A NOTE ON MUES’ CONJECTURE

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ABSTRACT. We prove that Mues’ conjecture holds for the second- and higher-order derivatives of a square and higher power of any transcendental meromorphic function.

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1. Introduction, definitions, and results. Let \( f \) be a transcendental meromorphic function defined in the open complex plane \( \mathbb{C} \). For a positive integer \( l \) we denote by \( N(r, \infty; f \mid \geq l) \) the counting function of the poles of \( f \) with multiplicities not less than \( l \), where a pole is counted according to its multiplicity. Also for \( \alpha \in \mathbb{C} \), we denote by \( N(r, \alpha; f \mid = 1) \) the counting function of simple zeros of \( f - \alpha \). We do not explain the standard definitions and notations of the value distribution theory as they are available in [1, 6].

In 1971, Mues [4] conjectured that for a positive integer \( k \) the following relation might be true:

\[
\sum_{a \neq \infty} \delta(a; f^{(k)}) \leq 1. \tag{1.1}
\]

Mues [4] himself proved the following theorem.

**Theorem 1.1.** If \( N(r, f) - \overline{N}(r, f) = o\{N(r, f)\} \), then for \( k \geq 2 \)

\[
\sum_{a \neq \infty} \delta(a; f^{(k)}) \leq 1. \tag{1.2}
\]

In this direction Ishizaki [3] proved the following result.

**Theorem 1.2.** If for some \( l \geq 2 \) \( N(r, \infty; f \mid \geq l) = o\{N(r, f)\} \), then for all \( k \geq l \)

\[
\sum_{a \neq \infty} \delta(a; f^{(k)}) \leq 1. \tag{1.3}
\]

Yang and Wang [7] also worked on Mues’ conjecture and proved the following theorem.

**Theorem 1.3.** There exists a positive number \( K = K(f) \) such that for every positive integer \( k \geq K \)

\[
\sum_{a \neq \infty} \delta(a; f^{(k)}) \leq 1. \tag{1.4}
\]
We see that in Theorem 1.3 the set of exceptional integers \( k \) is different for different function \( f \). In this paper, we show that if \( f \) is a square or a higher power of a meromorphic function, then the relation (1.1) holds for any integer \( k \geq 2 \). This result follows as a consequence of the following theorem because such a function has no simple zero.

**Theorem 1.4.** If \( N(r, \alpha; f \mid= 1) = S(r, f) \) for some \( \alpha \neq \infty \), then for \( k \geq 2 \)

\[
\sum_{a \neq \infty} \delta(a; f^{(k)}) \leq 1.
\] (1.5)

**2. Lemmas.** In this section, we state two lemmas which will be needed in the proof of Theorem 1.4.

**Lemma 2.1** (see [2]). Let \( A > 1 \), then there exists a set \( M(A) \) of upper logarithmic density at most \( \min\{\frac{2e^{A-1} - 1}{1 + e(A-1)\exp(e(1-A))}\} \) such that for \( k = 1, 2, 3, \ldots \)

\[
\limsup_{r \to \infty, r \notin M(A)} T(r, f) \leq 3eA.
\] (2.1)

**Lemma 2.2** (see [5]). For any integer \( k(\geq 0) \) and any positive number \( \varepsilon(> 0) \), we get

\[
(k - 2)\tilde{N}(r, f) + N(r, 0; f) \leq 2\tilde{N}(r, 0; f) + N(r, 0; f^{(k)}) + \varepsilon T(r, f) + S(r, f).
\] (2.2)

**3. Proof of Theorem 1.4.** Without loss of generality, we may choose \( \alpha = 0 \). Let \( g = f - \alpha \). Then \( f^{(k)} = g^{(k)} \) and

\[
N(r, 0; g \mid= 1) = N(r, \alpha; f \mid= 1) = S(r, f) = S(r, g).
\] (3.1)

Applying the second fundamental theorem to \( f^{(k)} \), we get for any \( q \) finite distinct complex numbers \( a_1, a_2, \ldots, a_q \)

\[
m(r, f^{(k)}) + \sum_{j=1}^{q} m(r, a_j; f^{(k)}) \leq 2T(r, f^{(k)}) - N(r, 0; f^{(k+1)}) - 2N(r, f^{(k)}) + N(r, f^{(k+1)}) + S(r, f^{(k)}),
\] (3.2)

that is,

\[
\sum_{j=1}^{q} m(r, a_j; f^{(k)}) \leq T(r, f^{(k)}) + \tilde{N}(r, f) - N(r, 0; f^{(k+1)}) + S(r, f^{(k)}).
\] (3.3)

By Lemma 2.2 and from (3.3) we get

\[
\sum_{j=1}^{q} m(r, a_j; f^{(k)}) \leq T(r, f^{(k)}) + \tilde{N}(r, f) + 2\tilde{N}(r, 0; f) - N(r, 0; f)
\]

\[
- (k - 1)\tilde{N}(r, f) + \varepsilon T(r, f) + S(r, f) + S(r, f^{(k)}).
\] (3.4)
Since $2\tilde{N}(r,0;f) - N(r,0;f) \leq N(r,0;f \mid = 1) = S(r,f)$ and $k \geq 2$, we get from (3.4)
\[
\sum_{j=1}^{q} m(r,a_j;f^{(k)}) \leq T(r,f^{(k)}) + \varepsilon T(r,f) + S(r,f) + S(r,f^{(k)}). \tag{3.5}
\]

Let $E$ be the exceptional set arising out of Lemma 2.2, the second fundamental theorem, and the condition $N(r,0;f \mid = 1) = S(r,f)$. We choose a sequence of positive numbers $\{r_n\}$ tending to infinity such that $r_n \notin E \cup M(A)$. Then from (3.5) we get, for $r = r_n$ in view of Lemma 2.1,
\[
\sum_{j=1}^{q} m(r_n,a_j;f^{(k)}) \leq T(r_n,f^{(k)}) + 3eA\varepsilon T(r_n,f^{(k)}) + o\{T(r_n,f^{(k)})\}, \tag{3.6}
\]
which gives
\[
\sum_{j=1}^{q} \delta(a_j;f^{(k)}) \leq 1 + 3eA\varepsilon. \tag{3.7}
\]
Since $\varepsilon(> 0)$ is arbitrary and $q$ is an arbitrary positive number, we get from (3.7)
\[
\sum_{a \neq \infty} \delta(a;f^{(k)}) \leq 1. \tag{3.8}
\]
This proves the theorem. \qed

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**REFERENCES**


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