GENERALIZED PERIODIC AND GENERALIZED BOOLEAN RINGS

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ABSTRACT. We prove that a generalized periodic, as well as a generalized Boolean, ring is either commutative or periodic. We also prove that a generalized Boolean ring with central idempotents must be nil or commutative. We further consider conditions which imply the commutativity of a generalized periodic, or a generalized Boolean, ring.

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Throughout, \( R \) denotes a ring, \( N \) the set of nilpotents, \( C \) the center, and \( E \) the set of idempotents of \( R \). A ring \( R \) is called periodic if for every \( x \) in \( R \), there exist distinct positive integers \( m,n \) such that \( x^m = x^n \). We now formally state the definitions of a generalized periodic ring and a generalized Boolean ring.

**Definition 1.** A ring \( R \) is called generalized periodic if for every \( x \) in \( R \) such that \( x \notin (N \cup C) \), we have \( x^n - x^m \in (N \cap C) \), for some positive integers \( m,n \) of opposite parity.

**Definition 2.** A ring \( R \) is called generalized Boolean if for every \( x \) in \( R \) such that \( x \notin (N \cup C) \), there exists an even positive integer \( n \) such that \( x - x^n \in (N \cap C) \).

**Theorem 3.** If \( R \) is a generalized periodic ring, then \( R \) is either commutative or periodic.

**Proof.** Let \( N \) and \( C \) denote the set of nilpotents and the center of \( R \), respectively. We distinguish three cases.

**Case 1** (\( N \subseteq C \)). Then \( x \notin C \) implies \( x \notin (N \cup C) \), and hence there exist distinct positive integers \( m,n \) such that \( x^m - x^n \in N \), with \( n > m \). Suppose \((x^m - x^n)^k = 0\). Then, as is readily verified,

\[
(x - x^{n-m+1})^k x^{k(m-1)} = 0, \tag{1}
\]

which, in turn, implies that

\[
(x - x^{n-m+1})^{km} = (x - x^{n-m+1})^k x^{k(m-1)} g(x)

= 0, \tag{2}
\]

where 

\[
g(\lambda) \in \mathbb{Z}[\lambda]. \tag{3}
\]

We have thus shown that

\[
x - x^{n-m+1} \in N, \quad \forall x \notin C, \quad (n - m + 1 > 1). \tag{4}
\]
Recall that, in our present case, we assumed that $N \subseteq C$, and hence by (4),

$$x - x^{n-m+1} \in C, \quad \forall x \notin C, \ (n - m + 1 > 1).$$  \hspace{1cm} (5)

Since (5) is trivially satisfied if $x \in C$, we see that

$$x - x^{n(x)} \in C, \quad \text{for some } n(x) > 1, \text{ where } x \in R \text{ (arbitrary).}$$  \hspace{1cm} (6)

Therefore, $R$ is commutative, by a well-known theorem of Herstein [3].

**Case 2** ($C \subseteq N$). Then $x \notin N$ implies $x \notin (N \cup C)$, and hence there exist distinct positive integers $m, n$ such that $x^n - x^m \in N$, with $n > m$. Repeating the argument used to prove (4), we see that

$$x - x^{n-m+1} \in N, \quad \forall x \notin N, \ (n - m + 1 > 1).$$  \hspace{1cm} (7)

Since (7) is trivially satisfied for all $x \in N$, we conclude that

$$x - x^{k(x)} \in N, \quad \text{for some } k(x) > 1, \text{ where } x \in R \text{ (arbitrary).}$$  \hspace{1cm} (8)

By a well-known theorem of Chacron [2], equation (8) implies that $R$ is periodic.

**Case 3** ($C \notin N$ and $N \notin C$). In this case, let

$$z \in C \setminus N, \quad u \in N \setminus C.$$

Equation (9) readily implies that $z + u \notin C$ and $z + u \notin N$, and hence (see Definition 1)

$$(z + u)^n - (z + u)^m \in N, \quad \text{for some integers } n > m \geq 1.$$  \hspace{1cm} (10)

Since $z$ commutes with the nilpotent element $u$, (10) implies that

$$z^n - z^m + u' \in N, \quad \text{where } u' \in N, \ u' \text{ commutes with } z.$$  \hspace{1cm} (11)

Hence $z^n - z^m \in N$, with $n > m \geq 1$. Now, a repetition of the argument used in the proof of (4) shows that

$$z - z^{n-m+1} \in N, \quad \forall z \in C \setminus N, \ (n - m + 1 > 1).$$  \hspace{1cm} (12)

Trivially,

$$x - x^k \in N, \quad \forall x \in N, \ \forall k \in \mathbb{Z}^+.$$  \hspace{1cm} (13)

Finally, if $x \notin (N \cup C)$, then

$$x^n - x^m \in N, \quad \text{for some integers } n > m \geq 1.$$  \hspace{1cm} (14)

Again, repeating the argument used in the proof of (4), we see that

$$x - x^{n-m+1} \in N, \quad \forall x \notin (N \cup C), \ (n - m + 1 > 1).$$  \hspace{1cm} (15)

Combining (12), (13), and (15), we conclude that

$$x - x^{k(x)} \in N, \quad \text{for some } k(x) > 1, \text{ where } x \in R \text{ (arbitrary).}$$  \hspace{1cm} (16)

Thus, by Chacron’s theorem [2], $R$ is periodic. This completes the proof. \hfill \square
Corollary 4. If $R$ is a generalized Boolean ring, then $R$ is either commutative or periodic.

This follows at once, since a generalized Boolean ring is necessarily a generalized periodic ring (see Definitions 1 and 2).

Before proving the next theorem, we prove the following lemma.

Lemma 5. Let $R$ be a generalized periodic ring. If $e$ is any nonzero central idempotent in $R$ and $a \in N$, then $ea \in C$.

Proof. The proof is by contradiction. Suppose the lemma is false, and let $\eta_0 \in N, e\eta_0 \notin C$.

Since $e \in C$ and $\eta_0 \in N$, therefore $e\eta_0$ is nilpotent. Let 

$$(e\eta_0)^{\alpha} \in C, \quad \forall \alpha \geq \alpha_0, \quad \alpha_0 \text{ minimal.}$$

(18)

Since $e\eta_0 \notin C$ (see (17)), therefore $\alpha_0 > 1$. Let $\eta = (e\eta_0)^{\alpha_0-1}$. Then,

$$\eta = (e\eta_0)^{\alpha_0-1} \in N, \quad \eta \notin C \text{ (by the minimality of } \alpha_0),$$

(19)

Equation (19) implies that $e + \eta \notin C$ and $e + \eta \notin N$, and hence (see Definition 1)

$$(e + \eta)^{m'} - (e + \eta)^{n'} \in C,$$

(20)

where $m'$, $n'$ are of opposite parity. Combining (20) and (19), we see that (keep in mind that $e\eta = \eta$; see (19))

$$(m' - n')e\eta \in C,$$

(21)

where $m' - n'$ is an odd integer. Equation (19) also implies that $(-e + \eta)$ is not in $(N \cup C)$, so

$$(-e + \eta)^{m''} - (-e + \eta)^{n''} \in N,$$

(22)

where $m''$, $n''$ are of opposite parity. Combining (19) and (22), we see that

$$(-e)^{m''} - (-e)^{n''} \in N,$$

(23)

and hence $2e \in N$, since $m''$ and $n''$ are of opposite parity. Therefore, $(2e)^{\gamma} = 0, \gamma \in \mathbb{Z}^+$, and thus $2^{\gamma}e = 0$, which implies that

$$2^{\gamma}e\eta \in C; \quad \gamma \in \mathbb{Z}^+. $$

(24)

Now, combining (21) and (24), keeping in mind that $(2^{\gamma}, m' - n') = 1$, we see that $e\eta \in C$, and hence, by (19), $\eta = e\eta \in C$, which contradicts (19). This contradiction proves the lemma.

As usual, $[x, y] = xy - yx$ denotes the commutator of $x$ and $y$.

We are now in a position to prove the following theorem.
Theorem 6. Suppose $R$ is a generalized periodic ring, and suppose that there exists an element $c$ in $C$, with $c \neq 0$, such that

$$c[x,y] = 0 \quad \text{implies} \quad [x,y] = 0, \quad \forall x,y \in R.$$  \hfill (25)

Then $R$ is commutative.

Proof. We distinguish two cases.

Case 1 ($c \in \mathbb{N}$). In this case, $c^k = 0$ for some positive integer $k$, and hence

$$c^k[x,y] = 0, \quad \forall x,y \in R.$$  \hfill (26)

Combining (25) and (26), we see that

$$c^k[x,y] = 0 \implies c^{k-1}[x,y] = 0 \implies c^{k-2}[x,y] = 0 \implies \cdots \implies [x,y] = 0.$$  \hfill (27)

Thus, $c^k[x,y] = 0$ implies $[x,y] = 0$, and hence $R$ is commutative.

Case 2 ($c \notin \mathbb{N}$). In view of Theorem 3, we may assume that $R$ is periodic. This implies, in particular, that $c^m$ is idempotent for some positive integer $m$. Furthermore, $c^m \neq 0$ (since $c \notin \mathbb{N}$ in our present case). The net result is (since $c \in C$ also)

$$c^m = e \quad \text{is a nonzero central idempotent in} \quad R.$$  \hfill (28)

Let $a \in \mathbb{N}$. By Lemma 5 and equation (28), we have $ea \in C$, and hence $[ea,x] = 0$ for all $x \in R$, which implies

$$[c^ma,x] = c^m[a,x] = 0, \quad \forall x \in R.$$  \hfill (29)

The argument used in Case 1 of Theorem 6 shows that

$$c^m[a,x] = 0 \quad \text{implies} \quad [a,x] = 0,$$  \hfill (30)

and hence (see (29))

$$[a,x] = 0 \quad \forall x \in R, \quad \forall a \in \mathbb{N}.$$  \hfill (31)

Thus, $R$ is a periodic ring with the property that $N \subseteq C$. By a well-known theorem of Herstein [4], it follows that $R$ is commutative, and the theorem is proved.

Corollary 7. Suppose that $R$ is a generalized periodic ring with identity $1$. Then, $R$ is commutative.

Corollary 7 follows at once by taking $c = 1$ in Theorem 6.

Since a generalized Boolean ring is also a generalized periodic ring, therefore we have the following corollary.

Corollary 8. A generalized Boolean ring with identity $1$ is necessarily commutative.

Another corollary is the following result, proved by the authors in [1].
**Corollary 9.** Suppose that $R$ is a generalized periodic ring containing a central element which is not a zero divisor. Then $R$ is commutative.

This follows at once, since the hypotheses of this corollary imply the hypotheses of Theorem 6.

**Theorem 10.** Suppose $R$ is a generalized periodic ring. Suppose, further, that there exists a nonzero central element $c$ such that

$$ca = 0 \implies a = 0, \forall a \in N. \quad (32)$$

Then $R$ is commutative.

**Proof.** In [1], the authors proved the following result:

If $R$ is a generalized periodic ring, then the nilpotents $N$ form an ideal and $R/N$ is commutative. \( (33) \)

Let $x, y \in R$. By (33), for all $\bar{x}, \bar{y}$ in $R/N$, $\bar{x}\bar{y} = \bar{y}\bar{x}$, and hence $[x, y] \in N$. Taking $a = [x, y] \in N$ in (32), we see that (32) yields

$$c[x, y] = 0 \implies [x, y] = 0, \forall x, y \in R. \quad (34)$$

The theorem now follows at once from Theorem 6. \( \square \)

**Theorem 11.** A generalized Boolean ring $R$ with central idempotents is necessarily nil ($R = N$) or commutative ($R = C$).

**Proof.** Since $R$ is also a generalized periodic ring, therefore by Theorem 3, $R$ is commutative or periodic. If $R$ is commutative, there is nothing to prove. So we may assume that $R$ is periodic. We now distinguish two cases.

**Case 1** ($C \subseteq N$). Recall that, by hypothesis, the set $E$ of idempotents is central, and hence $E \subseteq C \subseteq N$ (in the present case). Thus, $E \subseteq N$, and hence $E = \{0\}$. Therefore,

$$\text{zero is the only idempotent of } R. \quad (35)$$

Let $x \in R$. Since $R$ is periodic, therefore $x^k$ is idempotent for some positive integer $k$, and hence by (35), $x^k = 0$, which proves that $R$ is nil.

**Case 2** ($C \not\subseteq N$). Then, for some $c \in R$, we have

$$c \in C, \quad c \notin N. \quad (36)$$

Again, since $R$ is periodic, $c^m$ is idempotent for some positive integer $m$. Moreover, $c^m \neq 0$ (since $c \notin N$). The net result is (see (36))

$$e = c^m \text{ is a nonzero central idempotent of } R. \quad (37)$$

Now, suppose $a \in N$. Since $0 \neq e \in C$ and $a \in N$, therefore $e + a \notin N$. Suppose, for the moment, that $a \notin C$. Then $e + a \notin C$ (since $e \in C$), and hence $e + a \notin (N \cup C)$. Therefore, by Definition 2,

$$(e + a) - (e + a)^n \in (N \cap C), \quad \text{for some even integer } n \geq 2. \quad (38)$$
Since $R$ is also a generalized periodic ring, therefore by Lemma 5 (see (37))

$$ea^i \in C, \quad \forall i \in \{1, \ldots, n-1\}, \quad (0 \neq e = e^2, e \in C, a \in N).$$  

(39)

Combining (38) and (39), we see that

$$a - a^n \in C, \quad \forall a \in N \setminus C.$$  

(40)

Since (40) is trivially satisfied for $a \in (N \cap C)$, therefore

$$a - a^n \in C, \quad \forall a \in N, \quad n \geq 2.$$  

(41)

We claim that

$$N \subseteq C.$$  

(42)

The proof is by contradiction. Suppose (42) is false. Then, for some $a \in R$, we have

$$a \in N, \quad a \notin C.$$  

(43)

Since $a \in N$, there exists a positive integer $\sigma_0$ such that

$$a^\sigma \in C, \quad \forall \sigma \geq \sigma_0, \quad \sigma_0 \text{ minimal.}$$  

(44)

Moreover, since $a \notin C$ (see (43)), therefore $\sigma_0 > 1$. Now, applying (41) to the nilpotent element $a^{\sigma_0-1}$, we see that

$$a^{\sigma_0-1} - (a^{\sigma_0-1})^n \in C, \quad \text{for some } n = n(a^{\sigma_0-1}) \geq 2.$$  

(45)

Furthermore, since $(\sigma_0 - 1)n \geq (\sigma_0 - 1)2 \geq \sigma_0$ (since $\sigma_0 \geq 2$), (44) implies that

$$a^{\sigma_0-1} = a^{(\sigma_0-1)n} \in C.$$  

(46)

Combining (45) and (46), we conclude that $a^{\sigma_0-1} \in C$, which contradicts the minimality of $\sigma_0$ in (44). This contradiction proves (42). Since $R$ is a periodic ring satisfying (42), therefore, by a well-known theorem of Herstein [4], $R$ is commutative. This completes the proof.

**Corollary 12.** A generalized Boolean ring with central idempotents and commuting nilpotents is commutative.

This corollary recovers a result proved by the authors in [1].

**Corollary 13.** If $R$ is a generalized Boolean ring, and if $R$ is 2-torsion-free, then $R$ is nil or commutative.

**Proof.** We claim that all idempotents of $R$ are central. Suppose not, and suppose $e$ is a noncentral idempotent in $R$. Then $-e \notin (N \cup C)$, and hence (see Definition 2)

$$(-e) - (-e)^n \in C, \quad n \text{ even.}$$  

(47)

Thus, $2e \in C$, and hence $[2e, x] = 0$ for all $x$ in $R$. Since $R$ is 2-torsion-free, $2[e, x] = 0$ implies $[e, x] = 0$, and thus $e \in C$, a contradiction. This contradiction proves that all idempotents of $R$ are central, and hence $R$ is nil or commutative, by Theorem 11.

\[\square\]
Theorem 14. Let $R$ be a generalized Boolean ring in which every finite subring is either commutative or nil. Then $R$ is either commutative or nil.

Proof. By contradiction. Thus, suppose $R$ is a generalized Boolean ring such that every finite subring of $R$ is either commutative or nil. Suppose, further, that $R$ is not commutative and not nil either. By Theorem 11, there must exist a noncentral idempotent element $e$ in $R$, and hence $e \notin (C \cup N)$. Thus (see Definition 2), since $-e \notin (C \cup N)$,

$$(-e) - (-e)^n \in (N \cap C), \quad n \text{ even.} \quad (48)$$

This implies that $2e \in (N \cap C)$, and hence $(2e)^k = 2^k e = 0$, for some $k \in \mathbb{Z}^+$. Since $e \notin C$, we must have the following:

Either $ex - exe \neq 0$ for some $x \in R$, or $x'e - exe \neq 0$ for some $x' \in R$. \quad (49)

Suppose $u = ex - exe \neq 0$. Then,

$$eu = u \neq 0 = ue = u^2, \quad (u = ex - exe \neq 0). \quad (50)$$

Moreover,

$$2u = [2e, ex] = 0 \quad (\text{since } 2e \in C). \quad (51)$$

Furthermore, the subring generated by $e$ and $u$ is

$$\langle e, u \rangle = \{re + su \mid r, s \in \mathbb{Z} \}. \quad (52)$$

Since $2^k e = 0$ and $2u = 0$, the subring $\langle e, u \rangle$ is finite. Indeed,

$$\langle e, u \rangle = \{re + su \mid 1 \leq r \leq 2^k, 1 \leq s \leq 2 \}. \quad (53)$$

On the other hand, if $x'e - exe \neq 0$ for some $x' \in R$ (the only other possibility), then the subring, $\langle e, v \rangle$, generated by $e$ and $v = x'e - exe$ is (as is readily verified)

$$\langle e, v \rangle = \{re + sv \mid 1 \leq r \leq 2^k, 1 \leq s \leq 2 \}. \quad (54)$$

Again, $\langle e, v \rangle$ is a finite subring of $R$. Hence, in either case, we found a finite subring of $R$, which is neither commutative (since $e \notin C$), nor nil (since $e \notin N$), contradicting our hypothesis. This contradiction proves the theorem. \hfill \square

Remark 15. A careful examination of the proof of Theorem 14 shows that we only need to assume that “every subring $S$, with $|S| = 2^m$ for some positive integer $m$, is commutative or nil” in order for the ground generalized Boolean ring $R$ to be commutative or nil. Indeed, $|\langle e, u \rangle| = 2^k \cdot 2 = 2^{k+1}$, since the representation of any $x$ in this subring in the form $x = re + su; \ r, s \in \mathbb{Z}$, is unique. For, suppose $x = re + su$ and $x = r'e + s'u$. Then, $(r - r')e = (s' - s)u$. Recall that $2u = 0$, and $ue = 0$. Thus, if $s' - s$ is even, then $(r - r')e = 0$, and hence $re = r'e$, $su = s'u$. On the other hand, if $s' - s$ is odd, then $(r - r')e = u$, and hence $(r - r')ee = ue = 0$. Again, we obtain $re = r'e$, $su = s'u$.

We conclude with the following examples.
**Example 16.** Let

\[
R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in \text{GF}(4) \right\}.
\]

(55)

It is readily verified that the idempotents of \( R \) are central and

\[
x - x^7 = 0, \quad \forall x \in R \setminus (N \cup C),
\]

(56)

but \( R \) is neither nil nor commutative. Hence, Theorem 11 is not true if we drop the hypothesis that “\( n \) is even” in the definition of a generalized Boolean ring.

**Example 17.** Let

\[
R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in \text{GF}(3) \right\}.
\]

(57)

This example shows that we cannot drop the hypothesis that “\( N \) is commutative” in Corollary 12. (Note that \( R \) is not commutative.)

**Example 18.** Let

\[
R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} : 0, 1 \in \text{GF}(2) \right\}.
\]

(58)

This example shows that we cannot drop the hypothesis that “the idempotents are central” in Corollary 12. This example also shows that we cannot drop the hypothesis that “\( R \) is 2-torsion-free” in Corollary 13. Note that, in this ring \( R \), \( x - x^2 = 0 \) for all \( x \in R \setminus (N \cup C) \). Even more is true. This ring \( R \) also shows that we cannot drop the hypothesis that “\( 1 \in R \)” in Corollary 7, nor the hypothesis that “\( 1 \in R \)” in Corollary 8.

Returning to the ring \( R \) in Example 16, we see that this ring further shows that we cannot drop the hypothesis that “\( m \) and \( n \) are of opposite parity” in the definition of a generalized periodic ring in connection with Corollary 7, or the hypothesis that “\( n \) is even” in the definition of a generalized Boolean ring as far as Corollary 8 is concerned. (Recall that \( x - x^7 = 0 \) for all \( x \in R \setminus (N \cup C) \).)

**Example 19.** Let \( S \) be any noncommutative ring such that \( S^3 = (0) \). (For example, we may take \( S \) to be the ring of all \( 3 \times 3 \) strictly upper triangular matrices over a field \( F \).) Let \( R = \text{GF}(4) \oplus S \). It is readily verified that \( x^3 = x^6 \) for all \( x \) in \( R \), and hence \( R \) is indeed a generalized periodic ring. Moreover, the only idempotents of \( R \) are \((0,0)\) and \((1,0)\), and thus the idempotents of \( R \) are certainly central. Had \( R \) been a generalized Boolean ring, then, by Theorem 11, \( R \) would have to be either nil or commutative, which is certainly false here (recall that \( S \) is not commutative). This example shows that the set of generalized periodic rings is a wider class than that of generalized Boolean rings, and thus Theorem 11 does not hold for generalized periodic rings.
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