THE ABEL-TYPE TRANSFORMATIONS INTO \( G_w \)

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Abstract. The Abel-type matrix \( A_{\alpha,t} \) was introduced and studied as a mapping into \( \ell \) by Lemma (1999). The purpose of this paper is to study these transformations as mappings into \( G_w \). The necessary and sufficient conditions for \( A_{\alpha,t} \) to be \( G_w \) are established. The strength of \( A_{\alpha,t} \) in the \( G_w - G_w \) setting is investigated. Also, it is shown that \( A_{\alpha,t} \) is translative in the \( G_w - G_w \) sense for certain sequences.

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1. Introduction. The Abel-type power series method [1], denoted by \( A_\alpha \), \( \alpha > -1 \), is the following sequence-to-function transformation: if

\[
\sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k \text{ is convergent, for } 0 < x < 1,
\]

\[
\lim_{x \to 1} (1-x)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k x^k = L,
\]

then we say \( u \) is \( A_\alpha \)-summable to \( L \). The matrix analogue of \( A_\alpha \) is the \( A_{\alpha,t} \) matrix [2] whose \( nk \)th entry is given by

\[
a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1},
\]

where \( 0 < t_n < 1 \) for all \( n \) and \( \lim t_n = 1 \). Thus, the sequence \( u \) is transformed into the sequence \( A_{\alpha,t}u \) whose \( n \)th term is given by

\[
(A_{\alpha,t}u)_n = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} u_k t_n^k.
\]

The matrix \( A_{\alpha,t} \) is called the Abel-type matrix [2]. Throughout, \( \alpha > -1 \) and \( t \) will denote such a sequence: \( 0 < t_n < 1 \) for all \( n \), and \( \lim t_n = 1 \).

2. Basic notations and definitions. Let \( A = (a_{nk}) \) be an infinite matrix defining a sequence-to-sequence summability transformation given by

\[
(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k,
\]

(2.1)
where \((Ax)_n\) denotes the \(n\)th term of the image sequence \(Ax\). The sequence \(Ax\) is called the \(A\)-transform of the sequence \(x\). If \(X\) and \(Z\) are sets of complex number sequences, then the matrix \(A\) is called an \(X-Z\) matrix if the image \(Au\) of \(u\) under the transformation \(A\) is in \(Z\) whenever \(u\) is in \(X\).

Suppose that \(y\) is a complex sequence; then throughout we use the following basic notations and definitions:

\[
\ell = \left\{ y : \sum_{k=0}^{\infty} |y_k| \text{ is convergent} \right\},
\]

\[
d(A) = \left\{ y : \sum_{k=0}^{\infty} a_{nk}y_k \text{ is convergent for each } n \geq 0 \right\},
\]

\[
\ell(A) = \left\{ y : Ay \in \ell \right\},
\]

\[
G_w = \left\{ y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1 \right\},
\]

\[
c(A) = \left\{ y : y \text{ is summable by } A \right\},
\]

\[
G_w(A) = \left\{ y : Ay \in G_w \right\},
\]

\[
\Delta x_k = x_k - x_{k+1}.
\]

**Definition 2.1.** The summability matrix \(A\) is said to be \(G_w\)-translative for a sequence \(u\) in \(G_w(A)\) provided that each of the sequences \(Tu\) and \(Su\) is in \(G_w(A)\), where

\[
Tu = \{u_1, u_2, u_3, \ldots\} \quad \text{and} \quad Su = \{0, u_0, u_1, \ldots\}.
\]

**Definition 2.2.** The matrix \(A\) is said to be \(G_w\)-stronger than the matrix \(B\) provided \(G_w(B) \subseteq G_w(A)\).

\[3. \text{ The main results} \]

**Theorem 3.1.** The matrix \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix if and only if \((1 - t)^{\alpha+1} \in G_w\).

**Proof.** Suppose that \(x \in G_w\), then we show that \(Y \in G_w\), where \(Y\) is the \(A_{\alpha,t}\)-transform of the sequence \(x\). Since \(x \in G\), it follows that \(|x_k| \leq M_1 r^k\) for some \(r \in (0, w)\) and \(M_1 > 0\). Now we have

\[
|Y_n| = (1 - t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \right|,
\]

\[
|Y_n| \leq (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} |x_k| t_n^k
\]

\[
\leq M_1 (1 - t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} r^k t_n^k
\]

\[
\leq M_1 (1 - t_n)^{\alpha+1} (1 - rt_n)^{-(\alpha+1)}
\]

\[
\leq M_2 (1 - t_n)^{\alpha+1}, \quad \text{for some } M_2 > 0.
\]
Hence if \((1-t)^{\alpha+1} \in G_w\), then it follows that \(Y \in G_w\). Conversely, if \((1-t)^{\alpha+1}\) is not in \(G_w\), then the first column of \(A_{\alpha,t}\) is not in \(G_w\) because \(a_{n,0} = t_n(1-t_n)^{\alpha+1}\). Thus, \(A_{\alpha,t}\) is not a \(G_w\)-\(G_w\) matrix. \(\square\)

**Remark 3.2.** In the \(G_w\)-\(G_w\) setting, \(A_{\alpha,t}\) being a \(G_w\)-\(G_w\) matrix does not imply that \((1-t) \in G_w\). Also, \((1-t) \in G_w\) does not imply that \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix.

This can be demonstrated as follows.

1. Let \(t_n = 1 - (1/3)^n\), \(\alpha = 1\), and \(w = 1/4\). So, we have \((1-t_n)^{\alpha+1} = (1/9)^n\) and hence \((1-t)^{\alpha+1} \in G_w\). This implies that \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix by Theorem 3.1. But observe that \((1-t)\) is not \(G_w\). Hence, \(A_{\alpha,t}\) being a \(G_w\)-\(G_w\) matrix does not imply that \((1-t) \in G_w\).

2. Let \(t_n = 1 - (1/4)^n\), \(\alpha = -1/2\), and \(w = 1/3\). Then we have \((1-t) \in G_w\). But note that \((1-t_n)^{\alpha+1} = (1/2)^n\) and hence \((1-t)^{\alpha+1}\) is not in \(G_w\). This implies that \(A_{\alpha,t}\) is not a \(G_w\)-\(G_w\) matrix by Theorem 3.1. Hence, \((1-t) \in G_w\) does not imply that \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix.

**Corollary 3.3.** (1) If \(-1 < \alpha \leq 0\), then \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix implies that \((1-t) \in G_w\).

(2) If \(\alpha > 0\), then \((1-t) \in G_w\) implies that \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix.

**Proof.** (1) Since \(-1 < \alpha \leq 0\) implies that \((1-t_n) \leq (1-t_n)^{\alpha+1}\), it follows that \((1-t) \in G_w\) by Theorem 3.1.

(2) If \(\alpha > 0\), then we have \((1-t_n)^{\alpha+1} < (1-t_n)\) and hence by Theorem 3.1, \(A_{\alpha,t}\) a \(G_w\)-\(G_w\) matrix whenever \((1-t) \in G_w\). \(\square\)

**Corollary 3.4.** The matrix \(A_{\alpha,t}\) is a \(G\)-\(G_w\) matrix if and only if \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix.

**Proof.** Since \(G_w\) is a subset of \(G\), \(A_{\alpha,t}\) being a \(G\)-\(G_w\) matrix yields \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix. Conversely, if \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix, then by Theorem 3.1, we have \((1-t)^{\alpha+1} \in G_w\). Now using the same technique used in the proof of Theorem 3.1, we can easily show that \(A_{\alpha,t}\) is a \(G\)-\(G_w\) matrix. Thus, the corollary follows. \(\square\)

The next results indicate that the \(A_{\alpha,t}\) matrix is a strong method in the \(G_w\)-\(G_w\) setting. The \(A_{\alpha,t}\) matrix is \(G_w\)-stronger than the identity matrix.

**Theorem 3.5.** Suppose that \(-1 < \alpha \leq 0\) and \(A_{\alpha,t}\) is a \(G_w\)-\(G_w\) matrix; then \(G_w(A_{\alpha,t})\) contains the class of all sequences \(x\) whose partial sums are bounded.

**Proof.** The theorem follows using a similar argument as in the proof of [2, Theorem 8]. \(\square\)

**Remark 3.6.** Although Theorem 3.5 is stated for \(-1 < \alpha \leq 0\), it is also true for all \(\alpha > -1\) for some sequences, which we will demonstrate as follows. Let \(x\) be the unbounded sequence defined by

\[
x_k = (-1)^k \frac{k + \alpha + 1}{\alpha + 1}.
\]
Let $Y$ be the $A_{\alpha,t}$-transform of $x$. Then we have
\[
Y_n = \frac{(1 - t_n)^{\alpha+1}}{(1 + t_n)^{\alpha+2}} < (1 - t_n)^{\alpha+1}.
\] (3.3)

Thus, if $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix, then by Theorem 3.1, $(1 - t)^{\alpha+1} \in G_w$, so $x \in G_w(A_{\alpha,t})$.

**Corollary 3.7.** Suppose that $-1 < \alpha \leq 0$ and $A_{\alpha,t}$ is a $G_w$-$G_w$ matrix; then $G_w(A_{\alpha,t})$ contains the class of all sequences $x$ such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent.

Our next results deal with the $G_w$-translativity of the $A_{\alpha,t}$ matrix. We will show that the $A_{\alpha,t}$ matrix is $G_w$-translative for some sequences in $G_w(A_{\alpha,t})$.

**Theorem 3.8.** Every $G_w$-$G_wA_{\alpha,t}$ matrix is $G_w$-translative for each sequence $x \in G_w(A_{\alpha,t})$ for which $\{x_k/k\} \in G_w$, $k = 1, 2, 3, \ldots$.

**Proof.** Let $x \in G_w(A_{\alpha,t})$. Then we will show that
1. $T_x \in G_w(A_{\alpha,t})$ and
2. $S_x \in G_w(A_{\alpha,t})$.

We first show that (1) holds. Note that
\[
\left|\left(A_{\alpha,t}T_x\right)_n\right| = \left|\left(1 - t_n\right)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_{k+1} t_n^k\right|
\]
\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \left|\sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_{k+1} t_n^{k+1}\right|
\]
\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \left|\sum_{k=1}^{\infty} \binom{k - 1 + \alpha}{k - 1} x_k t_n^k\right|
\]
\[
= \frac{(1 - t_n)^{\alpha+1}}{t_n} \left|\sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \frac{k}{k + \alpha}\right|
\]
\[
\leq A_n + B_n,
\] (3.4)

where
\[
A_n = \frac{(1 - t_n)^{\alpha+1}}{t_n} \left|\sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_k t_n^k\right|,
\]
\[
B_n = \frac{|\alpha|(1 - t_n)^{\alpha+1}}{t_n} \left|\sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_k t_n^k\right|.
\] (3.5)

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show both $A$ and $B$ are in $G_w$, then (1) holds. But the conditions that $A \in G_w$ and $B \in G_w$ follow easily from the given hypothesis that $x \in G_w(A_{\alpha,t})$ and $\{x_k/k\} \in G_w$, respectively.
Next we will show that (2) holds. Observe that

\[
\left| (A_{\alpha,t}Sx)_n \right| = (1 - t_n)^{\alpha + 1} \left| \sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_{k-1} t_n^k \right|
\]

\[
= (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha + 1}{k + 1} x_k t_n^{k+1} \right|
\]

\[
= (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^{k+1} \left( \frac{k + \alpha + 1}{k + 1} \right) \right|
\]

\[
= (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^{k+1} \left( \frac{1 + \alpha}{k + 1} \right) \right|
\]

\[
\leq E_n + F_n,
\]

where

\[
E_n = (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \right|
\]

\[
F_n = (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^{k+1} \right|.
\]

Now the given hypothesis that \( x \in G_w(A_{\alpha,t}) \) and \( \{x_k/k\} \in G_w \) implies that both \( E \) and \( F \) are in \( G_w \). Consequently, (2) holds and hence the theorem follows.

**Theorem 3.9.** Suppose that \(-1 < \alpha \leq 0\); then every \( G_w \)-\( G_w \) matrix \( A_{\alpha,t} \) is \( G_w \)-translative for each \( A_{\alpha} \)-summable sequence \( x \) in \( G_w(A_{\alpha,t}) \).

**Proof.** Since the case \( \alpha = 0 \) can be easily proved using the technique used in the proof of [4, Theorem 4.1], here we only consider the case \(-1 < \alpha < 0\). Let \( x \in c(A_{\alpha}) \cap G_w(A_{\alpha,t}) \). Then we will show that

1. \( Tx \in G_w(A_{\alpha,t}) \) and
2. \( Sx \in G_w(A_{\alpha,t}) \).

We first show that (1) holds. Note that

\[
\left| (A_{\alpha,t}Tx)_n \right| = (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_{k+1} t_n^k \right|
\]

\[
= \left(1 - t_n\right)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha + 1}{k + 1} x_{k+1} t_n^{k+1} \right|
\]

\[
= \left(1 - t_n\right)^{\alpha + 1} \left| \sum_{k=1}^{\infty} \binom{k - 1 + \alpha}{k - 1} x_k t_n^k \right|
\]

\[
= \left(1 - t_n\right)^{\alpha + 1} \left| \sum_{k=1}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \left( \frac{k}{k + \alpha} \right) \right|
\]

\[
\leq A_n + B_n,
\]
where
\[ A_n = \frac{(1-t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=1}^{\infty} \frac{(k + \alpha)}{k} x_k t_n^k \right|, \]
\[ B_n = -\frac{\alpha(1-t_n)^{\alpha + 1}}{t_n} \left| x_1 t_n + \sum_{k=2}^{\infty} \frac{(k + \alpha)}{k + \alpha} t_n^k \right|. \]

(3.9)

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show that both \( A \) and \( B \) are in \( G_w \), then (1) holds. The condition \( A \in G_w \) follows from the hypothesis that \( x \in G_w(A_{\alpha,t}) \), and \( B \in G_w \) will be shown as follows. Observe that
\[ B_n = -\frac{\alpha(1-t_n)^{\alpha + 1}}{t_n} \left| x_1 t_n + \sum_{k=2}^{\infty} \frac{(k + \alpha)}{k + \alpha} t_n^k \right| \]
\[ \leq -\alpha |x_1| (1-t_n)^{\alpha + 1} + \frac{\alpha(1-t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=2}^{\infty} \frac{(k + \alpha)}{k + \alpha} t_n^k \right| \]
\[ \leq C_n + D_n, \]

where
\[ C_n = -\alpha |x_1| (1-t_n)^{\alpha + 1}, \]
\[ D_n = -\frac{\alpha(1-t_n)^{\alpha + 1}}{t_n} \left| \sum_{k=2}^{\infty} \frac{(k + \alpha)}{k + \alpha} t_n^k \right|. \]

(3.11)

By Theorem 3.1, the hypothesis that \( A_{\alpha,t} \) is \( G_w\)-G_w implies that \( C \in G_w \), hence there remains only to show \( D \in G_w \) to prove that (1) holds. Now using the same techniques used in the proof of [3, Theorem 2], we can show that
\[ D_n \leq \frac{M_1 M_2}{\alpha} (1-t_n) - \frac{M_1 M_2}{\alpha} (1-t_n)^{\alpha + 1}, \]

(3.12)

where \( M_1 \) and \( M_2 \) are some positive real numbers. Note that \( A_{\alpha,t} \) being a \( G_w\)-\( G_w \) matrix implies that \((1-t)^{\alpha + 1} \in G_w \) by Theorem 3.1, and \(-1 < \alpha < 0 \) yields \((1-t) \in G_w \). Consequently, we have \( D \in G_w \) and hence (1) holds. Next we show that (2) holds. We have
\[ \left| (A_{\alpha,t} S x)_n \right| = (1-t_n)^{\alpha + 1} \left| \sum_{k=1}^{\infty} \frac{(k + \alpha)}{k} x_k t_n^k \right| \]
\[ = (1-t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \frac{(k + \alpha + 1)}{k + 1} x_k t_n^{k+1} \right| \]
\[ = (1-t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \frac{(k + \alpha)}{k} x_k t_n^{k+1} \frac{(k + \alpha + 1)}{k + 1} \right| \]
\[ = (1-t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \frac{(k + \alpha)}{k} x_k t_n^{k+1} \left(1 + \frac{\alpha}{k + 1}\right) \right| \]
\[ \leq E_n + F_n, \]

(3.13)
where

\[ E_n = (1 - t_n)^{\alpha + 1} \left| \sum_{k=0}^{\infty} \binom{k + \alpha}{k} x_k t_n^k \right|, \]

\[ F_n = -(1 - t_n)^{\alpha + 1} \alpha \sum_{k=0}^{\infty} \binom{k + \alpha}{k} \frac{x_k}{k+1} t_n^{k+1}. \]

The hypothesis that \( x \in G_w(A_{\alpha,t}) \) implies that \( E \in G_w \) and by proceeding as in the proof of (1) above, we can easily show that \( F \in G_w \). Thus, (2) holds and hence our assertion follows.

**Theorem 3.10.** Suppose that \( \alpha > 0 \) and \( (1-t) \in G_w \); then every \( A_{\alpha,t} \) matrix is \( G_w \)-translative for each \( A_{\alpha} \)-summable sequence \( x \) in \( G_w(A_{\alpha,t}) \).

**Proof.** The theorem follows easily by using similar argument used in the proof of Theorem 3.9.

Our next result is a Tauberian theorem for \( A_{\alpha,t} \) matrix in the \( G_w-G_w \) setting.

**Theorem 3.11.** Let \( A_{\alpha,t} \) be a \( G_w-G_w \) matrix. If \( x \) is a sequence such that \( A_{\alpha,t}x \) and \( \Delta x \) are in \( G_w \), then \( x \) is in \( G_w \).

**Proof.** The theorem easily follows by an argument similar to the proof of [4, Theorem 2.1].

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