IJMMS 28:11 (2001) 663–672
PII. S016117120100672X
http://ijmms.hindawi.com

PRECOMPACTNESS AND TOTAL BOUNDEDNESS IN PRODUCTS OF METRIC SPACES

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(Received 16 February 2001)

ABSTRACT. We show that the canonical quantifications of uniform properties such as precompactness and total boundedness, which were already studied by Kuratowski and Hausdorff in the setting of complete metric spaces, can be generalized in the setting of products of metric spaces in an intuitively appealing way.

2000 Mathematics Subject Classification. 54E15, 18B30.

1. Introduction. In [2] Kuratowski introduced what he called the *measure of noncompactness* for complete metric spaces. The purely topological concept of compactness was quantified in the setting of metric spaces in order to measure the discrepancy a metric space may have from being compact. Since then several variants, such as Hausdorff's *ball measure of noncompactness*, have been introduced. For an extensive account on applications of these measures in the setting of Banach spaces, we refer to Banaś and Goebel [1]. Since all metric spaces are assumed to be complete, these measures deal in fact with total boundedness.

The introduction of *approach uniform spaces* (see Lowen and Windels [4]), established a more general setting for the quantification of uniform concepts. Approach uniform spaces, which are a unification of uniform spaces and metric spaces, express both qualitative and quantitative information. If these kinds of information are combined in a relevant (though canonical) way, then the numerical information can be used to express to what extent some qualitative aspect is or is not fulfilled. In [5], it is shown that total boundedness of uniform spaces and Hausdorff's ball measure on metric spaces mentioned above are special instances of a unifying concept, yielding a measure of total boundedness for approach uniform spaces.

Following the same philosophy, the introduction of approach uniform spaces allows for the quantification of other uniform concepts, such as precompactness, completeness and uniform connectedness.

Using the measure of total boundedness we regain a lot of information compared to the classical situation in uniform spaces. Nevertheless, we can do better. It is intuitively quite clear that if the measure of total boundedness of a set A in nonzero, then some elements in A contribute more to the non-total boundedness of A, than others. So we can consider functions from A to $[0,\infty]$, mapping every element in A to a number that equals (or is smaller than) the deviation from being totally bounded caused by that element. In the sequel we will call such a function itself *totally bounded*.

To that end, we need a numerification of filters, called *approach ideals*, which were investigated and extensively motivated in Lowen and Windels [6, 7]. The reader will

notice that, although we require these quite technical tools, the final results will be extremely canonical and generalize well-known facts.

2. Preliminaries. In order to present a self-contained text, we recall the main definitions and results from [6, 7].

Lowen et al. [3] introduced the concepts of an *approach ideal (a-ideal, for short)* as a canonical quantification of filters. An approach ideal on a set X, is an ideal $\mathfrak{F}$ of functions in $[0,\infty]^X$ such that $\infty \notin \mathfrak{F}$. If a collection $\mathfrak{G}$ of functions in $[0,\infty]^X$ not containing ∞, satisfies only the condition that $\forall \phi_1, \phi_2, \in \mathfrak{G}$, $\exists \phi_3 \in \mathfrak{G} : \phi_1 \vee \phi_2 \leq \phi_3$, then $\mathfrak{G}$ is called an *approach ideal basis*. The approach ideal generated by $\mathfrak{G}$ is denoted by $\langle \mathfrak{G} \rangle$. If $\psi \in [0,\infty]^X$, then $\psi := \langle \{\psi\} \rangle = \{\phi \mid \phi \leq \psi\}$ is an a-ideal. If $\mathfrak{F}$ and $\mathfrak{G}$ are a-ideals on X such that $\forall \phi \in \mathfrak{F}$, $\forall \psi \in \mathfrak{G} : \phi \vee \psi \neq \infty$, then $\mathfrak{F} \vee \mathfrak{G} := \{\phi \vee \psi \mid \phi \in \mathfrak{F},\ \psi \in \mathfrak{G}\}$ is an a-ideal. In particular, we define the *restriction* of $\mathfrak{F}$ to $\phi \in [0,\infty]^X$ by $\mathfrak{F} \mid \phi := \mathfrak{F} \vee \phi$. If $f : X \to Y$ is a function, $\mathfrak{F}$ is an a-ideal on X and $\mathfrak{G}$ is an a-ideal on Y, then $f(\mathfrak{F}) := \langle \{\phi_f \mid \forall y \in Y : \phi_f(y) = \inf_{x \in f^{-1}\{y\}} \phi(x),\ \phi \in \mathfrak{F}\} \rangle$ and $f^{-1}(\mathfrak{G}) := \langle \{\phi \circ f \mid \phi \in \mathfrak{G}\} \rangle$ are a-ideal on Y and X, respectively. An a-ideal $\mathfrak{F}$ is called *saturated* if $(\forall \varepsilon > 0,\ \forall N < \infty,\ \exists \phi_\varepsilon^N \in \mathfrak{F} : \phi \wedge N \leq \phi_\varepsilon^N + \varepsilon) \Rightarrow \phi \in \mathfrak{F}$. An a-ideal $\mathfrak{F}$ on X is said to be *prime* if $\forall \phi, \psi \in [0,\infty]^X$ we have $\phi \wedge \psi \in \mathfrak{F} \Rightarrow \phi \in \mathfrak{F}$ or $\psi \in \mathfrak{F}$. For every filter $\mathcal{F}$ and every a-ideal $\mathfrak{F}$ on X we write

$\mathbf{U}(\mathcal{F}) := \{\mathcal{U} \mid \mathcal{U}$ is an ultrafilter and $\mathcal{F} \subset \mathcal{U}\}$

$\mathbf{P}(\mathfrak{F}) := \{\mathfrak{P} \mid \mathfrak{P}$ is a prime a-ideal and $\mathfrak{F} \subset \mathfrak{P}\}$

$\mathbf{M}(\mathfrak{F}) := \{\mathfrak{P} \in \mathbf{P}(\mathfrak{F}) \mid \mathfrak{P}$ is minimal$\}$

$h(\mathfrak{F}) := \sup_{\phi \in \mathfrak{F}} \inf_{x \in X} \phi(x)$ (the *height* of $\mathfrak{F}$)

$m(\mathfrak{F}) := \sup_{\mathfrak{P} \in M(\mathfrak{F})} h(\mathfrak{P})$ (the *prime height* of $\mathfrak{F}$)

$\mathfrak{F}_\varepsilon := \langle \{\{\phi < \varepsilon\} \mid \phi \in \mathfrak{F}\} \rangle$ $(h < \varepsilon \leq \infty)$

$\mathfrak{F}^\varepsilon := \langle \{\{\phi < \alpha\} \mid \phi \in \mathfrak{F},\ \alpha > \varepsilon\} \rangle$ $(h \leq \varepsilon < \infty)$

$\mathcal{F}_\varepsilon := \{\phi \in [0,\infty]^X \mid \{\phi \leq \varepsilon\} \in \mathcal{F}\}$ $(\varepsilon < \infty)$

$\mathcal{F}^\varepsilon := \{\phi \in [0,\infty]^X \mid \forall \alpha > \varepsilon : \{\phi < \alpha\} \in \mathcal{F}\}$ $(\varepsilon < \infty)$

For any $A \subset X$ we write

$$\theta_A : X \longrightarrow [0,\infty] : \begin{cases} x \longmapsto 0 & \text{if } x \in A, \\ x \longmapsto \infty & \text{if } x \notin A. \end{cases} \tag{2.1}$$

Proposition 2.1. *Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are a-ideals on X, $\mathcal{F}$ is a filter on X, $A \subset X$, $f : X \to Y$ is a function, then we have the following:*

(a) $m(\mathfrak{F}) = \sup_{\phi \in \mathfrak{F}} \inf\{\alpha \mid \{\phi < \alpha\} \in \mathfrak{F}_\infty\}$

(b) $\mathcal{F}_\varepsilon = \langle \{\varepsilon + \theta_F \mid F \in \mathcal{F}\} \rangle$

(c) $\varepsilon < h(\mathfrak{F})$ *and* $A \in \mathfrak{F}_\infty \Rightarrow \varepsilon + \theta_A \in \mathfrak{F}$

(d) $\mathfrak{F} \subset \mathfrak{G} \Rightarrow h(\mathfrak{F}) \leq h(\mathfrak{G})$ *and* $m(\mathfrak{F}) \leq m(\mathfrak{G})$

(e) $h(\mathcal{F}_\varepsilon) = h(\mathcal{F}^\varepsilon) = \varepsilon$

(f) $m(\mathcal{F}_\varepsilon) = m(\mathcal{F}^\varepsilon) = \varepsilon$

(g) $h(\mathfrak{F}) \leq m(\mathfrak{F})$

(h) *If $\mathfrak{F}$ is prime, then* $h(\mathfrak{F}) = m(\mathfrak{F})$

(i) $\exists \mathfrak{P} \in M(\mathfrak{F}) : h(\mathfrak{F}) = h(\mathfrak{P})$

(j) $h(\mathfrak{F}) = h(f(\mathfrak{F}))$

(k) $m(f(\mathfrak{F})) \leq m(\mathfrak{F})$
(l) $\mathcal{F}$ *is an ultrafilter* $\Leftrightarrow \mathcal{F}_\varepsilon$ *is prime* $\Leftrightarrow \mathcal{F}^\varepsilon$ *is prime*
(m) $\mathfrak{F}$ *is prime* $\Leftrightarrow \mathfrak{F}_\infty$ *is an ultrafilter*
(n) $\mathfrak{F}$ *is prime* $\Rightarrow f(\mathfrak{F})$ *is prime*
(o) $\mathfrak{F} = \bigcap_{\mathfrak{P} \in M(\mathfrak{F})} \mathfrak{P}$
(p) $M(\mathfrak{F}) = \{\mathfrak{F} \vee \mathcal{U} \mid \mathcal{U} \in U(\mathfrak{F}_\infty)\}$.

Different useful topological and the like structures can be defined in the setting of a-ideals (cf. Lowen et al. [3]). We give one important example, which we will pursue in the sequel. If $\gamma, \xi \in [0,\infty]^{X\times X}$ then we define γ^{-1} by $\gamma^{-1}(x,y) := \gamma(y,x)$ and $\gamma \circ \xi$ by $\gamma \circ \xi(x,y) := \inf_{z\in X}(\gamma(x,z) + \xi(z,y))$.

Definition 2.2 (Lowen and Windels [4])**.** An *approach uniform system* on X is a saturated a-ideal Γ on $X \times X$ such that
(i) $\forall \gamma \in \Gamma$, $\forall x \in X : \gamma(x,x) = 0$
(ii) $\forall \gamma \in \Gamma$, $\forall \varepsilon > 0$, $\forall N < \infty$, $\exists \gamma_\varepsilon^N \in \Gamma : \gamma \wedge N \leq \gamma_\varepsilon^N \circ \gamma_\varepsilon^N + \varepsilon$
(iii) $\forall \gamma \in \Gamma : \gamma^{-1} \in \Gamma$.
Then the pair (X,Γ) is called an *approach uniform space*.

Conversely, the above constructions can be applied to construct approach uniform spaces from ordinary uniform spaces.

Example 2.3 (Lowen and Windels [4])**.** Let $(X,\mathcal{U})$ be a uniform space. Then $(X,\mathcal{U}_0)$ is an approach uniform space, called *the principal approach uniform space* associated with $\mathcal{U}$. This construction yields a coreflective embedding of *Unif* into *AUnif* (the category of approach uniform spaces and uniform contractions), the coreflection of any (X,Γ) being (X,Γ^0).

Example 2.4 (Lowen and Windels [4])**.** Let (X,Γ) be an approach uniform space. Then the collection $(\Gamma^\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ (defined above) is a collection of semi-uniformities on X satisfying the following supplementary conditions:
(T1) $\forall \varepsilon, \varepsilon' \in \mathbb{R}^+ : \Gamma^\varepsilon \circ \Gamma^{\varepsilon'} \supset \Gamma^{\varepsilon+\varepsilon'}$
(T2) $\forall \varepsilon \in \mathbb{R}^+ : \Gamma^\varepsilon = \cup_{\alpha>\varepsilon}\Gamma^\alpha$.

Conversely, if $(\mathcal{U}^\varepsilon)_{\varepsilon\in\mathbb{R}^+}$ is a collection of semi-uniformities on X satisfying the conditions (T1) and (T2), then $\Gamma := \{\gamma \mid \forall \varepsilon \in \mathbb{R}^+, \forall \alpha > \varepsilon : \{\gamma < \alpha\} \in \mathcal{U}^\varepsilon\}$ defines an approach uniform system Γ such that for every $\varepsilon \in \mathbb{R}^+$ we have that $\Gamma^\varepsilon = \mathcal{U}^\varepsilon$.

3. Cauchy approach ideals. The aim of this section is to generalize Cauchy filters. Recall that if $U \subset X \times X$ and $x \in X$, then we call $U(x) := \{y \in X \mid (x,y) \in U\}$ the *section* of U in x. If $A \subset X$ then we write $U(A) = \cup_{x\in A} U(x)$. We now define numeric sections of functions on $X \times X$.

Definition 3.1. Let $\gamma \in [0,\infty]^{X\times X}$, $\phi \in [0,\infty]^X$ and $x \in X$. Then we define $\gamma_x \in [0,\infty]^X$ by $\gamma_x(y) := \gamma(x,y)$ for every $y \in X$ and $\gamma_\phi \in [0,\infty]^X$ by $\gamma_\phi(y) := \inf_{z\in X}(\gamma(z,y) \vee \phi(z))$.

Recall that in a semi-uniform space $(X,\mathcal{U})$ a filter $\mathcal{F}$ is said to be $\mathcal{U}$-Cauchy if $\forall U \in \mathcal{U}$, $\exists x \in X : U(x) \in \mathcal{F}$. If $(X,\mathcal{U})$ is a uniform space then this is equivalent to the fact that

$\forall U \in \mathcal{U}$, $\exists F \in \mathcal{F} : F \times F \subset U$. The following definition is a possible numerification hereof.

Definition 3.2. Let (X,Γ) be an approach uniform space and let $\mathfrak{f}$ be an a-ideal on X. Then $\mathfrak{f}$ is called Γ-*Cauchy* if

$$\sup_{\gamma\in\Gamma} \inf_{x\in X} \sup_{\mathfrak{P}\in\mathbf{M}(\mathfrak{f})} \sup_{\phi\in\mathfrak{P}} \gamma_\phi(x) \le m(\mathfrak{f}). \tag{3.1}$$

Proposition 3.3. *Let (X,Γ) be an approach uniform space and let $\mathfrak{f}$ be a prime a-ideal on X. Then $\mathfrak{f}$ is Γ-Cauchy if and only if*

$$\sup_{\gamma\in\Gamma} \inf_{x\in X} \sup_{\phi\in\mathfrak{f}} \gamma\phi(x) \le h(\mathfrak{f}). \tag{3.2}$$

Proof. This follows at once from Definition 3.2 and Proposition 2.1(h). □

The following very useful characterization of Cauchy a-ideals was established in [7].

Theorem 3.4. *Let (X,Γ) be an approach uniform space and let $\mathfrak{f}$ be an a-ideal of bounded prime height m on X. Then $\mathfrak{f}$ is Γ-Cauchy if and only if $\mathfrak{f}_\infty$ is Γ^m-Cauchy.*

The following result describes the relationship between Cauchy filters and Cauchy a-ideals.

Proposition 3.5. *Let (X,Γ) be an approach uniform space, let $\mathcal{F}$ be a filter on X and let $\varepsilon < \infty$. Then $\mathcal{F}$ is Γ^ε-Cauchy if and only if $\mathcal{F}_\varepsilon$ is Γ-Cauchy.*

Proof. Using Theorem 3.4 and Proposition 2.1(f) we see that

$$\mathcal{F}_\varepsilon \text{ is } \Gamma\text{-Cauchy} \iff (\mathcal{F}_\varepsilon)_\infty \text{ is } \Gamma\text{-Cauchy} \iff \mathcal{F} \text{ is } \Gamma^\varepsilon\text{-Cauchy}, \tag{3.3}$$

which proves the claim. □

Cauchy prime a-ideals can be characterized analogously in terms of sections. The following proposition shows that $\mathfrak{f}$ is Γ-Cauchy if and only if for every "entourage" $\gamma \in \Gamma$ there is a section γ_x which belongs to $\mathfrak{f}$ "up to ε."

Proposition 3.6. *Let (X,Γ) be an approach uniform space and let $\mathfrak{f}$ be a prime a-ideal on X. Then the following are equivalent:*

(1) *$\mathfrak{f}$ is Γ-Cauchy*
(2) $\forall \gamma \in \Gamma$, $\forall \varepsilon > 0$, $\exists x \in X$, $\exists \phi \in \mathfrak{f} : \gamma_x \le \phi + \varepsilon$.

Proof. To show that (1)⇒(2), let $\mathfrak{f}$ be Γ-Cauchy. Then $\mathfrak{f}_\infty$ is Γ^h-Cauchy, where $h = h(\mathfrak{f})$ (Theorem 3.4). Let $\gamma \in \Gamma$ and $\varepsilon > 0$. Put $\alpha := h + \varepsilon/2$ and $\beta := (h - \varepsilon/2) \vee 0$. Then $\{\gamma < \alpha\} \in \Gamma^h$ and so there exists $x \in X$ such that $\{\gamma_x < \alpha\} = \{\gamma < \alpha\}_x \in \mathfrak{f}_\infty$. Then we have

$$\gamma_x \le \alpha + \theta_{\{\gamma_x<\alpha\}} = \left(\beta + \theta_{\{\gamma_x<\alpha\}}\right) + \varepsilon, \tag{3.4}$$

and by Proposition 2.1(c), $\beta + \theta_{\{\gamma_x<\alpha\}} \in \mathfrak{f}$.

The other implication is an immediate consequence of Proposition 3.3. □

Corollary 3.7. *Let $\mathfrak{F}$ be a prime Cauchy a-ideal on X and let $\mathfrak{G} \supset \mathfrak{F}$. Then $\mathfrak{G}$ is Cauchy too.*

Proof. It is immediate that condition (2) in Proposition 3.6 is stable with respect to finer a-ideals. □

4. Precompactness and total boundedness. As for semi-uniform spaces, or as in [5], we will have to make a distinction between precompactness and total boundedness. Total boundedness behaves nicely with respect to initial structures, whereas precompactness does not (see Proposition 4.19). On the other hand, compactness implies precompactness but in general not total boundedness (see Proposition 4.14 and Corollary 4.15). We recall the following definitions.

Definition 4.1. Let $(X,\mathcal{U})$ be a semi-uniform space and $A \subset X$. Then
(a) A is said to be ($\mathcal{U}$-)*precompact* if $\forall U \in \mathcal{U}$, $\exists F \in 2^{(X)} : U(F) \supset A$.
(b) A is said to be ($\mathcal{U}$-)*totally bounded* if $\forall U \in \mathcal{U}$, $\exists A_1, A_2, \dots, A_n \subset X$ such that $\forall i \in \{1,\dots,n\} : A_i \times A_i \subset U$ and $\cup_{i=1}^n A_i \supset A$.

It is a well-known fact that for uniform spaces these notions coincide. Also recall the following characterization of precompactness.

Lemma 4.2. *Let $(X,\mathcal{U})$ be a semi-uniform space and $A \subset X$. Then the following are equivalent:*
(1) *A is $\mathcal{U}$-precompact.*
(2) *Every ultrafilter on X containing A is $\mathcal{U}$-Cauchy.*

We will adopt canonical quantifications of Lemma 4.2 and Definition 4.1(b) as a criterion for precompactness and total boundedness for functions.

Definition 4.3. Let (X,Γ) be an approach uniform space and let $\phi \in [0,\infty]^X$. Then
(a) ϕ is said to be (Γ-)*precompact* if every prime a-ideal containing ϕ is Γ-Cauchy.
(b) ϕ is said to be (Γ-)*totally bounded* if $\forall \gamma \in \Gamma$, $\forall N < \infty$, $\forall \varepsilon > 0$, $\exists \phi_1, \dots, \phi_n \in [0,\infty]^X$ such that $\forall i \in \{1,\dots,n\}$:
- $\gamma(x,y) \wedge N \le \phi_i(x) \vee \phi_i(y) + \varepsilon$ for all $x, y \in X$.
- $\inf_{i=1}^n \phi \wedge N \le \phi + \varepsilon$.

In the definition for precompactness we can restrict ourselves to particular classes of prime a-ideals.

Theorem 4.4. *Let (X,Γ) be an approach uniform space and let $\phi \in [0,\infty]^X$. Then the following are equivalent:*
(1) *ϕ is precompact.*
(2) *Every saturated prime a-ideal containing ϕ is Γ-Cauchy.*
(3) *For every prime a-ideal $\mathfrak{P}$ we have that* $\sup_{\gamma\in\Gamma} \inf \sup_{\psi\in\mathfrak{F}} \gamma_\psi \le h(\mathfrak{P} \mid \phi)$.
(4) *For every prime a-ideal of zero height $\mathfrak{P}$ we have that* $\sup_{\gamma\in\Gamma} \inf \sup_{\psi\in\mathfrak{F}} \gamma_\psi \le h(\mathfrak{P} \mid \phi)$.
(5) *For every saturated prime a-idealilter $\mathfrak{P}$ we have that* $\sup_{\gamma\in\Gamma} \inf \sup_{\psi\in\mathfrak{F}} \gamma_\psi \le h(\mathfrak{P} \mid \phi)$.

(6) *For every saturated prime a-ideal of zero height* $\mathfrak{P}$ *we have that*

$$\sup_{\gamma\in\Gamma}\inf\sup_{\psi\in\mathfrak{F}}\gamma_\psi \le h(\mathfrak{P}\mid\phi). \tag{4.1}$$

Proof. We will show that (1)⇒(3)⇒(4)⇒(6)⇒(5)⇒(2)⇒(1). The fact that (3)⇒(4)⇒(6) is immediate.

To see that (1) implies (3), let $\mathfrak{F}$ be a prime a-ideal on X. If $h(\mathfrak{F}\mid\phi)=\infty$, then there is nothing to prove. So suppose $h(\mathfrak{F}\mid\phi)<\infty$ and thus $\mathfrak{F}\mid\phi$ exists. Since $\mathfrak{F}\subset\mathfrak{F}\mid\phi$, we have by assumption that

$$\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{F}}\gamma_\psi(x)\le\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{F}\mid\phi}\gamma_\psi(x)\le h(\mathfrak{F})\le h(\mathfrak{F}\mid\phi). \tag{4.2}$$

To see (5)⇒(2), notice that if $\mathfrak{F}$ is a saturated prime a-ideal containing ϕ, then the result follows immediately from the fact that $\mathfrak{F}=\mathfrak{F}\mid\phi$.

In order to show that (2)⇒(1), let $\mathfrak{F}$ be a prime a-ideal containing ϕ, $h=h(\mathfrak{F})$ say. Put $\mathfrak{G}:=(\mathfrak{F}_\infty)^h$. It is easy to see that $\mathfrak{G}$ is saturated and that $h(\mathfrak{G})=h$. Since $\mathfrak{G}_\infty=\mathfrak{F}_\infty$, we see that $\mathfrak{G}$ is prime (Proposition 2.1(m)). Moreover $\mathfrak{F}\subset\mathfrak{G}$, because if $\psi\in\mathfrak{F}$, then $\{\psi<\alpha\}\in\mathfrak{F}_\alpha=\mathfrak{F}_\infty$ for every $\alpha>h$, and so $\psi\in\mathfrak{G}$. In particular $\phi\in\mathfrak{G}$, so by assumption we have that $\mathfrak{G}$ is Γ-Cauchy, and thus

$$\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{F}}\gamma_\psi(x)\le\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{G}}\gamma_\psi(x)\le h(\mathfrak{G})=h(\mathfrak{F}). \tag{4.3}$$

Finally, to show (6)⇒(5), let $\mathfrak{F}$ be a saturated prime filter on X. Put $\mathfrak{G}:=(\mathfrak{F}_\infty)^0$. Then $\mathfrak{G}$ is saturated, prime and $h(\mathfrak{G})=0$. By assumption, $\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{F}}\gamma_\psi(x)\le h(\mathfrak{G}\mid\phi)$. Now we have $h(\mathfrak{G}\mid\phi)\le h(\mathfrak{F}\mid\phi)$, since $\mathfrak{G}\subset\mathfrak{F}$. Nevertheless, we have that $\sup_{\psi\in\mathfrak{F}}\gamma_\psi\le\sup_{\psi\in\mathfrak{G}}\gamma_\psi$. For suppose $\sup_{\psi\in\mathfrak{G}}\gamma_\psi(x)<M$, then $\forall\psi\in\mathfrak{G}$, $\exists z\in X:\gamma(z,x)\vee\psi(z)<M$. Let $\psi\in\mathfrak{F}$, then $\{\psi\le M\}\in\mathfrak{F}_M=\mathfrak{F}_\infty$ and thus there is some $z\in X$ for which $\gamma(z,x)\vee\theta_{\{\psi\le M\}}(z)\le M$ and so $\gamma(z,x)\vee\psi(z)\le M$, whence $\sup_{\psi\in\mathfrak{F}}\gamma_\psi(x)\le M$. Consequently,

$$\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{F}}\gamma_\psi\le\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{G}}\gamma_\psi\le h(\mathfrak{G}\mid\phi)\le h(\mathfrak{F}\mid\phi), \tag{4.4}$$

which we had to prove. □

There is a natural relationship between the precompactness (or total boundedness) of a function ϕ with respect to Γ, and the precompactness (or total boundedness) of the sets $\{\phi\le\varepsilon\}$ with respect to the semi-uniformity Γ^ε at every level $\varepsilon\in\mathbb{R}^+$.

Theorem 4.5. *Let* (X,Γ) *be an approach uniform space and* $\phi\in[0,\infty]^X$. *Then the following are equivalent:*

(1) ϕ *is* Γ*-precompact (totally bounded).*

(2) $\forall\alpha\in\mathbb{R}^+:\{\phi\le\alpha\}$ *is* Γ^α*-precompact (totally bounded).*

Proof. First we consider precompactness. To see that (1)⇒(2), let $\alpha\in\mathbb{R}^+$ and let $\mathcal{U}$ be an ultrafilter containing $\{\phi\le\alpha\}$. Since $\alpha+\theta_{\{\phi\le\alpha\}}\in\mathcal{U}_\alpha$ and $\phi\le\alpha+\theta_{\{\phi\le\alpha\}}$, we have $\phi\in\mathcal{U}_\alpha$. Since$(\mathcal{U}_\alpha)_\infty=\mathcal{U}$,$\mathcal{U}_\alpha$ is prime (Proposition 2.1(m)). By assumption, $\mathcal{U}_\alpha$ is Γ-Cauchy and therefore $\mathcal{U}$ is Γ^α-Cauchy (Proposition 3.5). Thus $\{\phi\le\alpha\}$ is Γ^α-precompact.

Conversely, to show that (2)⇒(1), let $\mathfrak{F}$ be a prime a-ideal containing ϕ. If $h(\mathfrak{F}) = \infty$, then there is nothing to prove, so suppose $h := h(\mathfrak{F}) < \infty$. Let $U \in \Gamma^h$, then $\{\gamma < \alpha\} \subset U$ for some $\gamma \in \Gamma$ and $\alpha > h$. Put $\beta := (\alpha + h)/2$. Then $\{\phi < \beta\}$ is Γ^β-precompact by assumption. Also $\{\phi \le \beta\} \in \mathfrak{F}_\infty$ and by Proposition 2.1(m), $\mathfrak{F}_\infty$ is an ultrafilter. Consequently, $\mathfrak{F}_\infty$ is a Γ^β-Cauchy filter. Since $\{\gamma < \alpha\} \in \Gamma^\beta$, there is some $x \in X$ such that $\{\gamma < \alpha\}_x \in \mathfrak{F}_\infty$, and thus $U(x) \in \mathfrak{F}_\infty$. Hence, $\mathfrak{F}_\infty$ is Γ^h-Cauchy, and applying Theorem 3.4, we obtain that $\mathfrak{F}$ is Γ-Cauchy. Consequently, ϕ is precompact.

Secondly, we consider total boundedness. To see that (1)⇒(2), let $\gamma \in \Gamma$ and $\alpha > \varepsilon$. Let $\phi_1, \dots, \phi_n \in [0,\infty]^X$ be such that

- $\gamma \wedge 2\alpha \le \phi_i \times \phi_i + (\alpha - \varepsilon)$
- $\inf_{i=1}^n \phi_i \wedge 2\alpha \le \phi + (\alpha - \varepsilon)$.

Put $A_i := \{\phi_i \le \alpha\}$ for every $i \in \{1,\dots,n\}$. If $\phi(z) \le \varepsilon$, then $\inf_{i=1}^n \phi_i(z) \le \alpha$ and thus $z \in \cup_{i=1}^n A_i$. Hence $\{\phi \le \varepsilon\} \subset \cup_{i=1}^n A_i$. Also, we have

$$(x,y) \in A_i \times A_i \Longrightarrow \phi_i(x) \vee \phi_i(y) \le \varepsilon \Longrightarrow \gamma(x,y) \le \alpha, \tag{4.5}$$

that is, $A_i \times A_i \subset \{\gamma \le \alpha\}$.

Conversely, to see that (2)⇒(1), first consider $\phi := \inf_{i=0}^n (\rho_j + \theta_{P_j})$ for some $P_0 \subset P_1 \subset \cdots \subset P_n$ and some net $\{0 = \rho_0, \rho_1, \dots, \rho_n = N\}$ $[0,N]$ in n equal subintervals of length less than ε. By assumption, P_j is Γ^{ρ_j}-totally bounded for every j. Let $\gamma \in \Gamma$, $\varepsilon > 0$, and $N < \infty$. Then for every j there exist $A_1^j, \dots, A_{n_j}^j \subset X$ such that for every $j \in \{0,\dots,n\}$

- $A_i^j \times A_i^j \subset \{\gamma < \rho_j + \varepsilon\}$
- $\cup_{i=1}^{n_j} A_i^j \supset P_j$.

Put $\phi_i^j := \rho_j + \theta_{A_i^j}$ for every $j \in \{0,\dots,n\}$, $i \in \{1,\dots,n_j\}$. Now suppose $\phi(x) = \rho_j$. Then $x \in P_j \backslash P_{j-1}$, and then there is some A_i^j containing x, that is, $\phi_i^j(x) = \rho_j = \phi(x)$. Consequently, $\inf_{i,j} \phi_i^j \le \phi$. Now suppose $\phi_i^j(x) \vee \phi_i^j(y) = M$. If $M = \infty$, then there is nothing to prove. If $M = \rho_j$, then $x, y \in A_i^j$ and thus $\gamma(x,y) \le \rho_j + \varepsilon$. So the theorem is shown for this particular choice for ϕ. Now let ϕ be arbitrary, $N < \infty$ and $\varepsilon > 0$. Choose a net $\{0 = \rho_0, \rho_1, \dots, \rho_n = N\}$ dividing $[0,N]$ in n equal subintervals of length less than $\varepsilon/2$ and consider $\tilde{\phi} = \inf_{j=0}^n (\rho_j + \theta_{\{\phi \le \rho_j\}})$. Let $\gamma \in \Gamma$. We just showed that there are $\phi_1, \dots, \phi_n$ such that $\gamma \wedge N \le \phi_i \times \phi_i + \varepsilon/2$ and $\inf \phi_i \wedge N \le \tilde{\phi} + \varepsilon/2 \le \phi + \varepsilon$. □

Corollary 4.6. *Let X be an approach uniform space and let $\phi \in [0,\infty]^X$. If ϕ is totally bounded, then ϕ is precompact.*

Proof. It is easy to verify this statement for semi-uniform spaces. Then the corollary follows from Theorem 4.5. □

However, precompactness is not the same as total boundedness.

Example 4.7. Consider $\mathbb{N}$ equipped with (the metric approach uniformity Γ determined by) the "postoffice-discrete" metric

$$d : \mathbb{N} \times \mathbb{N} \longrightarrow [0,\infty] : (x,y) \longmapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x = 0 \text{ or } y = 0, \text{ and } x \ne y, \\ 2 & \text{if } 0 \ne x \ne y \ne 0. \end{cases} \tag{4.6}$$

With respect to this approach uniform space, $1+\theta_{\mathbb{N}}$ is precompact but not totally bounded. The discrepancy becomes clear if we look at Γ^1 which is generated by $U = \{(x,y) \mid x = y \text{ or } x = 0 \text{ or } y = 0\}$. We see that $U(0) = \mathbb{N}$, hence $\mathbb{N}$ is precompact with respect to Γ^1. It is however not possible to cover $\mathbb{N}$ with finitely many $A \subset X$ such that $A \times A \subset U$, whence $\mathbb{N}$ is not totally bounded with respect to Γ^1. Consequently, $1+\theta_{\mathbb{N}}$ is Γ-precompact but not γ-totally bounded.

Definition 4.1 is an extension of the "measure of precompactness" μ_{pc} and the "measure of total boundedness" μ_{tb} introduced in Lowen and Windels [5].

Proposition 4.8. *Let (X,Γ) be an approach uniform space and $A \subset X$. Then for every $\varepsilon \in \mathbb{R}$, we have that $\mu_{pc}(A) \le \varepsilon (\mu_{tb}(A) \le \varepsilon)$ if and only if $\varepsilon + \theta_A$ is precompact (totally bounded). Consequently $\mu_{pc}(A) = \inf\{\alpha \mid \alpha + \theta_A$ is precompact$\}$ and $\mu_{tb}(A) = \inf\{\alpha \mid \alpha + \theta_A$ is totally bounded$\}$.*

Proof. By Theorem 4.5, $\varepsilon + \theta_A$ is precompact if A is Γ^α-precompact for every $\alpha \ge \varepsilon$, or, equivalently, if $\mu_{pc}(A) \le \varepsilon$. The proof for total boundedness goes along the same lines. □

Corollary 4.9. *A set A is precompact with respect to a uniform space if and only if θ_A is precompact with respect to the associated principal approach uniform space.*

Proof. This is a consequence of Proposition 4.8 and [5, Proposition 2.4]. □

We now show that precompactness shares some important features with the classical notion of precompactness.

Proposition 4.10. *Let (X,Γ) be an approach uniform space and let $\phi, \phi' \in [0,\infty]^X$. If ϕ is precompact (totally bounded) and $\phi \le \phi'$, then ϕ' is precompact (totally bounded).*

Proof. As far as precompactness is concerned, if $\mathfrak{F}$ is a prime a-ideal containing ϕ', then $\mathfrak{F}$ contains ϕ as well and therefore $\mathfrak{F}$ is Γ-Cauchy. The statement for total boundedness follows at once from the definition. □

Corollary 4.11. *Let X be an approach uniform space and let $A, B \subset X$. If $A \subset B$, then $\mu_{pc}(A) \le \mu_{pc}(B)$ and $\mu_{tb}(A) \le \mu_{tb}(B)$. Consequently, in a uniform space, any subset of a precompact set is precompact.*

Proof. This is an immediate consequence of Propositions 4.8 and 4.10. □

Proposition 4.12. *Let (X,Γ) be an approach uniform space and let $\phi_1, \phi_2, \dots, \phi_n \in [0,\infty]^X$ be precompact (totally bounded). Then $\inf_{i=1}^n \phi_i$ is precompact (totally bounded).*

Proof. For precompactness, let $\mathfrak{F}$ be a prime a-ideal containing $\inf_{i=1}^n \phi_i$. Then $\phi_i \in \mathfrak{F}$ for some $i \in \{1,\dots,n\}$, and by assumption, $\mathfrak{F}$ is Γ-Cauchy.

For total boundedness, let for every $i \in \{1,\dots,n\}$, F_i be a finite collection of functions in $[0,\infty]^X$ satisfying the conditions in Definition 4.3(b) with respect to ϕ_i. Then $\cup_{i=1}^n F_i$ is a finite collection satisfying the same conditions with respect to $\inf_{i=1}^n \phi_i$. □

Corollary 4.13. *Let X be an approach uniform space and let $A_1, \dots, A_n \subset X$. Then $\mu_{pc}(\cup_{i=1}^n A_i) = \sup_{i=1}^n \mu_{pc}(A_i)$ and $\mu_{tb}(\cup_{i=1}^n A_i) = \sup_{i=1}^n \mu_{tb}(A_i)$. Consequently, in a uniform space, a finite union of precompact sets is precompact.*

Proof. This is a consequence of Propositions 4.8 and 4.12. □

In [6] a function $\phi \in [0,\infty]^X$ is defined to be compact if for every prime ideal $\mathfrak{f}$ containing ϕ holds that $\inf(\phi \vee \alpha\mathfrak{f}) \le h(\mathfrak{f})$. Here $\alpha\mathfrak{f}$ is the adherence of $\mathfrak{f}$ defined as $\alpha\mathfrak{f}(x) = \sup_{\gamma\in\Gamma}\sup_{\psi\in\mathfrak{f}}\gamma_\psi(x)$.

Proposition 4.14. *Let (X,Γ) be an approach uniform space and let $\phi \in [0,\infty]^X$. If ϕ is compact, then ϕ is precompact.*

Proof. Let $\mathfrak{f}$ be a prime a-ideal containing ϕ. Then

$$\begin{aligned}\sup_{\gamma\in\Gamma}\inf_{x\in X}\sup_{\psi\in\mathfrak{f}}\gamma_\psi(x) &\le \inf_{x\in X}\sup_{\gamma\in\Gamma}\sup_{\psi\in\mathfrak{f}}\gamma_\psi(x) = \inf_{x\in X}\alpha\mathfrak{f}(x)\\ &\le \inf(\phi\vee\alpha\mathfrak{f}) \le h(\mathfrak{f}).\end{aligned} \tag{4.7}$$

□

Corollary 4.15. *Let X be an approach uniform space and let $A \subset X$. Then $\mu_{pc}(A) \le \mu_c(A)$. Consequently, in a uniform space, a compact set is precompact.*

Proof. Let $\varepsilon > 0$. If $\mu_c(A) \le \varepsilon$, then by [6, Proposition 3.3], $\varepsilon + \theta_A$ is compact. Consequently, $\varepsilon + \theta_A$ is precompact, which by Proposition 4.8 yields that $\mu_{pc}(A) \le \varepsilon$. □

If ϕ is compact, then ϕ need not be totally bounded. This is illustrated by the following example.

Example 4.16. Consider $\mathbb{N}$ equipped with the postoffice-discrete metric as in Example 4.7. It is not difficult to verify that for every filter $\mathcal{F}$ on $\mathbb{N}$, $\alpha\mathcal{F}(0) \le 1$. Hence, by [6, Proposition 3.4], $1+\theta_{\mathbb{N}}$ is compact. Nonetheless, Example 4.7 shows that $1+\theta_{\mathbb{N}}$ is not totally bounded.

Proposition 4.17. *Let (X,Γ) and (Y,Ψ) be approach uniform spaces and let $f : (X,\Gamma) \to (Y,\Psi)$ be a surjective uniform contraction. If ϕ is Γ-precompact (totally bounded), then ϕ_f is Ψ-precompact (totally bounded).*

Proof. The proof concerning precompactness is exemplary. It is easy to see that $\{\phi_f \le \alpha\} = f\{\phi \le \alpha\}$. Hence, by Theorem 4.5,

$$\begin{aligned}\phi \text{ is } \Gamma\text{-precompact} &\iff \forall\alpha\in\mathbb{R}^+ : \{\phi\le\alpha\} \text{ is } \Gamma^\alpha\text{-precompact}\\ &\implies \forall\alpha\in\mathbb{R}^+ : f\{\phi\le\alpha\} \text{ is } \Psi^\alpha\text{-precompact}\\ &\iff \forall\alpha\in\mathbb{R}^+ : \{\phi_f\le\alpha\} \text{ is } \Psi^\alpha\text{-precompact}\\ &\iff \phi_f \text{ is } \Psi\text{-precompact.}\end{aligned} \tag{4.8}$$

□

Corollary 4.18. *Let $f : X \to Y$ be a uniformly continuous surjection between uniform spaces and let $A \subset X$. Then $\mu_{pc}(f(A)) \le \mu_{pc}(A)$ and $\mu_{tb}(f(A)) \le \mu_{tb}(A)$. Consequently, the uniformly continuous image of a precompact set is precompact.*

Proposition 4.19. *Let $(f_i : (X,\Gamma) \to (Y_i,\Psi_i))_{i\in I}$ be an initial source in AUnif and let $\phi \in [0,\infty]^X$. Then ϕ is Γ-totally bounded if and only if $\forall i \in I : \phi_{f_i}$ is Ψ_i-totally bounded.*

Proof. We have

$$\begin{aligned}\phi \text{ is } \Gamma\text{-totally bounded} &\iff \forall \alpha \in \mathbb{R}^+ : \{\phi \le \alpha\} \text{ is } \Gamma^\alpha\text{-totally bounded}\\ &\iff \forall i \in I,\ \forall \alpha \in \mathbb{R}^+ : f_i\{\phi \le \alpha\} \text{ is } \Psi_i^\alpha\text{-totally bounded}\\ &\iff \forall i \in I,\ \forall \alpha \in \mathbb{R}^+ : \{\phi_{f_i} \le \alpha\} \text{ is } \Psi_i^\alpha\text{-totally bounded}\\ &\iff \forall i \in I : \phi_{f_i} \text{ is } \Psi_i\text{-totally bounded.}\end{aligned} \tag{4.9}$$

□

Corollary 4.20. *Let $(f_i : (X,\Gamma) \to (Y_i,\Psi_i))_{i\in I}$ be an initial source in AUnif and let $A \subset X$. Then $\mu_{tb}(A) = \sup_{i\in I} \mu_{tb}(A_i)$. Consequently, a product of uniform spaces is totally bounded if and only if each factor space is totally bounded.*

References

[1] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics, vol. 60, Marcel Dekker, New York, 1980. MR 82f:47066. Zbl 0441.47056.

[2] C. Kuratowski, *Sur les espaces complets*, Fund. Math. **15** (1930), 301–309 (French).

[3] R. Lowen, M. Sioen, and C. Verbeeck, *Generalizing filter convergence by ideals*, in preparation.

[4] R. Lowen and B. Windels, *AUnif: a common supercategory of pMET and Unif*, Int. J. Math. Math. Sci. **21** (1998), no. 1, 1–18. MR 98j:54014. Zbl 0890.54024.

[5] ———, *Quantifying completion*, Int. J. Math. Math. Sci. **23** (2000), no. 11, 729–739. MR 2001h:54047. Zbl 0962.54023.

[6] ———, *Quantification of topological concepts using ideals*, Int. J. Math. Math. Sci. **26** (2001), no. 9, 561–579. CMP 1 853 573.

[7] B. Windels, *On the quantification of uniform properties. II*, Studia Sci. Math. Hungar. **37** (2001), no. 1-2, 169–184. CMP 1 834 329.

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