ON $\theta$-PRECONTINUOUS FUNCTIONS

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ABSTRACT. We introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous (or almost weakly continuous) functions and contains the class of almost precontinuous functions. It is shown that the $\theta$-precontinuous image of a $p$-closed space is quasi $H$-closed.

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1. Introduction. A subset $A$ of a topological space $X$ is said to be preopen [14] or nearly open [26] if $A \subset \text{Int}(\text{Cl}(A))$. A function $f : X \to Y$ is called precontinuous [14] if the preimage $f^{-1}(V)$ of each open set $V$ of $Y$ is preopen in $X$. Precontinuity was called near continuity by Pták [26] and also called almost continuity by Frolík [9] and Husain [10]. In 1985, Janković [12] introduced almost weak continuity as a weak form of precontinuity. Popa and Noiri [23] introduced weak precontinuity and showed that almost weak continuity is equivalent to weak precontinuity. Paul and Bhattacharyya [21] called weakly precontinuous functions quasi precontinuous and obtained the further properties of quasi precontinuity. Recently, Nasef and Noiri [16] have introduced and investigated the notion of almost precontinuity. Quite recently, Jafari and Noiri [11] investigated the further properties of almost precontinuous functions.

In this paper, we introduce a new class of functions called $\theta$-precontinuous functions which is contained in the class of weakly precontinuous functions and contains the class of almost precontinuous functions. We obtain basic properties of $\theta$-precontinuous functions. It is shown in the last section that the $\theta$-precontinuous images of $p$-closed (resp., $\beta$-connected) spaces are quasi $H$-closed (resp., semi-connected).

2. Preliminaries. Throughout, by $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) we denote topological spaces. Let $S$ be a subset of $X$. We denote the interior and the closure of $S$ by $\text{Int}(S)$ and $\text{Cl}(S)$, respectively. A subset $S$ is said to be preopen [14] (resp., semi-open [13], $\alpha$-open [17]) if $S \subset \text{Int}(\text{Cl}(S))$ (resp., $S \subset \text{Cl}(\text{Int}(S))$, $S \subset \text{Int}(\text{Cl}(\text{Int}(S)))$). The complement of a preopen set is called preclosed. The intersection of all preclosed sets containing $S$ is called the preclosure [8] of $S$ and is denoted by $p\text{Cl}(S)$. The preinterior of $S$ is defined by the union of all preopen sets contained in $S$ and is denoted by $p\text{Int}(S)$. The family of all preopen sets of $X$ is denoted by $\text{PO}(X)$. We set $\text{PO}(X, x) = \{U : x \in U \text{ and } U \in \text{PO}(X)\}$. A point $x$ of $X$ is called a $\theta$-cluster point of $S$ if $\text{Cl}(U) \cap S \neq \emptyset$ for every open set $U$ of $X$ containing $x$. The set of all $\theta$-cluster points of $S$ is called the $\theta$-closure of $S$ and is denoted by $\text{Cl}_\theta(S)$. A subset $S$ is said to be $\theta$-closed [27] if $S = \text{Cl}_\theta(S)$. The complement of a $\theta$-closed set is said to be $\theta$-open. A point $x$ of $X$
is called a \textit{pre $\theta$-cluster} point of $S$ if $\text{pCl}(U) \cap S \neq \emptyset$ for every preopen set $U$ of $X$ containing $x$. The set of all pre-$\theta$-cluster points of $S$ is called the \textit{pre $\theta$-closure} of $S$ and is denoted by $\text{pCl}_{\theta}(S)$. A subset $S$ is said to be \textit{pre $\theta$-closed} [20] if $S = \text{pCl}_{\theta}(S)$. The complement of a pre-$\theta$-closed set is said to be \textit{pre $\theta$-open}.

**Definition 2.1.** A function $f : X \to Y$ is said to be \textit{precontinuous} [14] (resp., \textit{almost precontinuous} [16], \textit{weakly precontinuous} [23] or \textit{quasi precontinuous} [21]) if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(U) \subset V$ (resp., $f(U) \subset \text{Int}(\text{Cl}(V))$, $f(U) \subset \text{Cl}(V)$).

**Definition 2.2.** A function $f : X \to Y$ is said to be \textit{almost weakly continuous} [12] if $f^{-1}(V) \subset \text{Int}(\text{Cl}(f^{-1}(\text{Cl}(V))))$ for every open set $V$ of $Y$.

**Definition 2.3.** A function $f : X \to Y$ is said to be \textit{strongly $\theta$-precontinuous} [19] if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset V$.

**Definition 2.4.** A function $f : X \to Y$ is said to be \textit{$\theta$-precontinuous} if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$.

**Remark 2.5.** By the above definitions and Theorem 3.3 below, we have the following implications and none of these implications is reversible by [19, Example 2.2], [11, Example 2.9], and Examples 2.6 and 5.11 below.

\begin{align*}
\text{strongly $\theta$-precontinuous} & \Rightarrow \text{precontinuous} \Rightarrow \text{almost precontinuous} \\
& \Rightarrow \text{$\theta$-precontinuous} \Rightarrow \text{weakly precontinuous}.
\end{align*}

**Example 2.6.** This example is due to Arya and Deb [4]. Let $X$ be the set of all real numbers. The topology $\tau$ on $X$ is the cocountable topology. Let $Y = \{a,b,c\}$, $\sigma = \{\emptyset, Y, \{a\}, \{c\}, \{a,c\}\}$. We define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(x) = a$ if $x$ is rational; $f(x) = b$ if $x$ is irrational. Then $f$ is a $\theta$-precontinuous function which is not almost precontinuous.

3. Characterizations

**Theorem 3.1.** For a function $f : X \to Y$ the following properties are equivalent:

(1) $f$ is $\theta$-precontinuous;
(2) $\text{pCl}_{\theta}(f^{-1}(B)) \subset f^{-1}(\text{Cl}_{\theta}(B))$ for every subset $B$ of $Y$;
(3) $f(p\text{Cl}_{\theta}(A)) \subset \text{Cl}_{\theta}(f(A))$ for every subset $A$ of $X$.

**Proof.** (1)$\Rightarrow$(2). Let $B$ be any subset of $Y$. Suppose that $x \notin f^{-1}(\text{Cl}_{\theta}(B))$. Then $f(x) \notin \text{Cl}_{\theta}(B)$ and there exists an open set $V$ containing $f(x)$ such that $\text{Cl}(V) \cap B = \emptyset$. Since $f$ is $\theta, p, c$, there exists $U \in \text{PO}(X,x)$ such that $f(p\text{Cl}(U)) \subset \text{Cl}(V)$. Therefore, we have $f(p\text{Cl}(U)) \cap B = \emptyset$ and $\text{Cl}(U) \cap f^{-1}(B) = \emptyset$. This shows that $x \notin \text{pCl}(f^{-1}(B))$. Thus, we obtain $\text{pCl}_{\theta}(f^{-1}(B)) \subset f^{-1}(\text{Cl}_{\theta}(B))$. 

(2)⇒(3). Let A be any subset of X. Then we have pCl$_\theta$(A) ⊂ pCl$_\theta$(f$^{-1}$(f(A))) ⊂ f$^{-1}$(Cl$_\theta$(f(A))) and hence f(pCl$_\theta$(A)) ⊂ Cl$_\theta$(f(A)).

(3)⇒(2). Let B be a subset of Y. We have f(pCl$_\theta$(f$^{-1}$(B))) ⊂ Cl$_\theta$(f$^{-1}$(B))) ⊂ Cl$_\theta$(B) and hence pCl$_\theta$(f$^{-1}$(B)) ⊂ f$^{-1}$(Cl$_\theta$(B)).

(2)⇒(1). Let x ∈ X and V be an open set of Y containing f(x). Then we have Cl(V) ∩ (Y − Cl(V)) = ∅ and f(x) /∈ Cl$_\theta$(Y − Cl(V)). Hence, x /∈ f$^{-1}$(Cl$_\theta$(Y − Cl(V))) and x /∈ pCl$_\theta$(f$^{-1}$(Y − Cl(V))). There exists U ∈ PO(X,x) such that pCl(U) ∩ f$^{-1}$(Y − Cl(V)) = ∅; hence f(pCl(U)) ⊂ Cl(V). Therefore, f is $\theta$.p.c.

**Theorem 3.2.** For a function f : X → Y the following properties are equivalent:

1. f is $\theta$-precontinuous;
2. f$^{-1}$(V) ⊂ pInt$_\theta$(f$^{-1}$(Cl(V))) for every open set V of Y;
3. pCl$_\theta$(f$^{-1}$(V)) ⊂ f$^{-1}$(Cl(V)) for every open set V of Y.

**Proof.** (1)⇒(2). Suppose that V is any open set of Y and x ∈ f$^{-1}$(V). Then f(x) ∈ V and there exists U ∈ PO(X,x) such that f(pCl(U)) ⊂ Cl(V). Therefore, x ∈ U ⊂ pCl(U) ⊂ f$^{-1}$(Cl(V)). This shows that x ∈ pInt$_\theta$(f$^{-1}$(Cl(V))). Therefore, we obtain f$^{-1}$(V) ⊂ pInt$_\theta$(f$^{-1}$(Cl(V))).

(2)⇒(3). Suppose that V is any open set of Y and x /∈ f$^{-1}$(Cl(V)). Then f(x) /∈ Cl(V) and there exists an open set W containing f(x) such that W ∩ V = ∅; hence Cl(W) ∩ V = ∅. Therefore, we have f$^{-1}$(Cl(W)) ∩ f$^{-1}$(V) = ∅. Since x ∈ f$^{-1}$(W), by (2) x ∈ pInt$_\theta$(f$^{-1}$(Cl(W))). There exists U ∈ PO(X,x) such that pCl(U) ⊂ f$^{-1}$(Cl(W)). Thus we have pCl(U) ∩ f$^{-1}$(V) = ∅ and hence x /∈ pCl$_\theta$(f$^{-1}$(V)). This shows that pCl$_\theta$(f$^{-1}$(V)) ⊂ f$^{-1}$(Cl(V)).

(3)⇒(1). Suppose that x ∈ X and V is any open set of Y containing f(x). Then V ∩ (Y − Cl(V)) = ∅ and f(x) /∈ Cl(Y − Cl(V)). Therefore, x /∈ f$^{-1}$(Cl(Y − Cl(V))) and by (3) x /∈ pCl$_\theta$(f$^{-1}$(Y − Cl(V))). There exists U ∈ PO(X,x) such that pCl(U) ∩ f$^{-1}$(Y − Cl(V)) = ∅. Therefore, we obtain f(pCl(U)) ⊂ Cl(V). This shows that f is $\theta$.p.c.

**Theorem 3.3.** For a function f : X → Y the following properties hold:

1. if f is almost precontinuous, then it is $\theta$-precontinuous;
2. if f is $\theta$-precontinuous, then it is weakly precontinuous.

**Proof.** Statement (2) is obvious. We will show statement (1). Suppose that x ∈ X and V is any open set of Y containing f(x). Since f is almost precontinuous, f$^{-1}$(Int(Cl(V))) is preopen and f$^{-1}$(Cl(V)) is preclosed in X by [16, Theorem 3.1]. Now, set U = f$^{-1}$(Int(Cl(V))). Then we have U ∈ PO(X,x) and pCl(U) ⊂ f$^{-1}$(Cl(V)). Therefore, we obtain f(pCl(U)) ⊂ Cl(V). This shows that f is $\theta$.p.c.

**Corollary 3.4.** Let Y be a regular space. Then, for a function f : X → Y the following properties are equivalent:

1. f is strongly $\theta$-precontinuous;
2. f is precontinuous;
3. f is almost precontinuous;
4. f is $\theta$-precontinuous;
5. f is weakly precontinuous.

**Proof.** This is an immediate consequence of [19, Theorem 3.2].
**Definition 3.5.** A topological space $X$ is said to be pre-regular [20] if for each preclosed set $F$ and each point $x \in X - F$, there exist disjoint preopen sets $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

**Lemma 3.6** (see [20]). A topological space $X$ is pre-regular if and only if for each $U \in \text{PO}(X)$ and each point $x \in U$, there exists $V \in \text{PO}(X,x)$ such that $x \in V \supseteq \text{pCl}(V) \subseteq U$.

**Theorem 3.7.** Let $X$ be a pre-regular space. Then $f : X \rightarrow Y$ is $\theta.p.c.$ if and only if it is weakly precontinuous.

**Proof.** Suppose that $f$ is weakly precontinuous. Let $x \in X$ and $V$ be any open set of $Y$ containing $f(x)$. Then, there exists $U \in \text{PO}(X,x)$ such that $f(U) \subseteq \text{Cl}(V)$. Since $X$ is pre-regular, there exists $U_s \in \text{PO}(X,x)$ such that $x \in U_s \subseteq \text{pCl}(U_s) \subseteq U$. Therefore, we obtain $f(\text{pCl}(U_s)) \subset \text{Cl}(V)$. This shows that $f$ is $\theta.p.c.$

**Theorem 3.8.** Let $f : X \rightarrow Y$ be a function and $g : X \times Y \rightarrow X \times Y$ the graph function of $f$ defined by $g(x) = (x, f(x))$ for each $x \in X$. Then $g$ is $\theta.p.c.$ if and only if $f$ is $\theta.p.c.$

**Proof.**

**Necessity.** Suppose that $g$ is $\theta.p.c.$ Let $x \in X$ and $V$ be an open set of $Y$ containing $f(x)$. Then $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since $g$ is $\theta.p.c.$, there exists $U \in \text{PO}(X,x)$ such that $g(\text{pCl}(U)) \subset \text{Cl}(X \times V)$. It follows that $\text{Cl}(X \times V) = X \times \text{Cl}(V)$. Therefore, we obtain $f(\text{pCl}(U)) \subset \text{Cl}(V)$. This shows that $f$ is $\theta.p.c.$

**Sufficiency.** Let $x \in X$ and $W$ be any open set of $X \times Y$ containing $g(x)$. There exist open sets $U_1 \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \subseteq U_1 \times V \subseteq W$. Since $f$ is $\theta.p.c.$, there exists $U_2 \in \text{PO}(X,x)$ such that $f(\text{pCl}(U_2)) \subset \text{Cl}(V)$. Let $U = U_1 \cap U_2$, then $U \in \text{PO}(X,x)$. Therefore, we obtain $g(\text{pCl}(U)) \subset \text{Cl}(U_1) \times f(\text{pCl}(U_2)) \subset \text{Cl}(U_1) \times \text{Cl}(V) \subset \text{Cl}(W)$. This shows that $g$ is $\theta.p.c.$

4. Some properties

**Lemma 4.1** (see [15]). Let $A$ and $X_0$ be subsets of a space $X$.

1. If $A \in \text{PO}(X)$ and $X_0$ is semi-open in $X$, then $A \cap X_0 \in \text{PO}(X_0)$.
2. If $A \in \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $A \in \text{PO}(X)$.

**Lemma 4.2** (see [7]). Let $A$ and $X_0$ be subsets of a space $X$ such that $A \subseteq X_0 \subseteq X$. Let $\text{pCl}_{X_0}(A)$ denote the preclosure of $A$ in the subspace $X_0$.

1. If $X_0$ is semi-open in $X$, then $\text{pCl}_{X_0}(A) \subseteq \text{pCl}(A)$.
2. If $A \in \text{PO}(X_0)$ and $X_0 \in \text{PO}(X)$, then $\text{pCl}(A) \subseteq \text{pCl}_{X_0}(A)$.

**Theorem 4.3.** If $f : X \rightarrow Y$ is $\theta.p.c.$ and $X_0$ is a semi-open subset of $X$, then the restriction $f/X_0 : X_0 \rightarrow Y$ is $\theta.p.c.$

**Proof.** For any $x \in X_0$ and any open neighborhood $V$ of $f(x)$, there exists $U \in \text{PO}(X,x)$ such that $f(\text{pCl}(U)) \subset \text{Cl}(V)$ since $f$ is $\theta.p.c.$ Put $U_0 = U \cap X_0$, then by Lemmas 4.1 and 4.2 $U_0 \in \text{PO}(X_0,x)$ and $\text{pCl}_{X_0}(U_0) \subset \text{pCl}(U_0)$. Therefore, we obtain

$$f/X_0(\text{pCl}_{X_0}(U_0)) = f(\text{pCl}_{X_0}(U_0)) \subset f(\text{pCl}(U_0)) \subseteq f(\text{Cl}(U)) \subset \text{Cl}(V).$$  \hspace{1cm} (4.1)

This shows that $f/X_0$ is $\theta.p.c.$
Theorem 4.4. A function \( f : X \to Y \) is \( \theta.p.c. \) if for each \( x \in X \) there exists \( X_0 \in \text{PO}(X, x) \) such that the restriction \( f/X_0 : X_0 \to Y \) is \( \theta.p.c. \).

Proof. Let \( x \in X \) and \( V \) be any open neighborhood of \( f(x) \). There exists \( X_0 \in \text{PO}(X, x) \) such that \( f/X_0 : X_0 \to Y \) is \( \theta.p.c. \). Thus, there exists \( U \in \text{PO}(X, x) \) such that \( (f/X_0)(\text{pCl}(X, x)) \subset \text{Cl}(V) \). By Lemmas 4.1 and 4.2, \( U \in \text{PO}(X, x) \) and \( \text{pCl}(U) \subset \text{pCl}(X, x) \). Hence, we have \( f(\text{pCl}(U)) = (f/X_0)(\text{pCl}(U)) \subset (f/X_0)(\text{pCl}(X, x)) \subset \text{Cl}(V) \). This shows that \( f \) is \( \theta.p.c. \).

Corollary 4.5. Let \( \{U_\lambda : \lambda \in \Lambda\} \) be an \( \alpha \)-open cover of a topological space \( X \). A function \( f : X \to Y \) is \( \theta.p.c. \) if and only if the restriction \( f/U_\lambda : U_\lambda \to Y \) is \( \theta.p.c. \) for each \( \lambda \in \Lambda \).

Proof. This is an immediate consequence of Theorems 4.3 and 4.4.

Let \( \{X_\alpha : \alpha \in \mathcal{A}\} \) be a family of topological spaces, \( A_\alpha \) a nonempty subset of \( X_\alpha \) for each \( \alpha \in \mathcal{A} \) and \( X = \Pi \{X_\alpha : \alpha \in \mathcal{A}\} \) denote the product space, where \( \mathcal{A} \) is nonempty.

Lemma 4.6 (see [8]). Let \( n \) be a positive integer and \( A = \Pi_{j=1}^n A_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} X_\alpha \).

1. \( A \in \text{PO}(X) \) if and only if \( A_{\alpha_j} \in \text{PO}(X_{\alpha_j}) \) for each \( j = 1, 2, \ldots, n \).
2. \( \text{pCl}(A) \in \Pi_{\alpha \in \mathcal{A}} \text{pCl}(A_\alpha) \).

Theorem 4.7. If a function \( f_\alpha : X_\alpha \to Y_\alpha \) is \( \theta.p.c. \) for each \( \alpha \in \mathcal{A} \). Then the product function \( f : \Pi X_\alpha \to \Pi Y_\alpha \), defined by \( f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\} \) for each \( x = \{x_\alpha\} \), is \( \theta.p.c. \).

Proof. Let \( x = \{x_\alpha\} \in \Pi X_\alpha \) and \( W \) be any open set of \( \Pi Y_\alpha \) containing \( f(x) \). Then, there exists an open set \( V_{\alpha_j} \) of \( Y_{\alpha_j} \) such that

\[
 f(x) = \{f_\alpha(x_\alpha)\} \in \Pi_{j=1}^n V_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} Y_\alpha \subset W. 
\] (4.2)

Since \( f_\alpha \) is \( \theta.p.c. \) for each \( \alpha \), there exists \( U_{\alpha_j} \in \text{PO}(X_{\alpha_j}, x_{\alpha_j}) \) such that \( f_{\alpha_j}(\text{pCl}(U_{\alpha_j})) \subset \text{Cl}(V_{\alpha_j}) \) for \( j = 1, 2, \ldots, n \). Now, put \( U = \Pi_{j=1}^n U_{\alpha_j} \times \Pi_{\alpha \neq \alpha_j} X_\alpha \). Then, it follows from Lemma 4.6 that \( U \in \text{PO}(\Pi X_\alpha, x) \). Moreover, we have

\[
 f(\text{pCl}(U)) = f(\Pi_{j=1}^n \text{pCl}(U_{\alpha_j}) \times \Pi_{\alpha \neq \alpha_j} X_\alpha) 
\ni \Pi_{j=1}^n f_{\alpha_j}(\text{pCl}(U_{\alpha_j})) \times \Pi_{\alpha \neq \alpha_j} Y_\alpha 
\ni \Pi_{j=1}^n \text{Cl}(V_{\alpha_j}) \times \Pi_{\alpha \neq \alpha_j} Y_\alpha \subset \text{Cl}(W). 
\] (4.3)

This shows that \( f \) is \( \theta.p.c. \).

5. Preservation property

Definition 5.1. A topological space \( X \) is said to be

1. \( p\)-closed [7] (resp., \( p\)-Lindelöf) if every cover of \( X \) by preopen sets has a finite (resp., countable) subfamily whose preclosures cover \( X \),
2. countably \( p\)-closed if every countable cover of \( X \) by preopen sets has a finite subfamily whose preclosures cover \( X \);
3. \( \text{quasi } H\)-closed [25] (resp., \( \text{almost } \)Lindelöf [6]) if every cover of \( X \) by open sets has a finite (resp., countable) subfamily whose closures cover \( X \),
4. lightly compact [5] if every countable cover of \( X \) by open sets has a finite subfamily whose closures cover \( X \).
\textbf{Definition 5.2.} A subset $K$ of a space $X$ is said to be
\begin{enumerate}
\item[(1)] $p$-closed relative to $X$ \cite{7} if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of $K$ by preopen sets of $X$, there exists a finite subset $\mathcal{A}_+ \subseteq \mathcal{A}$ such that $K \subseteq \bigcup \{\text{precl}(V_\alpha) : \alpha \in \mathcal{A}_+\}$,
\item[(2)] quasi $H$-closed relative to $X$ \cite{25} if for every cover $\{V_\alpha : \alpha \in \mathcal{A}\}$ of $K$ by open sets of $X$, there exists a finite subset $\mathcal{A}_+ \subseteq \mathcal{A}$ such that $K \subseteq \bigcup \{\text{cl}(V_\alpha) : \alpha \in \mathcal{A}_+\}$.
\end{enumerate}

\textbf{Theorem 5.3.} If $f : X \to Y$ is a $\theta \cdot p\cdot c.$ function and $K$ is $p$-closed relative to $X$, then $f(K)$ is quasi $H$-closed relative to $Y$.

\textbf{Proof.} Suppose that $f : X \to Y$ is $\theta \cdot p\cdot c.$ and $K$ is $p$-closed relative to $X$. Let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be a cover of $f(K)$ by open sets of $Y$. For each point $x \in K$, there exists $\alpha(x) \in \mathcal{A}$ such that $f(x) \in V_{\alpha(x)}$. Since $f$ is $\theta \cdot p\cdot c.$, there exists $U_x \in \text{PO}(X,x)$ such that $f(\text{pCl}(U_x)) \subseteq \text{cl}(V_{\alpha(x)})$. The family $\{U_x : x \in K\}$ is a cover of $K$ by preopen sets of $X$ and hence there exists a finite subset $\mathcal{A}_+ \subseteq \mathcal{A}$ such that $K \subseteq \bigcup_{x \in K} \text{cl}(V_{\alpha(x)})$. Therefore, we obtain $f(K) \subseteq \bigcup_{x \in K} \text{cl}(V_{\alpha(x)})$. This shows that $f(K)$ is quasi $H$-closed relative to $Y$. \hfill $\Box$

\textbf{Corollary 5.4.} Let $f : X \to Y$ be a $\theta \cdot p\cdot c.$ surjection. Then, the following properties hold:
\begin{enumerate}
\item[(1)] If $X$ is $p$-closed, then $Y$ is quasi $H$-closed.
\item[(2)] If $X$ is $p$-Lindelöf, then $Y$ is almost Lindelöf.
\item[(3)] If $X$ is countably $p$-closed, then $Y$ is lightly compact.
\end{enumerate}

A subset $S$ of a topological space $X$ is said to be $\beta$-open \cite{1} or semipreopen \cite{3} if $S \subseteq \text{cl}(\text{Int}(\text{cl}(S)))$. It is well known that $\alpha$-openness implies both preopenness and semi-openness which imply $\beta$-openness. The complement of a semipreopen set is said to be \textit{semipreconnected} \cite{3}. The intersection of all semipreconnected sets of $X$ containing a subset $S$ is the \textit{semipreclosure} of $S$ and is denoted by $\text{spCl}(S)$ \cite{3}.

\textbf{Definition 5.5.} A topological space $X$ is said to be
\begin{enumerate}
\item[(1)] $\beta$-connected \cite{24} or \textit{semipreconnected} \cite{2} if $X$ cannot be expressed as the union of two nonempty disjoint $\beta$-open sets,
\item[(2)] semi-connected \cite{22} if $X$ cannot be expressed as the union of two nonempty disjoint semi-open sets.
\end{enumerate}

\textbf{Remark 5.6.} We have the following implications:
$$\beta\text{-connected} \implies \text{semi-connected} \implies \text{connected}. \quad (5.1)$$

But, the converses need not be true as the following simple examples show.

\textbf{Example 5.7.} \begin{enumerate}
\item[(1)] Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$. Then $(X,\tau)$ is connected but not semi-connected.
\item[(2)] Let $X = \{a,b,c\}$ and $\tau = \{X,\emptyset,\{b,c\}\}$. Then $(X,\tau)$ is semi-connected but not $\beta$-connected.
\end{enumerate}

\textbf{Lemma 5.8.} For a topological space $X$, the following properties are equivalent:
\begin{enumerate}
\item[(1)] $X$ is $\beta$-connected or semipreconnected.
\item[(2)] The intersection of two nonempty semipreopen subsets of $X$ is always nonempty.
\item[(3)] The intersection of two nonempty preopen subsets of $X$ is always nonempty.
\end{enumerate}
(4) \( \text{pCl}(V) = X \) for every nonempty preopen subset \( V \) of \( X \).

(5) \( \text{spCl}(V) = X \) for every nonempty semipreopen subset \( V \) of \( X \).

**Proof.** The proofs of equivalences of (1), (2), and (3) are given in [2, Theorem 6.4]. The other properties (4) and (5), which are stated in [18], are easily equivalent to (3) and (2), respectively. \( \square \)

**Theorem 5.9.** If \( f : X \to Y \) is a \( \theta \).p.c. surjection and \( X \) is \( \beta \)-connected, then \( Y \) is semi-connected.

**Proof.** Let \( V \) be any nonempty open set of \( Y \). Let \( y \in V \). Since \( f \) is surjective, there exists \( x \in X \) such that \( f(x) = y \). Since \( f \) is \( \theta \).p.c., there exists \( U \in \text{PO}(X, x) \) such that \( f(\text{pCl}(U)) \subset \text{Cl}(V) \). Since \( X \) is \( \beta \)-connected, by Lemma 5.8 \( \text{pCl}(U) = X \) and hence \( \text{Cl}(V) = Y \) since \( f \) is surjective. Therefore, it follows from [22, Theorem 4.3] that \( Y \) is semi-connected. \( \square \)

**Remark 5.10.** The following example shows that the image of \( \beta \)-connectedness under weakly precontinuous surjections is not necessarily semi-connected.

**Example 5.11.** Let \( X \) be the set of real numbers, \( \tau = \{ \emptyset \} \cup \{ V \subset X : 0 \in V \} \), \( Y = \{ a, b, c \} \), and \( \sigma = \{ Y, \emptyset, \{ a \}, \{ b \}, \{ a, b \} \} \). Define a function \( f : (X, \tau) \to (Y, \sigma) \) as follows: \( f(x) = a \) if \( x < 0 \); \( f(x) = c \) if \( x = 0 \); \( f(x) = b \) if \( x > 0 \). Then \( f \) is a weakly precontinuous surjection which is not \( \theta \).p.c. The topological space \( (X, \tau) \) is \( \beta \)-connected by Lemma 5.8. By Example 5.7(1), \( (Y, \sigma) \) is connected but not semi-connected. \( \square \)

**References**


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