CONSTRUCTING IRREDUCIBLE POLYNOMIALS WITH PRESCRIBED LEVEL CURVES OVER FINITE FIELDS

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ABSTRACT. We use Eisenstein's irreducibility criterion to prove that there exists an absolutely irreducible polynomial $P(X, Y) \in GF(q)[X, Y]$ with coefficients in the finite field $GF(q)$ with q elements, with prescribed level curves $X_c := \{(x, y) \in GF(q)^2 \mid P(x, y) = c\}$.

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1. Introduction. Let $GF(q)$ be the finite field with q elements. Assume that for any $c \in GF(q)$, a subset X_c (possibly empty) of the finite affine plane $GF(q)^2$ is given, such that $X_c \cap X_d \neq \emptyset$ for any $c \neq d$ and

$$GF(q)^2 = \bigcup_{c \in GF(q)} X_c. \quad (1.1)$$

In this paper, we use Eisenstein's irreducibility criterion to build absolutely irreducible polynomials

$$P(X, Y) \in GF(q)[X, Y] \quad (1.2)$$

such that for any $c \in GF(q)$ the level curve $\{(x, y) \in GF(q)^2 \mid P(x, y) = c\}$ coincides with X_c . Note that $P(X, Y) \in GF(q)[X, Y]$ is called absolutely irreducible if it is irreducible over the algebraic closure of $GF(q)$.

If we define a function $f : GF(q)^2 \rightarrow GF(q)$ taking a constant value c on the set X_c for any $c \in GF(q)$, it is easy to see that this is equivalent to the fact that there exists an absolutely irreducible polynomial which interpolates the function f .

It is of course well known that there exists a polynomial that interpolates the function f (see [3, Section 7.5] for a general discussion on this topic). Thus, our result can be viewed as a stronger version of this basic fact, going back to Weber [4].

The basic facts about bivariate polynomial interpolation over finite fields that we will need are summarized in the following theorem.

THEOREM 1.1. *Any function $f : GF(q)^2 \rightarrow GF(q)$ can be interpolated by some polynomial in two variables. Moreover, there exists a unique polynomial $F(X, Y) \in GF(q)[X, Y]$ of degree less than q in both X and Y that interpolates the function f , that is, satisfying $F(a, b) = f(a, b)$ for any $(a, b) \in GF(q)^2$. Also, any two interpolating polynomials for f are congruent modulo the ideal of $GF(q)[X, Y]$ generated by $X^q - X$ and $Y^q - Y$.*

Our main result is the following theorem.

THEOREM 1.2. *Let $f : GF(q)^2 \rightarrow GF(q)$ be a function. Then there exists an absolutely irreducible polynomial $P(X, Y) \in GF(q)[X, Y]$ that interpolates the function f .*

2. Proof of the main result. Let $f : GF(q)^2 \rightarrow GF(q)$ be an arbitrary function. By [Theorem 1.1](#), there exists a unique interpolating polynomial $H(X, Y) \in GF(q)[X, Y]$ for f , of degree at most $q - 1$ in both X and Y . We order $H(X, Y)$ in terms of the powers of Y

$$H(X, Y) = c_0(X) + c_1(X)Y + \cdots + c_{q-1}(X)Y^{q-1}, \quad (2.1)$$

where $c_0(X), c_1(X), \dots, c_{q-1}(X) \in GF(q)[X, Y]$ are of degree at most $q - 1$.

Clearly, if we add $Y^q - Y$ to $H(X, Y)$, we still get an interpolating polynomial for f , say $K(X, Y)$, that is, monic in Y . Thus, it will be perfectly legitimate to start with an interpolating polynomial of the form

$$K(X, Y) = Y^q + d_{q-1}(X)Y^{q-1} + \cdots + d_1(X)Y + d_0(X), \quad (2.2)$$

where $d_0(X), d_1(X), \dots, d_{q-1}(X) \in GF(q)[X, Y]$ are of degree at most $q - 1$.

It is well known (see [\[3, Corollary 2.11\]](#)) that there are irreducible polynomials of any degree over a finite field $GF(q)$. Fix such an irreducible polynomial $h(X) \in GF(q)[X]$ of degree 2. Clearly $h(X)$ has two roots in the algebraic closure of $GF(q)$, each of them generating the quadratic extension of $GF(q)$. Let α be a root of $h(X)$ in $\overline{GF(q)}$, the algebraic closure of $GF(q)$.

Our construction is based on replacing each polynomial coefficient $d_i(X)$ of [\(2.2\)](#) with a polynomial of the form

$$e_i(X) = d_i(X) + (X^q - X)u_i(X), \quad (2.3)$$

where $u_i(X) \in GF(q)[X]$, such that each $e_i(X)$ is divisible by $h(X)$ for $i = 0, \dots, q - 1$, while $e_0(X)$ is not divisible by $h(X)^2$. Clearly, the polynomial $F(X, Y)$ we get by performing these replacements

$$F(X, Y) = Y^q + e_{q-1}(X)Y^{q-1} + \cdots + e_1(X)Y + e_0(X) \quad (2.4)$$

will still be an interpolating polynomial for f , by [Theorem 1.1](#). We will then see that $F(X, Y)$ follows to be absolutely irreducible.

We prove that for some choice of $u_i(X) \in GF(q)[X]$ in [\(2.3\)](#), $e_i(X)$ is divisible by $h(X)$, that is,

$$d_i(X) + (X^q - X)u_i(X) \equiv 0 \pmod{h(X)} \quad (2.5)$$

is solvable. Indeed, from the way we defined $h(X)$, $X^q - X$ is relatively prime to $h(X)$. Thus, [\(2.5\)](#) is a linear congruence modulo $h(X)$ in the Euclidean ring $GF(q)[X]$ in which the coefficient $X^q - X$ of the unknown $u_i(X)$ is relatively prime to the modulus $h(X)$. This being the case, a solution $u_i(X)$ of [\(2.5\)](#) exists, and is uniquely determined up to a multiple of $h(X)$. It follows that we can select a solution $u_i(X)$ of [\(2.5\)](#) which is

a polynomial of degree one. This will completely take care of the cases $i = 1, \dots, q-1$. For the special case $i = 0$ we are looking for a solution $u_0(X)$ of (2.5) satisfying the additional requirement

$$d_0(X) + (X^q - X)u_0(X) \not\equiv 0 \pmod{h(X)^2}. \quad (2.6)$$

This can be done as follows. If the solution $u_0(X)$ of the $i = 0$ case of (2.5) already satisfies (2.6) there is nothing to prove. Otherwise, if $u_0(X)$ satisfies

$$d_0(X) + (X^q - X)u_0(X) \equiv 0 \pmod{h(X)^2}, \quad (2.7)$$

just replace $u_0(X)$ with $u_0(X) + h(X)$. This last polynomial will satisfy both (2.5) and (2.6).

The last step in our proof will consist in showing that the polynomial $F(X, Y)$ constructed above is absolutely irreducible.

The key ingredient of this last step is *Eisenstein's irreducibility criterion* (see [2, Theorem 6.15]), to the effect that if $P(X) = \gamma_n X^n + \gamma_{n-1} X^{n-1} + \dots + \gamma_1 X + \gamma_0$ is a polynomial with coefficients in some unique factorization domain $\mathbb{R}$, if we can find some irreducible element $p \in \mathbb{R}$ which divides $\gamma_0, \dots, \gamma_{n-1}$, does not divide γ_n , while p^2 does not divide γ_0 , then $P(X)$ is an irreducible element of $\mathbb{R}[X]$.

We view $F(X, Y)$ as a (monic) polynomial in Y with coefficients in the unique factorization domain $GF(q)[X]$, that is, $F(X, Y) \in (\overline{GF(q)}[X])[Y]$.

Pick up the irreducible

$$p(X) := X - \alpha \in \overline{GF(q)}[X]. \quad (2.8)$$

Since α is a root of $h(X)$, by the way we constructed $F(X, Y)$ it follows that $p(X)$ divides the polynomial coefficients $e_0(X), e_1(X), \dots, e_{q-1}(X) \in GF(q)[X]$ and $p(X)^2$ does not divide the free coefficient $e_0(X)$. Also, the coefficient of the highest power of Y in (2.4) is 1. Thus, we can apply now Eisenstein's criterion to conclude that $F(X, Y)$ is an irreducible element of the polynomial ring $(\overline{GF(q)}[X])[Y] \cong \overline{GF(q)}[X, Y]$. In other words, the interpolating polynomial $F(X, Y)$ for f is absolutely irreducible. This concludes the proof of our main theorem. $\square$

By our construction, the degrees of the polynomial coefficients $e_1(X), \dots, e_{q-1}(X)$ of $F(X, Y)$ are at most $q+1$, the degree of $e_0(X)$ is at most $q+2$, while $F(X, Y)$ is monic of degree q in Y .

Theorem 1.2 may be seen as a useful tool in the theory of curves over finite fields, since it allows a fairly elementary and efficient construction of equations of absolutely irreducible plane curves over $GF(q)$ with a given set $Z \subset GF(q)^2$ of $GF(q)$ -rational points (we may, for example, apply our construction to the special case in which the level curves are $X_0 = Z$, $X_1 = GF(q)^2 \setminus Z$, and $X_c = \emptyset$ for any $c \in GF(q) \setminus \{0, 1\}$). Finally, our interpolation result (with a construction based on a different method, though less direct) still holds true for the case of more than two variables (the proof of this will appear in [1]).

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