings of type $(S)_+$, as developed in Browder [7] and Berkowits and Mustonen [5], to solve (3.1) it suffices to prove the following estimate:

$$
\exists R > 0 \text{ such that } \forall t \in [0,1], \forall u \in \partial B(OR) \text{ such that } S_t(u) \neq 0.
$$

(3.13)

To prove (3.13), we assume by contradiction that

$$
\forall n \in \mathbb{N}, \exists t_n \in [0,1], \exists u_n \in W_0^{1,p}(\Omega) \text{ with } \|u_n\|_{1,p} = n \text{ such that } S_{t_n}(u_n) = 0.
$$

(3.14)

Let $w_n = u_n/n$. We can extract from $(w_n)$ a subsequence, still denoted by $(w_n)$, which converges weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in $\Omega$ to $w \in W_0^{1,p}(\Omega)$. We can also suppose that $t_n$ converges to $t \in [0,1]$.

In the same manner as in the proof of Theorem 2.3, to obtain a contradiction, we use Lemmas 2.4, 2.6, and 2.7 (which do not involve the operator $A$) together with the following two lemmas.

**Lemma 3.5.** $w \neq 0$.

**Proof.** By (3.14) we have

$$
\left\langle t_nA\left(\frac{u_n}{n^{p-1}}\right) - (1-t_n)\Delta_p w_n, w_n\right\rangle = (1-t_n)\alpha \int_\Omega |w_n|^p \, dx
$$

$$
+ t_n \left[ \int_\Omega g_n(x) w_n(x) \, dx + \frac{1}{n^{p-1}} \langle h, w_n \rangle \right].
$$

(3.15)
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Since

\[
\left| \frac{t_n A(u_n)}{n^{p-1}} - t_n (-\Delta_p w_n), w_n \right| \leq n^{1-p} \int_{\Omega} \sum_{i=1}^{N} A_i(x, u_n, n \nabla w_n) - n^{p-1} |\nabla w_n|^{p-2} \frac{\partial w_n}{\partial x_i} \cdot \frac{\partial w_n}{\partial x_i} \, dx,
\]

using (3.5) and the fact that \( \|w_n\|_{1,p} = 1 \), we obtain

\[
\left| \frac{t_n A(u_n)}{n^{p-1}} - t_n (-\Delta_p w_n), w_n \right| \leq c(n) \left[ \|\nabla w_n\|_{L^p(\Omega)}^{p'/p} + \|w_n\|_{L^p(\Omega)}^{p'/p} + \|K\|_{L^{p'}(\Omega)} \right] \|w_n\|_{1,p}^{\frac{n-1}{n-\infty}} 0.
\]

Therefore

\[
1 = (1 - t) \alpha \int_{\Omega} |w|^p \, dx + t \int_{\Omega} g(x) w(x) \, dx,
\]

which clearly implies \( w \neq 0 \). \qed

**Lemma 3.6.** \( w \) is a solution of

\[
-\Delta_p w = m|w|^{p-2} w \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega,
\]

where \( m(x) = ((1 - t) \alpha + t \tilde{g}(x)) \) and \( \tilde{g} \) is defined in Lemma 2.7.

**Proof.** We first show that \( w \) is a solution of

\[
-\Delta_p w = (1 - t) \alpha |w|^{p-2} w + t g \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega.
\]

Since \((-\Delta_p)(w_n)\) is bounded in \( W^{-1,p'}(\Omega) \), there exists a subsequence, still denoted by \((w_n)\), and a distribution \( T \in W^{-1,p'}(\Omega) \), such that \((-\Delta_p)(w_n)\) converges weakly to \( T \) in \( W^{-1,p'}(\Omega) \). In particular

\[
\lim_{n \to +\infty} \langle -\Delta_p w_n, w \rangle = \langle T, w \rangle.
\]

We also have

\[
\langle -\Delta_p w_n, w_n - w \rangle = (1 - t_n) \alpha \int_{\Omega} |w_n|^{p-2} w_n (w_n - w) \, dx
\]

\[
+ t_n \left[ \int_{\Omega} g_n(x) (w_n - w) \, dx + \frac{1}{n^{p-1}} \langle h, w_n - w \rangle \right]
\]

\[
- \left\langle t_n \left[ \frac{A(u_n)}{n^{p-1}} + \Delta_p w_n \right], w_n - w \right\rangle,
\]

\[
(3.20)
\]
and since, by (3.5),
\[
\left| \left\langle t_n \left[ \frac{A(u_n)}{n^{p-1}} + \Delta_p w_n \right], w_n - w \right\rangle \right| \leq c(n) \left[ \| \nabla w_n \|_{L^p(\Omega)}^{p/p'} + \| w_n \|_{L^p(\Omega)}^{p/p'} + \| K \|_{L^p(\Omega)} \right] \| w_n - w \|_{1,p} \rightarrow 0, \quad n \rightarrow +\infty,
\]
we deduce
\[
\lim_{n \rightarrow +\infty} \left\langle -\Delta_p w_n, w_n - w \right\rangle = 0. \tag{3.24}
\]

The rest of the proof of Lemma 3.6 uses the fact that \((-\Delta_p)\) is of type (M) and is similar to the proof of Lemma 2.8.

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\textbf{References}


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