PARTIAL SUMS OF CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. The object of the present paper is to consider the starlikeness and convexity of partial sums of certain analytic functions in the open unit disk.

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1. Introduction. Let \( A \) denote the class of functions \( f(z) \) of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( S^*(\alpha) \) be the subclass of \( A \) consisting of functions \( f(z) \) which satisfy

\[
\Re \left[ \frac{zf'(z)}{f(z)} \right] > \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \). A function \( f(z) \) in \( S^*(\alpha) \) is said to be starlike of order \( \alpha \) in \( U \). Furthermore, let \( K(\alpha) \) denote the subclass of \( A \) consisting of all functions \( f(z) \) which satisfy

\[
\Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > \alpha \quad (z \in U)
\]

for some \( \alpha (0 \leq \alpha < 1) \). A function \( f(z) \) belonging to \( K(\alpha) \) is said to be convex of order \( \alpha \) in \( U \). We note that \( f(z) \in S^*(\alpha) \) if and only if \( zf'(z) \in K(\alpha) \) and denote by \( S^*(0) \equiv S^* \) and \( K(0) \equiv K \). For \( f(z) \in A \), we introduce the partial sum of \( f(z) \) by

\[
f_n(z) = z + \sum_{k=2}^{n} a_k z^k.
\]

Remark 1.1. It is well known that

(i) \( f(z) = z / (1 - z)^2 = z + \sum_{k=2}^{\infty} k z^k \) is the extremal function for the class \( S^* \). But \( f_2(z) = z + 2z^2 \in S^* \).

(ii) \( f(z) = z / (1 - z) = z + \sum_{k=2}^{\infty} z^k \) is the extremal function for the class \( K \). But \( f_2(z) = z + z^2 \notin K \).

For the partial sums \( f_n(z) \) of \( f(z) \in S^* \), Szegö [2] showed the following theorem.
Theorem 1.2. (i) \( f(z) \in S^* \) implies that \( f_n(z) \in S^* \) for \( |z| < 1/4 \). The result is sharp.
(ii) \( f(z) \in S^* \) implies that \( f_n(z) \in K \) for \( |z| < 1/8 \). The result is sharp.

Further, Padmanabhan [1] proved the following theorem.

Theorem 1.3. If \( f(z) \) is 2-valently starlike in \( U \), then \( f_n(z) \) is 2-valently starlike for \( |z| < 1/6 \). The result is sharp.

2. Function \( F_n(z) \). We define the function \( F_n(z) \) which is the partial sum of \( f(z) \in A \) by

\[
F_n(z) = z + a_n z^n. \tag{2.1}
\]

Theorem 2.1. The function \( F_n(z) \) satisfies

\[
\frac{1 - n |a_n| r^{n-1}}{1 - |a_n| r^{n-1}} \leq \text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] \leq \frac{1 + n |a_n| r^{n-1}}{1 + |a_n| r^{n-1}} \tag{2.2}
\]

for \( 0 \leq r < \frac{n}{1 - |a_n|} \leq 1 \). Therefore, \( F_n(z) \in S^*(\alpha) \) for \( 0 \leq r < \frac{n - \alpha}{n(1 - \alpha)} |a_n| \leq 1 \).

Proof. Note that

\[
\frac{z F_n'(z)}{F_n(z)} = \frac{z + n a_n z^n}{z + a_n z^n} = n - \frac{n - 1}{1 + a_n z^{n-1}}. \tag{2.3}
\]

It follows from (2.3) that

\[
\text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] = n - (n - 1) \frac{1 + |a_n| r^{n-1} \cos \theta}{1 + |a_n| r^{n-1} - 1} \tag{2.4}
\]

Since, the right-hand side of (2.4) is increasing for \( \cos \theta \) if \( |a_n| < 1 \), we obtain (2.2). Further, we also see that

\[
\text{Re} \left[ \frac{z F_n'(z)}{F_n(z)} \right] \geq \frac{1 - n |a_n| r^{n-1}}{1 - |a_n| r^{n-1}} > \alpha \tag{2.5}
\]

for \( 0 \leq r < \frac{n - \alpha}{n(1 - \alpha)} |a_n| \leq 1 \). This completes the proof of the theorem.

Next, we derive the following theorem.

Theorem 2.2. The function \( F_n(z) \) satisfies

\[
\frac{1 - n^2 |a_n| r^{n-1}}{1 - n |a_n| r^{n-1}} \leq \text{Re} \left[ 1 + \frac{z F_n''(z)}{F_n(z)} \right] \leq \frac{1 + n^2 |a_n| r^{n-1}}{1 + n |a_n| r^{n-1}} \tag{2.6}
\]

for \( 0 \leq r < \frac{n - 1}{n |a_n|} \leq 1 \). Therefore, \( F_n(z) \in K \) for \( 0 \leq r < \frac{n - 1}{n(1 - \alpha)} |a_n| \leq 1 \).

Proof. Noting that

\[
1 + \frac{z F_n''(z)}{F_n(z)} = n - \frac{n - 1}{1 + n a_n z^{n-1}}, \tag{2.7}
\]

we have

\[
\text{Re} \left[ 1 + \frac{z F_n''(z)}{F_n(z)} \right] = n - (n - 1) \frac{1 + n |a_n| r^{n-1} \cos \theta}{1 + n^2 |a_n| r^{2(n-1)} + 2n |a_n| r^{n-1} \cos \theta}, \tag{2.8}
\]

which derives (2.6).
By virtue of Theorems 2.1 and 2.2, we have the following conjecture.

**Conjecture 2.3.** For the partial sum \( f_n(z) \) of \( f(z) \) belonging to the class \( A \),

(i) \( f_n(z) \in S^\alpha(\alpha) \) for \( 0 \leq r < \frac{\alpha}{4} \),

(ii) \( f_n(z) \in K(\alpha) \) for \( 0 \leq r < \frac{\alpha}{4} \).

3. The partial sums of certain analytic functions. In this section, we consider the partial sums of functions \( f(z) = z/(1 - z) \) and \( f(z) = z/(1 - z)^2 \).

**Theorem 3.1.** Let \( f_3(z) = z + z^2 + z^3 \) be the partial sum of \( f(z) = z/(1 - z) \) which is the extremal function of the class \( K \). Then \( f_3(z) \in S^\alpha(626/961) \) for \( 0 \leq r < \beta \) \( (1/7 < \beta < 1/6) \), where \( \beta \) is the positive root of

\[
x^4 - 8x^3 + 9x^2 - 8x + 1 = 0 \quad \left( 0 < x < \frac{1}{\sqrt{3}} \right). \tag{3.1}
\]

**Proof.** We consider \( \alpha \) such that

\[
\text{Re} \left[ \frac{z f_3'(z)}{f_3(z)} \right] = \text{Re} \left[ 3 - \frac{2 + z}{1 + z^2 + z^3} \right] > \alpha \tag{3.2}
\]

for \( 0 \leq r < \beta \). This implies that

\[
\text{Re} \left[ \frac{2 + z}{1 + z^2 + z^3} \right] = 1 + \frac{(1 - r^2)(1 + r^2 + r \cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2) \cos \theta} < 3 - \alpha, \tag{3.3}
\]

that is,

\[
\text{Re} \left[ \frac{(1 - r^2)(1 + r^2 + r \cos \theta)}{1 - r^2 + r^4 + 4r^2 \cos^2 \theta + 2r(1 + r^2) \cos \theta} \right] < 2 - \alpha. \tag{3.4}
\]

Let the function \( g(t) \) be given by

\[
g(t) = \frac{(1 - r^2)(1 + r^2 + rt)}{1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2) t} \quad (t = \cos \theta). \tag{3.5}
\]

Then, we have

\[
g'(t) = \frac{r(r + 1)(r - 1)(1 + 5r^2 + r^4 + 4r^2 t^2 + 8r(1 + r^2) t)}{(1 - r^2 + r^4 + 4r^2 t^2 + 2r(1 + r^2) t)^2}. \tag{3.6}
\]

Letting

\[
h(t) = 1 + 5r^2 + r^4 + 4r^2 t^2 + 8r(1 + r^2) t, \tag{3.7}
\]

we see that (i) \( h(t) < 0 \Rightarrow g'(t) > 0 \), (ii) \( h(t) > 0 \Rightarrow g'(t) < 0 \), and (iii) \( h(t) = 0 \) for \( t = (-2(1 + r^2) \pm \sqrt{3(1 + r^2 + r^4)}) / 2r \).

If we write

\[
t_1 = \frac{-2(1 + r^2) + \sqrt{3(1 + r^2 + r^4)}}{2r} < 0, \tag{3.8}
\]

then, \( 0 \leq r \leq \beta \) implies that \( t_1 \leq -1 \), so that, \( h(t) \geq 0 \). This gives us that

\[
g(t) \leq g(-1) = \frac{1 - r + r^3 - r^4}{1 - 2r + 3r^2 - 2r^3 + r^4} = \frac{g_1(r)}{g_2(r)}. \tag{3.9}
\]
It is easy to check that $g_1(r)$ is decreasing for $r \ (0 \leq r < 1/\sqrt{3})$. Therefore,
\[
\frac{8 - 2\sqrt{3}}{9} = g_1\left(\frac{1}{\sqrt{3}}\right) < g_1(r) \leq g_1(0) = 1. \tag{3.10}
\]
Also, $g_2(r)$ is decreasing for $r \ (0 \leq r < \beta)$, because $g_2'(0) = -2 < 0$ and $g_2'(1/6) = -31/27 < 0$. This gives that
\[
\frac{961}{1296} = g_2\left(\frac{1}{6}\right) < g_2(r) \leq g_2(0) = 1. \tag{3.11}
\]
Consequently, we conclude that
\[
g(t) \leq g(-1) = \frac{g_1(r)}{g_2(r)} < \frac{1296}{961} = 2 - \alpha, \tag{3.12}
\]
that is, $\alpha = 626/961 = 0.651 \ldots$. Thus, we have
\[
\text{Re} \left[ \frac{zf_3''(z)}{f_3'(z)} \right] > \alpha \quad (\alpha = \frac{626}{961}) \tag{3.13}
\]
for $0 \leq r < \beta$.

Finally, we obtain the following theorem.

**Theorem 3.2.** Let $f_3(z) = z + 2z^2 + 3z^3$ be the partial sum of the Koebe function $f(z) = z/(1-z)^2$ which is the extremal function for the class $S^*$. Then $f_3(z) \in K(3191/15876)$ for $0 \leq r < \beta \ (1/14 < \beta < 113)$, where $\beta$ is the positive root of
\[
81x^4 - 162x^3 + 72x^2 - 18x + 1 = 0 \quad (0 \leq x < \frac{1}{3}). \tag{3.14}
\]

**Proof.** Since
\[
\text{Re} \left[ 1 + \frac{zf_3''(z)}{f_3'(z)} \right] = \text{Re} \left[ 3 - \frac{2(1+2z)}{1+4z+9z^2} \right] > \alpha \tag{3.15}
\]
implies that
\[
\text{Re} \left[ \frac{1+2z}{1+4z+9z^2} \right] = \frac{1}{2} + \frac{4r(1-9r^2)\cos \theta + 1 - 81r^4}{2(1-2r^2 + 81r^4 + 8r(1+9r^2)\cos \theta + 36r^2\cos^2 \theta)} < \frac{3-\alpha}{2}, \tag{3.16}
\]
we have to check that
\[
\frac{(1-9r^2)(1+9r^2+4r\cos \theta)}{1-2r^2 + 81r^4 + 8r(1+9r^2)\cos \theta + 36r^2\cos^2 \theta} < 2 - \alpha. \tag{3.17}
\]
If we let
\[
h(t) = \frac{(1-9r^2)(1+9r^2+4rt)}{1-2r^2 + 81r^4 + 8r(1+9r^2)t + 36r^2t^2}, \tag{3.18}
\]
then, we have
\[
h(t) \leq h(-1) = \frac{1-4r + 36r^3 - 81r^4}{1-8r + 34r^2 - 72r^3 + 81r^4} \equiv \frac{g_1(r)}{g_2(r)}. \tag{3.19}
\]
Noting that $0 < g_1(r) < 1$, and $g_2(r) > g_2(1/13) = 15876/28561$, we have

$$h(t) \leq h(-1) < \frac{1}{g_2(r)} < \frac{28561}{15876} = 2 - \alpha,$$

which implies that $\alpha = 3191/15876 = 0.200\ldots$ \hfill \(\square\)

**REFERENCES**


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