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THE ABEL-TYPE TRANSFORMATIONS INTO G_w

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ABSTRACT. The Abel-type matrix $A_{\alpha,t}$ was introduced and studied as a mapping into ℓ by Lemma (1999). The purpose of this paper is to study these transformations as mappings into G_w. The necessary and sufficient conditions for $A_{\alpha,t}$ to be G_w-G_w are established. The strength of $A_{\alpha,t}$ in the G_w-G_w setting is investigated. Also, it is shown that $A_{\alpha,t}$ is translative in the G_w-G_w senses for certain sequences.

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1. Introduction. The Abel-type power series method [1], denoted by A_α, $\alpha > -1$, is the following sequence-to-function transformation: if

$$\begin{gathered}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k x^k \text{ is convergent}, \quad \text{for } 0<x<1,\\ \lim_{x\to 1}(1-x)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k x^k = L,\end{gathered} \tag{1.1}$$

then we say u is A_α-summable to L. The matrix analogue of A_α is the $A_{\alpha,t}$ matrix [2] whose nkth entry is given by

$$a_{nk} = \binom{k+\alpha}{k} t_n^k (1-t_n)^{\alpha+1}, \tag{1.2}$$

where $0 < t_n < 1$ for all n and $\lim t_n = 1$. Thus, the sequence u is transformed into the sequence $A_{\alpha,t}u$ whose nth term is given by

$$(A_{\alpha,t}u)_n = (1-t_n)^{\alpha+1}\sum_{k=0}^{\infty}\binom{k+\alpha}{k}u_k t_n^k. \tag{1.3}$$

The matrix $A_{\alpha,t}$ is called the Abel-type matrix [2]. Throughout, $\alpha > -1$ and t will denote such a sequence: $0 < t_n < 1$ for all n, and $\lim t_n = 1$.

2. Basic notations and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence-to-sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \tag{2.1}$$

where $(Ax)_n$ denotes the nth term of the image sequence Ax. The sequence Ax is called the A-transform of the sequence x. If X and Z are sets of complex number sequences, then the matrix A is called an X-Z matrix if the image Au of u under the transformation A is in Z whenever u is in X.

Suppose that y is a complex sequence; then throughout we use the following basic notations and definitions:

$$\begin{gathered}
\ell = \left\{ y : \sum_{k=0}^{\infty} |y_k| \text{ is convergent} \right\}, \\
d(A) = \left\{ y : \sum_{k=0}^{\infty} a_{nk} y_k \text{ is convergent for each } n \geq 0 \right\}, \\
\ell(A) = \{ y : Ay \in \ell \}, \\
G_w = \{ y : y_k = O(r^k) \text{ for some } r \in (0,w),\ 0 < w < 1 \}, \\
c(A) = \{ y : y \text{ is summable by } A \}, \\
G_w(A) = \{ y : Ay \in G_w \}, \\
\Delta x_k = x_k - x_{k+1}.
\end{gathered} \tag{2.2}$$

Definition 2.1. The summability matrix A is said to be G_w-translative for a sequence u in $G_w(A)$ provided that each of the sequences T_u and S_u is in $G_w(A)$, where $T_u = \{u_1, u_2, u_3, \ldots\}$ and $S_u = \{0, u_0, u_1, \ldots\}$.

Definition 2.2. The matrix A is said to be G_w-stronger than the matrix B provided $G_w(B) \subseteq G_w(A)$.

3. The main results

Theorem 3.1. *The matrix $A_{\alpha,t}$ is a G_w-G_w matrix if and only if $(1-t)^{\alpha+1} \in G_w$.*

Proof. Suppose that $x \in G_w$, then we show that $Y \in G_w$, where Y is the $A_{\alpha,t}$-transform of the sequence x. Since $x \in G$, it follows that $|x_k| \leq M_1 r^k$ for some $r \in (0,w)$ and $M_1 > 0$. Now we have

$$\begin{aligned}
|Y_n| &= (1-t_n)^{\alpha+1} \left| \sum_{k=0}^{\infty} \binom{k+\alpha}{k} x_k t_n^k \right|, \\
|Y_n| &\leq (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} |x_k| t_n^k \\
&\leq M_1 (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} \binom{k+\alpha}{k} r^k t_n^k \\
&\leq M_1 (1-t_n)^{\alpha+1} (1-rt_n)^{-(\alpha+1)} \\
&\leq M_2 (1-t_n)^{\alpha+1}, \quad \text{for some } M_2 > 0.
\end{aligned} \tag{3.1}$$

Hence if $(1-t)^{\alpha+1} \in G_w$, then it follows that $Y \in G_w$. Conversely, if $(1-t)^{\alpha+1}$ is not in G_w, then the first column of $A_{\alpha,t}$ is not in G_w because $a_{n,0} = t_n(1-t_n)^{\alpha+1}$. Thus, $A_{\alpha,t}$ is not a G_w-G_w matrix. □

Remark 3.2. In the G_w-G_w setting, $A_{\alpha,t}$ being a G_w-G_w matrix does not imply that $(1-t) \in G_w$. Also, $(1-t) \in G_w$ does not imply that $A_{\alpha,t}$ is a G_w-G_w matrix.

This can be demonstrated as follows.

(1) Let $t_n = 1-(1/3)^n$, $\alpha = 1$, and $w = 1/4$. So, we have $(1-t_n)^{\alpha+1} = (1/9)^n$ and hence $(1-t)^{\alpha+1} \in G_w$. This implies that $A_{\alpha,t}$ is a G_w-G_w matrix by Theorem 3.1. But observe that $(1-t)$ is not G_w. Hence, $A_{\alpha,t}$ being a G_w-G_w matrix does not imply that $(1-t) \in G_w$.

(2) Let $t_n = 1-(1/4)^n$, $\alpha = -1/2$, and $w = 1/3$. Then we have $(1-t) \in G_w$. But note that $(1-t_n)^{\alpha+1} = (1/2)^n$ and hence $(1-t)^{\alpha+1}$ is not in G_w. This implies that $A_{\alpha,t}$ is not a G_w-G_w matrix by Theorem 3.1. Hence, $(1-t) \in G_w$ does not imply that $A_{\alpha,t}$ is a G_w-G_w matrix.

Corollary 3.3. (1) *If $-1 < \alpha \le 0$, then $A_{\alpha,t}$ is a G_w-G_w matrix implies that $(1-t) \in G_w$.*

(2) *If $\alpha > 0$, then $(1-t) \in G_w$ implies that $A_{\alpha,t}$ is a G_w-G_w matrix.*

Proof. (1) Since $-1 < \alpha \le 0$ implies that $(1-t_n) \le (1-t_n)^{\alpha+1}$, it follows that $(1-t) \in G_w$ by Theorem 3.1.

(2) If $\alpha > 0$, then we have $(1-t_n)^{\alpha+1} < (1-t_n)$ and hence by Theorem 3.1, $A_{\alpha,t}$ a G_w-G_w matrix whenever $(1-t) \in G_w$. □

Corollary 3.4. *The matrix $A_{\alpha,t}$ is a G-G_w matrix if and only if $A_{\alpha,t}$ is a G_w-G_w matrix.*

Proof. Since G_w is a subset of $G, A_{\alpha,t}$ being a G-G_w matrix yields $A_{\alpha,t}$ is a G_w-G_w matrix. Conversely, if $A_{\alpha,t}$ is a G_w-G_w matrix, then by Theorem 3.1, we have $(1-t)^{\alpha+1} \in G_w$. Now using the same technique used in the proof of Theorem 3.1, we can easily show that $A_{\alpha,t}$ is a G-G_w matrix. Thus, the corollary follows. □

The next results indicate that the $A_{\alpha,t}$ matrix is a strong method in the G_w-G_w setting. The $A_{\alpha,t}$ matrix is G_w-stronger than the identity matrix.

Theorem 3.5. *Suppose that $-1 < \alpha \le 0$ and $A_{\alpha,t}$ is a G_w-G_w matrix; then $G_w(A_{\alpha,t})$ contains the class of all sequences x whose partial sums are bounded.*

Proof. The theorem follows using a similar argument as in the proof of [2, Theorem 8]. □

Remark 3.6. Although Theorem 3.5 is stated for $-1 < \alpha \le 0$, it is also true for all $\alpha > -1$ for some sequences, which we will demonstrate as follows. Let x be the unbounded sequence defined by

$$x_k = (-1)^k \frac{k+\alpha+1}{\alpha+1}. \tag{3.2}$$

Let Y be the $A_{\alpha,t}$-transform of x. Then we have

$$Y_n = \frac{(1-t_n)^{\alpha+1}}{(1+t_n)^{\alpha+2}} < (1-t_n)^{\alpha+1}. \tag{3.3}$$

Thus, if $A_{\alpha,t}$ is a G_w-G_w matrix, then by Theorem 3.1, $(1-t)^{\alpha+1} \in G_w$, so $x \in G_w(A_{\alpha,t})$.

Corollary 3.7. *Suppose that $-1 < \alpha \le 0$ and $A_{\alpha,t}$ is a G_w-G_w matrix; then $G_w(A_{\alpha,t})$ contains the class of all sequences x such that $\sum_{k=0}^{\infty} x_k$ is conditionally convergent.*

Our next results deal with the G_w-translativity of the $A_{\alpha,t}$ matrix. We will show that the $A_{\alpha,t}$ matrix is G_w-translative for some sequences in $G_w(A_{\alpha,t})$.

Theorem 3.8. *Every G_w-$G_w A_{\alpha,t}$ matrix is G_w-translative for each sequence $x \in G_w(A_{\alpha,t})$ for which $\{x_k/k\} \in G_w$, $k = 1,2,3,\ldots$.*

Proof. Let $x \in G_w(A_{\alpha,t})$. Then we will show that

(1) $T_x \in G_w(A_{\alpha,t})$ and

(2) $S_x \in G_w(A_{\alpha,t})$.

We first show that (1) holds. Note that

$$\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k \frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\le A_n + B_n,
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
A_n &= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|, \\
B_n &= \frac{|\alpha|(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|.
\end{aligned} \tag{3.5}$$

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show both A and B are in G_w, then (1) holds. But the conditions that $A \in G_w$ and $B \in G_w$ follow easily from the given hypothesis that $x \in G_w(A_{\alpha,t})$ and $\{x_k/k\} \in G_w$, respectively.

Next we will show that (2) holds. Observe that

$$\begin{aligned}
\left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right| \\
&= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right| \\
&\le E_n + F_n,
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
E_n &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|, \\
F_n &= (1-t_n)^{\alpha+1}|\alpha|\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+}\right|.
\end{aligned} \tag{3.7}$$

Now the given hypothesis that $x \in G_w(A_{\alpha,t})$ and $\{x_k/k\} \in G_w$ implies that both E and F are in G_w. Consequently, (2) holds and hence the theorem follows. □

Theorem 3.9. *Suppose that $-1 < \alpha \le 0$; then every G_w-G_w matrix $A_{\alpha,t}$ is G_w-translative for each A_α-summable sequence x in $G_w(A_{\alpha,t})$.*

Proof. Since the case $\alpha = 0$ can be easily proved using the technique used in the proof of [4, Theorem 4.1], here we only consider the case $-1 < \alpha < 0$. Let $x \in c(A_\alpha) \cap G_w(A_{\alpha,t})$. Then we will show that

(1) $T_x \in G_w(A_{\alpha,t})$ and

(2) $S_x \in G_w(A_{\alpha,t})$.

We first show that (1) holds. Note that

$$\begin{aligned}
\left|(A_{\alpha,t}T_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_{k+1}t_n^{k+1}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k-1+\alpha}{k-1}x_k t_n^k\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\frac{k}{k+\alpha}\right| \\
&= \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\left(1-\frac{\alpha}{k+\alpha}\right)\right| \\
&\le A_n + B_n,
\end{aligned} \tag{3.8}$$

where

$$A_n = \frac{(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|,$$
$$B_n = -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|. \tag{3.9}$$

The use of the triangle inequality is legitimate as the radii of convergence of the two power series are at least 1. Now if we show that both A and B are in G_w, then (1) holds. The condition $A \in G_w$ follows from the hypothesis that $x \in G_w(A_{\alpha,t})$, and $B \in G_w$ will be shown as follows. Observe that

$$\begin{aligned} B_n &= -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|x_1 t_n + \sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right| \\ &\le -\alpha|x_1|(1-t_n)^{\alpha+1} + \frac{-\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right| \\ &\le C_n + D_n, \end{aligned} \tag{3.10}$$

where

$$C_n = -\alpha|x_1|(1-t_n)^{\alpha+1},$$
$$D_n = -\frac{\alpha(1-t_n)^{\alpha+1}}{t_n}\left|\sum_{k=2}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+\alpha}t_n^k\right|. \tag{3.11}$$

By Theorem 3.1, the hypothesis that $A_{\alpha,t}$ is G_w-G_w implies that $C \in G_w$, hence there remains only to show $D \in G_w$ to prove that (1) holds. Now using the same techniques used in the proof of [3, Theorem 2], we can show that

$$D_n \le \frac{M_1M_2}{\alpha}(1-t_n) - \frac{M_1M_2}{\alpha}(1-t_n)^{\alpha+1}, \tag{3.12}$$

where M_1 and M_2 are some positive real numbers. Note that $A_{\alpha,t}$ being a G_w-G_w matrix implies that $(1-t)^{\alpha+1} \in G_w$ by Theorem 3.1, and $-1 < \alpha < 0$ yields $(1-t) \in G_w$. Consequently, we have $D \in G_w$ and hence (1) holds. Next we show that (2) holds. We have

$$\begin{aligned} \left|(A_{\alpha,t}S_x)_n\right| &= (1-t_n)^{\alpha+1}\left|\sum_{k=1}^{\infty}\binom{k+\alpha}{k}x_{k-1}t_n^k\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha+1}{k+1}x_k t_n^{k+1}\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(\frac{k+\alpha+1}{k+1}\right)\right| \\ &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^{k+1}\left(1+\frac{\alpha}{k+1}\right)\right| \\ &\le E_n + F_n, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} E_n &= (1-t_n)^{\alpha+1}\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}x_k t_n^k\right|, \\ F_n &= -(1-t_n)^{\alpha+1}\alpha\left|\sum_{k=0}^{\infty}\binom{k+\alpha}{k}\frac{x_k}{k+1}t_n^{k+}\right|. \end{aligned} \tag{3.14}$$

The hypothesis that $x \in G_w(A_{\alpha,t})$ implies that $E \in G_w$ and by proceeding as in the proof of (1) above, we can easily show that $F \in G_w$. Thus, (2) holds and hence our assertion follows. □

Theorem 3.10. *Suppose that $\alpha > 0$ and $(1-t) \in G_w$; then every $A_{\alpha,t}$ matrix is G_w-translative for each A_α-summable sequence x in $G_w(A_{\alpha,t})$.*

Proof. The theorem follows easily by using similar argument used in the proof of Theorem 3.9. □

Our next result is a Tauberian theorem for $A_{\alpha,t}$ matrix in the G_w-G_w setting.

Theorem 3.11. *Let $A_{\alpha,t}$ be a G_w-G_w matrix. If x is a sequence such that $A_{\alpha,t}x$ and Δx are in G_w, then x is in G_w.*

Proof. The theorem easily follows by an argument similar to the proof of [4, Theorem 2.1]. □

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