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A NOTE ON MUES' CONJECTURE

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ABSTRACT. We prove that Mues' conjecture holds for the second- and higher-order derivatives of a square and higher power of any transcendental meromorphic function.

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1. Introduction, definitions, and results. Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. For a positive integer l we denote by $N(r,\infty;f\mid\geq l)$ the counting function of the poles of f with multiplicities not less than l, where a pole is counted according to its multiplicity. Also for $\alpha\in\mathbb{C}$, we denote by $N(r,\alpha;f\mid=1)$ the counting function of simple zeros of $f-\alpha$. We do not explain the standard definitions and notations of the value distribution theory as they are available in [1, 6].

In 1971, Mues [4] conjectured that for a positive integer k the following relation might be true:

$$\sum_{a\neq\infty}\delta(a;f^{(k)})\leq 1. \tag{1.1}$$

Mues [4] himself proved the following theorem.

THEOREM 1.1. *If $N(r,f)-\bar{N}(r,f)=o\{N(r,f)\}$, then for $k\geq 2$*

$$\sum_{a\neq\infty}\delta(a;f^{(k)})\leq 1. \tag{1.2}$$

In this direction Ishizaki [3] proved the following result.

THEOREM 1.2. *If for some $l(\geq 2)$ $N(r,\infty;f\mid\geq l)=o\{N(r,f)\}$, then for all $k\geq l$*

$$\sum_{a\neq\infty}\delta(a;f^{(k)})\leq 1. \tag{1.3}$$

Yang and Wang [7] also worked on Mues' conjecture and proved the following theorem.

THEOREM 1.3. *There exists a positive number $K=K(f)$ such that for every positive integer $k\geq K$*

$$\sum_{a\neq\infty}\delta(a;f^{(k)})\leq 1. \tag{1.4}$$

We see that in Theorem 1.3 the set of exceptional integers k is different for different function f. In this paper, we show that if f is a square or a higher power of a meromorphic function, then the relation (1.1) holds for any integer $k \geq 2$. This result follows as a consequence of the following theorem because such a function has no simple zero.

Theorem 1.4. *If $N(r,\alpha;f \mid = 1) = S(r,f)$ for some $\alpha \neq \infty$, then for $k \geq 2$*

$$\sum_{a \neq \infty} \delta(a;f^{(k)}) \leq 1. \tag{1.5}$$

2. Lemmas. In this section, we state two lemmas which will be needed in the proof of Theorem 1.4.

Lemma 2.1 (see [2]). *Let $A > 1$, then there exists a set $M(A)$ of upper logarithmic density at most* $\min\{(2e^{A-1}-1)^{-1}, (1+e(A-1)\exp(e(1-A)))\}$ *such that for $k = 1,2,3,\ldots$*

$$\limsup_{r\to\infty,\ r\notin M(A)} \frac{T(r,f)}{T(r,f^{(k)})} \leq 3eA. \tag{2.1}$$

Lemma 2.2 (see [5]). *For any integer $k(\geq 0)$ and any positive number $\varepsilon(>0)$, we get*

$$(k-2)\bar{N}(r,f) + N(r,0;f) \leq 2\bar{N}(r,0;f) + N(r,0;f^{(k)}) + \varepsilon T(r,f) + S(r,f). \tag{2.2}$$

3. Proof of Theorem 1.4. Without loss of generality, we may choose $\alpha = 0$. Let $g = f - \alpha$. Then $f^{(k)} = g^{(k)}$ and

$$N(r,0;g \mid = 1) = N(r,\alpha;f \mid = 1) = S(r,f) = S(r,g). \tag{3.1}$$

Applying the second fundamental theorem to $f^{(k)}$, we get for any q finite distinct complex numbers $a_1, a_2, \ldots, a_q$

$$\begin{aligned} &m(r,f^{(k)}) + \sum_{j=1}^{q} m(r,a_j;f^{(k)}) \\ &\quad \leq 2T(r,f^{(k)}) - N(r,0;f^{(k+1)}) - 2N(r,f^{(k)}) + N(r,f^{(k+1)}) + S(r,f^{(k)}), \end{aligned} \tag{3.2}$$

that is,

$$\sum_{j=1}^{q} m(r,a_j;f^{(k)}) \leq T(r,f^{(k)}) + \bar{N}(r,f) - N(r,0;f^{(k+1)}) + S(r,f^{(k)}). \tag{3.3}$$

By Lemma 2.2 and from (3.3) we get

$$\begin{aligned} \sum_{j=1}^{q} m(r,a_j;f^{(k)}) &\leq T(r,f^{(k)}) + \bar{N}(r,f) + 2\bar{N}(r,0;f) - N(r,0;f) \\ &\quad - (k-1)\bar{N}(r,f) + \varepsilon T(r,f) + S(r,f) + S(r,f^{(k)}). \end{aligned} \tag{3.4}$$

Since $2\tilde{N}(r,0;f)-N(r,0;f)\le N(r,0;f\mid=1)=S(r,f)$ and $k\ge 2$, we get from (3.4)

$$\sum_{j=1}^{q} m(r,a_j;f^{(k)})\le T(r,f^{(k)})+\varepsilon T(r,f)+S(r,f)+S(r,f^{(k)}). \tag{3.5}$$

Let E be the exceptional set arising out of Lemma 2.2, the second fundamental theorem, and the condition $N(r,0;f\mid=1)=S(r,f)$. We choose a sequence of positive numbers $\{r_n\}$ tending to infinity such that $r_n\notin E\cup M(A)$. Then from (3.5) we get, for $r=r_n$ in view of Lemma 2.1,

$$\sum_{j=1}^{q} m(r_n,a_j;f^{(k)})\le T(r_n,f^{(k)})+3eA\varepsilon T(r_n,f^{(k)})+o\{T(r_n,f^{(k)})\}, \tag{3.6}$$

which gives

$$\sum_{j=1}^{q}\delta(a_j;f^{(k)})\le 1+3eA\varepsilon. \tag{3.7}$$

Since $\varepsilon(>0)$ is arbitrary and q is an arbitrary positive number, we get from (3.7)

$$\sum_{a\neq\infty}\delta(a;f^{(k)})\le 1. \tag{3.8}$$

This proves the theorem. □

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