ITERATIVE SOLUTIONS OF K -POSITIVE DEFINITE OPERATOR EQUATIONS IN REAL UNIFORMLY SMOOTH BANACH SPACES

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ABSTRACT. Let X be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \rightarrow X$ be a K -positive definite operator. Under suitable conditions we establish that the iterative method by Bai (1999) converges strongly to the unique solution of the equation $Tx = f$, $f \in X$. The results presented in this paper generalize the corresponding results of Bai (1999), Chidume and Aneke (1993), and Chidume and Osilike (1997).

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1. Introduction and preliminaries. Let X be a real Banach space with a dual space X^* . The normalized duality mapping $J : X \rightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X. \quad (1.1)$$

It is known that X is uniformly smooth (equivalently, X^* is uniformly convex) if and only if J is single-valued and uniformly continuous on any bounded subset of X .

Chidume and Aneke [3] introduced the concept of K -positive definite operators and established the existence of the unique solution of the equation $Tx = f$ for that operator in real separable Banach spaces. Meanwhile they constructed, in L_p (or l_p) spaces with $p \geq 2$, an iteration method which converges strongly to the unique solution, provided that T and K commute. Chidume and Osilike [5] gave a new iteration scheme, in separable q -uniformly smooth Banach spaces, which converges strongly to the unique solution of the equation $Tx = f$, $f \in X$.

Recently, Bai [1] constructed a more general iteration procedure and improved the results of [3, 5] to separable uniformly smooth real Banach spaces.

Very recently, Zhou et al. [7] established the following excellent result, which is a generalization of the main result of Chidume and Aneke [3].

LEMMA 1.1 (see [7]). *Let X be a real Banach space and let T be a K -positive definite operator with $D(T) = D(K)$. Then there exists a constant $\alpha > 0$ such that*

$$\|Tx\| \leq \alpha \|Kx\|, \quad x \in D(T). \quad (1.2)$$

Moreover, the operator T is closed, $R(T) = X$, and the equation $Tx = f$ for each $f \in X$, has a unique solution.

The purpose of this paper is to study the convergence problem of the iteration procedure introduced in [1] for K -positive definite operators in real uniformly smooth real

Banach spaces. Our results extend the corresponding results due to Bai [1], Chidume and Aneke [3], and Chidume and Osilike [5].

In what follows, we will also need the following concepts and results.

DEFINITION 1.2 (see [3, 7]). Let X be a real Banach space and X_1 a subspace of X . An operator T with domain $D(T) \supseteq X_1$ is called *continuously X_1 -invertible* if T , as an operator restricted to X_1 , has a bounded inverse on $R(T)$. A linear unbounded operator T with domain $D(T)$ in X and range $R(T)$ in X is called *K -positive definite* if there exist a continuously $D(T)$ -invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that

$$\langle Tu, j(Ku) \rangle \geq c \|Ku\|^2, \quad u \in D(T), \quad j(Ku) \in J(Ku). \quad (1.3)$$

Let X be a real Banach space. Recall that the modulus of smoothness of X is defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x, y \in X, \|x\| = 1, \|y\| \leq t \right\}, \quad t \geq 0. \quad (1.4)$$

X is said to be *uniformly smooth* if $\lim_{t \rightarrow 0} \rho_X(t)/t = 0$. Let $p > 1$ be a real number. X is called *p -uniformly smooth* if there exists a constant $r > 0$ such that

$$\rho_X(t) \leq r t^p, \quad t > 0. \quad (1.5)$$

Hilbert spaces, L_p (or l_p) spaces, $1 < p < \infty$, and the Sobolev spaces W_p^m , $1 < p < \infty$, are all p -uniformly smooth. It is well known that the class of p -uniformly smooth real Banach spaces is a proper subclass of that of uniformly smooth real ones.

LEMMA 1.3 (see [4, 6]). *Let X be a real uniformly smooth Banach space. Then*

(i) *there exist some positive constants A and B such that*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x) \rangle + A \max \{ \|x\| + \|y\|, B \} \rho_X(\|y\|), \quad x, y \in X. \quad (1.6)$$

(ii) *there exists a continuous nondecreasing function $b : [0, \infty) \rightarrow [0, \infty)$ such that*

$$b(0) = 0, \quad b(ct) \leq cb(t), \quad c \geq 1; \\ \|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x) \rangle + \max \{ \|x\|, 1 \} \|y\| b(\|y\|), \quad x, y \in X. \quad (1.7)$$

LEMMA 1.4 (see [2]). *Suppose that $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$, and $\{\omega_n\}_{n=0}^\infty$ are nonnegative sequences such that*

$$\alpha_{n+1} \leq (1 - \omega_n) \alpha_n + \beta_n \omega_n, \quad n \geq 0, \quad (1.8)$$

with $\{\omega_n\}_{n=0}^\infty \subset [0, 1]$, $\sum_{n=0}^\infty \omega_n = \infty$ and $\lim_{n \rightarrow \infty} \beta_n = 0$. Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

LEMMA 1.5 (see [6]). *Let X be a real Banach space. Then*

- (i) $\rho_X(0) = 0$, $\rho_X(t) \leq t$, $t > 0$;
- (ii) $\rho_X(t)$ is convex, continuous, and nondecreasing on $[0, \infty)$;
- (iii) $\rho_X(t)/t$ is nondecreasing on $(0, \infty)$.

2. Main results

THEOREM 2.1. *Let X be a real uniformly smooth Banach space and let $T : D(T) \subseteq X \rightarrow X$ be a K -positive definite operator with $D(T) = D(K)$. Define a sequence $\{x_n\}_{n=0}^\infty$ iteratively from any $f \in X$ and $x_0 \in D(T)$ by*

$$y_n = x_n + b_n v_n, \quad x_{n+1} = y_n + a_n u_n, \quad n \geq 0; \quad (2.1)$$

$$v_n = K^{-1}f - K^{-1}Tx_n, \quad u_n = K^{-1}f - K^{-1}Ty_n, \quad n \geq 0, \quad (2.2)$$

where $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$ are arbitrary nonnegative sequences such that

$$\sum_{n=0}^{\infty} (a_n + b_n) = \infty; \quad (2.3)$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0; \quad (2.4)$$

$$\max \{a_n, b_n\} \leq \frac{1}{2c}, \quad n \geq 0; \quad (2.5)$$

$$\alpha A \max \{ (1 + \alpha a_n) \|Kv_0\|, (1 + \alpha b_n) \|Kv_0\|, B \} \leq 2c \|Kv_0\|, \quad n \geq 0, \quad (2.6)$$

where c , α , A and B are the constants appearing in (1.2), (1.3), and (1.6), respectively. Then the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the unique solution of the equation $Tx = f$.

PROOF. It follows from Lemma 1.1 that the equation $Tx = f$ has a unique solution in X . Note that T and K are linear. From (2.1) and (2.2) we have

$$Kv_{n+1} = f - Tx_{n+1} = Ku_n - a_n Tu_n, \quad n \geq 0; \quad (2.7)$$

$$Ku_n = f - Ty_n = Kv_n - b_n Tv_n, \quad n \geq 0. \quad (2.8)$$

In view of (2.8) and (1.2), (1.3), and (1.6), we conclude that

$$\begin{aligned} \|Ku_n\|^2 &= \|Kv_n - b_n Tv_n\|^2 \\ &\leq \|Kv_n\|^2 - 2b_n \langle Tv_n, J(Kv_n) \rangle \\ &\quad + A \max \{ \|Kv_n\| + b_n \|Tv_n\|, B \} \rho_X(b_n \|Tv_n\|) \\ &\leq (1 - 2cb_n) \|Kv_n\|^2 \\ &\quad + A \max \{ (1 + \alpha b_n) \|Kv_n\|, B \} \rho_X(\alpha b_n \|Kv_n\|) \end{aligned} \quad (2.9)$$

for all $n \geq 0$. Using (2.7) and (1.2), (1.3), and (1.6), we have

$$\begin{aligned} \|Kv_{n+1}\|^2 &= \|Ku_n - a_n Tu_n\|^2 \\ &\leq \|Ku_n\|^2 - 2a_n \langle Tu_n, J(Ku_n) \rangle \\ &\quad + A \max \{ \|Ku_n\| + a_n \|Tu_n\|, B \} \rho_X(a_n \|Tu_n\|) \\ &\leq (1 - 2ca_n) \|Ku_n\|^2 \\ &\quad + A \max \{ (1 + \alpha a_n) \|Ku_n\|, B \} \rho_X(\alpha a_n \|Ku_n\|) \end{aligned} \quad (2.10)$$

for all $n \geq 0$. Set $M = \|Kv_0\|$. We claim that

$$\max\{\|Kv_n\|, \|Ku_n\|\} \leq M, \quad n \geq 0. \quad (2.11)$$

By virtue of (2.6), (2.9), and Lemma 1.5, we get that

$$\begin{aligned} \|Ku_0\|^2 &\leq (1 - 2cb_0)\|Kv_0\|^2 \\ &\quad + A \max\{(1 + \alpha b_0)\|Kv_0\|, B\} \rho_X(\alpha b_0\|Kv_0\|) \\ &\leq (1 - 2cb_0)M^2 + A \max\{(1 + \alpha b_0)M, B\} \alpha b_0 M \\ &\leq M^2. \end{aligned} \quad (2.12)$$

That is, (2.11) is true for $n = 0$. Suppose that (2.11) holds for some $n \geq 0$. Using (2.10), (2.6), and Lemma 1.5, we infer that

$$\begin{aligned} \|Kv_{n+1}\|^2 &\leq (1 - 2ca_n)\|Ku_n\|^2 \\ &\quad + A \max\{(1 + \alpha a_n)\|Ku_n\|, B\} \rho_X(\alpha a_n\|Ku_n\|) \\ &\leq (1 - 2ca_n)M^2 + A \max\{(1 + \alpha a_n)M, B\} \alpha a_n M \\ &\leq M^2. \end{aligned} \quad (2.13)$$

From (2.6), (2.9), (2.13), and Lemma 1.5, we have

$$\begin{aligned} \|Ku_{n+1}\|^2 &\leq (1 - 2cb_{n+1})\|Kv_{n+1}\|^2 \\ &\quad + A \max\{(1 + \alpha b_{n+1})\|Kv_{n+1}\|, B\} \rho_X(\alpha b_{n+1}\|Kv_{n+1}\|) \\ &\leq (1 - 2cb_{n+1})M^2 + A \max\{(1 + \alpha b_{n+1})M, B\} \alpha b_{n+1} M \\ &\leq M^2. \end{aligned} \quad (2.14)$$

Therefore (2.11) holds for all $n \geq 0$. Since X is uniformly smooth, by (2.4) and Lemma 1.5 we conclude that there exist nonnegative sequences $\{s_n\}_{n=0}^\infty$ and $\{t_n\}_{n=0}^\infty$ such that $\rho_X(\alpha Ma_n) = s_n a_n$, $\rho_X(\alpha Mb_n) = t_n b_n$ for all $n \geq 0$ and

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = 0. \quad (2.15)$$

It follows from (2.5), (2.9), (2.10), and (2.11) that

$$\begin{aligned} \|Kv_{n+1}\|^2 &\leq (1 - 2ca_n)(1 - 2cb_n)\|Kv_n\|^2 \\ &\quad + (1 - 2ca_n)A \max\{(1 + \alpha b_n)\|Kv_n\|, B\} \rho_X(\alpha b_n\|Kv_n\|) \\ &\quad + A \max\{(1 + \alpha a_n)\|Ku_n\|, B\} \rho_X(\alpha a_n\|Ku_n\|) \\ &\leq [1 - 2c(a_n + b_n) + 4c^2 a_n b_n]\|Kv_n\|^2 \\ &\quad + A \max\{(1 + \alpha)M, B\} (\rho_X(\alpha Ma_n) + \rho_X(\alpha Mb_n)) \\ &\leq [1 - c(a_n + b_n)]\|Kv_n\|^2 + L(a_n s_n + b_n t_n) \end{aligned} \quad (2.16)$$

for all $n \geq 0$, where $L = A \max\{(1 + \alpha)M, B\}$. Let

$$\alpha_n = \|Kv_n\|^2, \quad \omega_n = c(a_n + b_n), \quad \beta_n = \frac{L}{c} r_n, \quad n \geq 0, \quad (2.17)$$

where

$$r_n = \begin{cases} 0, & a_n + b_n = 0, \\ \frac{a_n}{a_n + b_n} s_n + \frac{b_n}{a_n + b_n} t_n, & a_n + b_n \neq 0. \end{cases} \quad (2.18)$$

It follows from (2.15) that $\lim_{n \rightarrow \infty} r_n = 0$. That is, $\lim_{n \rightarrow \infty} \beta_n = 0$. Thus (2.15) can be rewritten in the form

$$\alpha_{n+1} \leq (1 - \omega_n) \alpha_n + \omega_n \beta_n, \quad n \geq 0. \quad (2.19)$$

Note that (2.3) and (2.5) mean that $\sum_{n=0}^{\infty} \omega_n = \infty$, $\omega_n \in [0, 1]$. Consequently, Lemma 1.4 ensures that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. That is,

$$\|Kv_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.20)$$

It follows from (2.2) and (2.20) that

$$\|Tx_n - f\| = \|Kv_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.21)$$

Note that T has a bounded inverse. Thus (2.21) means that $x_n \rightarrow T^{-1}f$, the unique solution of $Tx = f$. This completes the proof. $\square$

THEOREM 2.2. *Let X , T , K , f , $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$ be as in Theorem 2.1. Suppose that $\{a_n\}$ and $\{b_n\}_{n=0}^{\infty}$ are any nonnegative sequences such that (2.3), (2.4), and (2.5) and*

$$\max\{b(\alpha a_n), b(\alpha b_n)\} \leq \frac{2c}{\max\{1, \|Kv_0\|\}}, \quad n \geq 0, \quad (2.22)$$

where $b(t)$ is as in (1.7), α and c are the constants appearing in (1.3) and (1.2), respectively. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $Tx = f$.

PROOF. Set $M = \max\{1, \|Kv_0\|\}$. As in the proof of Theorem 3 in [1] we have

$$\|Kv_{n+1}\|^2 \leq (1 - c(a_n + b_n))\|Kv_n\|^2 + M^3 \alpha(a_n b(\alpha a_n) + b_n b(\alpha b_n)), \quad n \geq 0. \quad (2.23)$$

Let

$$\alpha_n = \|Kv_n\|^2, \quad \omega_n = c(a_n + b_n), \quad \beta_n = \frac{\alpha}{c} M^3 r_n, \quad n \geq 0, \quad (2.24)$$

where

$$r_n = \begin{cases} 0, & a_n + b_n = 0, \\ \frac{a_n}{a_n + b_n} b(\alpha a_n) + \frac{b_n}{a_n + b_n} b(\alpha b_n), & a_n + b_n \neq 0. \end{cases} \quad (2.25)$$

It is easily seen that $\lim_{n \rightarrow \infty} \beta_n = 0$. The rest of the argument now follows as in the proof of Theorem 2.1 to yield that $x_n \rightarrow T^{-1}f$ as $n \rightarrow \infty$. This completes the proof. $\square$

REMARK 2.3. Theorems 2.1 and 2.2 extend Theorem 3.3 of Bai [1], Theorem 2 of Chidume and Aneke [3] and Theorem of Chidume and Osilike [5], respectively, in the following ways:

- (a) Condition (2.3) is much weaker than $\sum_{n=0}^{\infty} a_n = \infty$ of [1].
- (b) L_p (or l_p) spaces, $p \geq 2$, in [3] and q -uniformly smooth Banach space, $q > 1$, in [5] are replaced by the more general uniformly smooth Banach spaces.
- (c) The commutativity condition of T and K in [3] is dropped.
- (d) The iteration methods in [3, 5] are special cases of our iteration method.

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