FUZZY BCI-SUBALGEBRAS WITH INTERVAL-VALUED MEMBERSHIP FUNCTIONS

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Abstract. The purpose of this paper is to define the notion of an interval-valued fuzzy BCI-subalgebra (briefly, an i-v fuzzy BCI-subalgebra) of a BCI-algebra. Necessary and sufficient conditions for an i-v fuzzy set to be an i-v fuzzy BCI-subalgebra are stated. A way to make a new i-v fuzzy BCI-subalgebra from old one is given. The images and inverse images of i-v fuzzy BCI-subalgebras are defined, and how the images or inverse images of i-v fuzzy BCI-subalgebras become i-v fuzzy BCI-subalgebras is studied.

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1. Introduction. The notion of BCK-algebras was proposed by Iam and Iséki in 1966. In the same year, Iséki [2] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras. Fuzzy sets were initiated by Zadeh [3]. In [4], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. In [4], Zadeh also constructed a method of approximate inference using his i-v fuzzy sets. In [1], Biswas defined interval-valued fuzzy subgroups (i.e., i-v fuzzy subgroups) of Rosenfeld’s nature, and investigated some elementary properties. In this paper, using the notion of interval-valued fuzzy set by Zadeh, we introduce the concept of an interval-valued fuzzy BCI-subalgebra (briefly, i-v fuzzy BCI-subalgebra) of a BCI-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy BCI-subalgebra. We prove that every BCI-subalgebra of a BCI-algebra X can be realized as an i-v level BCI-subalgebra of an i-v fuzzy BCI-subalgebra of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy BCI-subalgebras become i-v fuzzy BCI-subalgebras.

2. Preliminaries. In this section, we include some elementary aspects that are necessary for this paper.

Recall that a BCI-algebra is an algebra \((X, *, 0)\) of type \((2, 0)\) satisfying the following axioms:

(I) \(((x * y) * (x * z)) * (z * y) = 0,\)

(II) \((x * (x * y)) * y = 0,\)
(III) $x \ast x = 0$, and
(IV) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$,
for every $x, y, z \in X$.

Note that the equality $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$ holds in a BCI-algebra. A non-empty subset $S$ of a BCI-algebra $X$ is called a BCI-subalgebra of $X$ if $x \ast y \in S$ whenever $x, y \in S$. A mapping $f : X \rightarrow Y$ of BCI-algebras is called a homomorphism if $f(x \ast y) = f(x) \ast f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts. Let $X$ be a set. A fuzzy set in $X$ is a function $\mu : X \rightarrow [0, 1]$. Let $f$ be a mapping from a set $X$ into a set $Y$. Let $\nu$ be a fuzzy set in $Y$. Then the inverse image of $\nu$, denoted by $f^{-1}[\nu]$, is the fuzzy set in $X$ defined by $f^{-1}[\nu](x) = \nu(f(x))$ for all $x \in X$. Conversely, let $\mu$ be a fuzzy set in $X$. The image of $\mu$, written as $f[\mu]$, is a fuzzy set in $Y$ defined by

$$f[\mu](y) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu(z) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases} \quad (2.1)$$

for all $y \in Y$, where $f^{-1}(y) = \{ x \mid f(x) = y \}$.

An interval-valued fuzzy set (briefly, i-v fuzzy set) $A$ defined on $X$ is given by

$$A = \{(x, [\mu^L_A(x), \mu^U_A(x)]) \}, \quad \forall x \in X \quad \text{(briefly, denoted by } A = [\mu^L_A, \mu^U_A]), \quad (2.2)$$

where $\mu^L_A$ and $\mu^U_A$ are two fuzzy sets in $X$ such that $\mu^L_A(x) \leq \mu^U_A(x)$ for all $x \in X$.

Let $\bar{\mu}_A(x) = [\mu^L_A(x), \mu^U_A(x)], \forall x \in X$ and let $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$. If $\mu^L_A(x) = \mu^U_A(x) = c$, where $0 \leq c \leq 1$, then we have $\bar{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0, 1]$. Thus $\bar{\mu}_A(x) \in D[0, 1], \forall x \in X$, and therefore the i-v fuzzy set $A$ is given by

$$A = \{(x, \bar{\mu}_A(x)) \}, \quad \forall x \in X, \text{ where } \bar{\mu}_A : X \rightarrow D[0, 1]. \quad (2.3)$$

Now let us define what is known as refined minimum (briefly, rmin) of two elements in $D[0, 1]$. We also define the symbols $\geq$, $\leq$, and $=$ in case of two elements in $D[0, 1]$. Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0, 1]$. Then

$$\text{rmin}(D_1, D_2) = \left[ \min\{a_1, a_2\}, \min\{b_1, b_2\} \right];$$
$$D_1 \geq D_2 \quad \text{if and only if } a_1 \geq a_2, b_1 \geq b_2; \quad (2.4)$$

and similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

**Definition 2.1.** A fuzzy set $\mu$ in a BCI-algebra $X$ is called a fuzzy BCI-subalgebra of $X$ if $\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

**3. Interval-valued fuzzy BCI-subalgebras.** In what follows, let $X$ denote a BCI-algebra unless otherwise specified. We begin with the following two propositions.
**Proposition 3.1.** Let \( f \) be a homomorphism from a BCI-algebra \( X \) into a BCI-algebra \( Y \). If \( \nu \) is a fuzzy BCI-subalgebra of \( Y \), then the inverse image \( f^{-1}[\nu] \) of \( \nu \) is a fuzzy BCI-subalgebra of \( X \).

**Proof.** For any \( x, y \in X \), we have

\[
f^{-1}[\nu](x * y) = \nu(f(x * y)) = \nu(f(x) * f(y)) \\
\geq \min\{\nu(f(x)), \nu(f(y))\} \\
= \min\{f^{-1}[\nu](x), f^{-1}[\nu](y)\}.
\]

Hence \( f^{-1}[\nu] \) is a fuzzy BCI-subalgebra of \( X \). \(\square\)

**Proposition 3.2.** Let \( f : X \to Y \) be a homomorphism between BCI-algebras \( X \) and \( Y \). For every fuzzy BCI-subalgebra \( \mu \) of \( X \), the image \( f[\mu] \) of \( \mu \) is a fuzzy BCI-subalgebra of \( Y \).

**Proof.** We first prove that

\[
f^{-1}(y_1) * f^{-1}(y_2) \subseteq f^{-1}(y_1 * y_2)
\]

for all \( y_1, y_2 \in Y \). For, if \( x \in f^{-1}(y_1) * f^{-1}(y_2) \), then \( x = x_1 * x_2 \) for some \( x_1 \in f^{-1}(y_1) \) and \( x_2 \in f^{-1}(y_2) \). Since \( f \) is a homomorphism, it follows that \( f(x) = f(x_1) * f(x_2) = y_1 * y_2 \) so that \( x \in f^{-1}(y_1 * y_2) \). Hence (3.2) holds. Now let \( y_1, y_2 \in Y \) be arbitrarily given. Assume that \( y_1 * y_2 \in \text{Im}(f) \). Then \( f[\mu](y_1 * y_2) = 0 \). But if \( y_1 * y_2 \notin \text{Im}(f) \), that is, \( f^{-1}(y_1 * y_2) = \emptyset \), then \( f^{-1}(y_1) = \emptyset \) or \( f^{-1}(y_2) = \emptyset \) by (3.2). Thus \( f[\mu](y_1) = 0 \) or \( f[\mu](y_2) = 0 \), and so

\[
f[\mu](y_1 * y_2) = 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}.
\]

Suppose that \( f^{-1}(y_1 * y_2) \neq \emptyset \). Then we should consider the two cases:

\[
f^{-1}(y_1) = \emptyset \quad \text{or} \quad f^{-1}(y_2) = \emptyset,
\]

\[
f^{-1}(y_1) \neq \emptyset \quad \text{and} \quad f^{-1}(y_2) \neq \emptyset.
\]

For the case (3.4), we have \( f[\mu](y_1) = 0 \) or \( f[\mu](y_2) = 0 \), and so

\[
f[\mu](y_1 * y_2) \geq 0 = \min\{f[\mu](y_1), f[\mu](y_2)\}.
\]

Case (3.5) implies, from (3.2), that

\[
f[\mu](y_1 * y_2) = \sup_{z \in f^{-1}(y_1 * y_2)} \mu(z) \geq \sup_{z \in f^{-1}(y_1) * f^{-1}(y_2)} \mu(z) \\
= \sup_{x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)} \mu(x_1 * x_2).
\]
Since $\mu$ is a fuzzy BCI-subalgebra of $X$, it follows from the definition of a fuzzy BCI-subalgebra that

$$f[\mu](y_1 \ast y_2) \geq \sup_{x_1 \in f^{-1}(y_1), \ x_2 \in f^{-1}(y_2)} \min\{\mu(x_1), \mu(x_2)\}$$

$$= \sup_{x_1 \in f^{-1}(y_1)} \left( \min\left\{ \sup_{x_2 \in f^{-1}(y_2)} \mu(x_1), \mu(x_2) \right\} \right)$$

$$= \sup_{x_1 \in f^{-1}(y_1)} \left( \min\{\mu(x_1), f[\mu](y_2)\} \right)$$

$$= \min_{x_1 \in f^{-1}(y_1)} \left\{ \sup_{x_2 \in f^{-1}(y_2)} \mu(x_1), f[\mu](y_2) \right\}$$

$$= \min_{x_1 \in f^{-1}(y_1)} \{ f[\mu](y_1), f[\mu](y_2) \}.$$

Hence $f[\mu](y_1 \ast y_2) \geq \min\{f[\mu](y_1), f[\mu](y_2)\}$ for all $y_1, y_2 \in Y$. This completes the proof.

**Definition 3.3.** An i-v fuzzy set $A$ in $X$ is called an interval-valued fuzzy BCI-subalgebra (briefly, i-v fuzzy BCI-subalgebra) of $X$ if

$$\tilde{\mu}_A(x \ast y) \geq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \quad \forall x, y \in X.$$  \hspace{1cm} (3.9)

**Example 3.4.** Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>$\ast$</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>a</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let an i-v fuzzy set $A$ defined on $X$ be given by

$$\tilde{\mu}_A(x) = \begin{cases} [0.2, 0.8] & \text{if } x \in \{0, b\}, \\ [0.1, 0.7] & \text{otherwise.} \end{cases}$$  \hspace{1cm} (3.10)

It is easy to check that $A$ is an i-v fuzzy BCI-subalgebra of $X$.

**Lemma 3.5.** If $A$ is an i-v fuzzy BCI-subalgebra of $X$, then $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ for all $x \in X$. 

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Proof. For every \( x \in X \), we have
\[
\bar{\mu}_A(0) = \bar{\mu}_A(x \ast x) \geq \min \{ \bar{\mu}_A(x), \bar{\mu}_A(x) \}
\]
\[
= \min \{ [\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(x), \mu^U_A(x)] \}
\]
\[
= [\mu^L_A(x), \mu^U_A(x)] = \bar{\mu}_A(x),
\]
this completes the proof. \( \square \)

Theorem 3.6. Let \( A \) be an \( i \cdot v \) fuzzy BCI-subalgebra of \( X \). If there is a sequence \( \{ x_n \} \) in \( X \) such that
\[
\lim_{n \to \infty} \bar{\mu}_A(x_n) = [1, 1],
\]
then \( \bar{\mu}_A(0) = [1, 1] \).

Proof. Since \( \bar{\mu}_A(0) \geq \bar{\mu}_A(x) \) for all \( x \in X \), we have \( \bar{\mu}_A(0) \geq \bar{\mu}_A(x_n) \) for every positive integer \( n \). Note that
\[
[1, 1] \geq \bar{\mu}_A(x_n) \geq \lim_{n \to \infty} \bar{\mu}_A(x_n) = [1, 1].
\]
Hence \( \bar{\mu}_A(0) = [1, 1] \). \( \square \)

Theorem 3.7. An \( i \cdot v \) fuzzy set \( A = [\mu^L_A, \mu^U_A] \) in \( X \) is an \( i \cdot v \) fuzzy BCI-subalgebra of \( X \) if and only if \( \mu^L_A \) and \( \mu^U_A \) are fuzzy BCI-subalgebras of \( X \).

Proof. Suppose that \( \mu^L_A \) and \( \mu^U_A \) are fuzzy BCI-subalgebras of \( X \). Let \( x, y \in X \). Then
\[
\bar{\mu}_A(x \ast y) = [\mu^L_A(x \ast y), \mu^U_A(x \ast y)]
\]
\[
\geq \min \{ \mu^L_A(x), \mu^L_A(y) \}, \min \{ \mu^U_A(x), \mu^U_A(y) \}
\]
\[
= \min \{ [\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(y), \mu^U_A(y)] \}
\]
\[
= \min \{ \mu^L_A(x), \mu^U_A(x) \}, \min \{ \mu^U_A(x), \mu^U_A(y) \}.
\]
Hence \( A \) is an \( i \cdot v \) fuzzy BCI-subalgebra of \( X \).

Conversely, assume that \( A \) is an \( i \cdot v \) fuzzy BCI-subalgebra of \( X \). For any \( x, y \in X \), we have
\[
[\mu^L_A(x \ast y), \mu^U_A(x \ast y)] = \bar{\mu}_A(x \ast y) \geq \min \{ \bar{\mu}_A(x), \bar{\mu}_A(y) \}
\]
\[
= \min \{ [\mu^L_A(x), \mu^U_A(x)], [\mu^L_A(y), \mu^U_A(y)] \}
\]
\[
= [\min \{ \mu^L_A(x), \mu^U_A(y) \}, \min \{ \mu^U_A(x), \mu^U_A(y) \}].
\]
It follows that \( \mu^L_A(x \ast y) \geq \min \{ \mu^L_A(x), \mu^L_A(y) \} \) and \( \mu^U_A(x \ast y) \geq \min \{ \mu^U_A(x), \mu^U_A(y) \} \).
Hence \( \mu^L_A \) and \( \mu^U_A \) are fuzzy BCI-subalgebras of \( X \). \( \square \)

Theorem 3.8. Let \( A \) be an \( i \cdot v \) fuzzy set in \( X \). Then \( A \) is an \( i \cdot v \) fuzzy BCI-subalgebra of \( X \) if and only if the nonempty set
\[
U(A; [\delta_1, \delta_2]) := \{ x \in X \mid \bar{\mu}_A(x) \geq [\delta_1, \delta_2] \}
\]
is a BCI-subalgebra of \( X \) for every \( [\delta_1, \delta_2] \in D[0, 1] \).
We then call \( \tilde{U}(A;[\delta_1, \delta_2]) \) the \( i-v \) level BCI-subalgebra of \( A \).

**Proof.** Assume that \( A \) is an \( i-v \) fuzzy BCI-subalgebra of \( X \) and let \([\delta_1, \delta_2] \in D[0,1]\) be such that \( x, y \in \tilde{U}(A;[\delta_1, \delta_2]) \). Then

\[
\mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \} \geq \min \{ \min \{ \mu_A(x), \mu_A(y) \} \} = [\delta_1, \delta_2],
\]

and so \( x \ast y \in \tilde{U}(A;[\delta_1, \delta_2]) \). Thus \( \tilde{U}(A;[\delta_1, \delta_2]) \) is a BCI-subalgebra of \( X \).

Conversely, assume that \( \tilde{U}(A;[\delta_1, \delta_2]) \) is a BCI-subalgebra of \( X \) for every \([\delta_1, \delta_2] \in D[0,1]\). Suppose there exist \( x_0, y_0 \in X \) such that

\[
\mu_A(x_0 \ast y_0) < \min \{ \mu_A(x_0), \mu_A(y_0) \}.
\]

Let \( \mu_A(x_0) = [y_1, y_2], \mu_A(y_0) = [y_3, y_4], \) and \( \mu_A(x_0 \ast y_0) = [\delta_1, \delta_2] \). Then

\[
[\delta_1, \delta_2] < \min \{ [y_1, y_2], [y_3, y_4] \} = \min \{ y_1, y_3 \}, \min \{ y_2, y_4 \}.\]

Hence \( \delta_1 < \min \{ y_1, y_3 \} \) and \( \delta_2 < \min \{ y_2, y_4 \} \). Taking

\[
[\lambda_1, \lambda_2] = \frac{1}{2}(\mu_A(x_0 \ast y_0) + \min \{ \mu_A(x_0), \mu_A(y_0) \}),
\]

we obtain

\[
[\lambda_1, \lambda_2] = \frac{1}{2}([\delta_1, \delta_2] + \min \{ y_1, y_3 \}, \min \{ y_2, y_4 \}) = \left[ \frac{1}{2}(\delta_1 + \min \{ y_1, y_3 \}), \frac{1}{2}(\delta_2 + \min \{ y_2, y_4 \}) \right].
\]

It follows that

\[
\min \{ y_1, y_3 \} > \lambda_1 = \frac{1}{2}(\delta_1 + \min \{ y_1, y_3 \}) > \delta_1,
\]

\[
\min \{ y_2, y_4 \} > \lambda_2 = \frac{1}{2}(\delta_2 + \min \{ y_2, y_4 \}) > \delta_2,
\]

so that \( [\min \{ y_1, y_3 \}, \min \{ y_2, y_4 \}] > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \mu_A(x_0 \ast y_0) \). Therefore, \( x_0 \ast y_0 \notin \tilde{U}(A;[\lambda_1, \lambda_2]) \). On the other hand,

\[
\tilde{\mu}_A(x_0) = [y_1, y_2] \geq [\min \{ y_1, y_3 \}, \min \{ y_2, y_4 \}] > [\lambda_1, \lambda_2],
\]

\[
\tilde{\mu}_A(y_0) = [y_3, y_4] \geq [\min \{ y_1, y_3 \}, \min \{ y_2, y_4 \}] > [\lambda_1, \lambda_2],
\]

and so \( x_0, y_0 \in \tilde{U}(A;[\lambda_1, \lambda_2]) \). It contradicts that \( \tilde{U}(A;[\lambda_1, \lambda_2]) \) is a BCI-subalgebra of \( X \). Hence \( \tilde{\mu}_A(x \ast y) \geq \min \{ \tilde{\mu}_A(x), \tilde{\mu}_A(y) \} \) for all \( x, y \in X \). This completes the proof.

**Theorem 3.9.** Every BCI-subalgebra of \( X \) can be realized as an \( i-v \) level BCI-subalgebra of an \( i-v \) fuzzy BCI-subalgebra of \( X \).

**Proof.** Let \( Y \) be a BCI-subalgebra of \( X \) and let \( A \) be an \( i-v \) fuzzy set on \( X \) defined by

\[
\tilde{\mu}_A(x) = \begin{cases} 
[\alpha_1, \alpha_2] & \text{if } x \in Y, \\
[0,0] & \text{otherwise},
\end{cases}
\]

where \( \alpha_1, \alpha_2 \in (0,1] \) with \( \alpha_1 < \alpha_2 \). It is clear that \( \tilde{U}(A;[\alpha_1, \alpha_2]) = Y \). We show that \( A \)
is an i-v fuzzy BCI-subalgebra of $X$. Let $x,y \in X$. If $x,y \in Y$, then $x \ast y \in Y$ and so

$$\bar{\mu}_A(x \ast y) = [\alpha_1, \alpha_2] = \min \{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \quad (3.25)$$

If $x,y \notin Y$, then $\bar{\mu}_A(x) = [0,0] = \bar{\mu}_A(y)$ and thus

$$\bar{\mu}_A(x \ast y) \geq [0,0] = \min \{[0,0], [0,0]\} = \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \quad (3.26)$$

If $x \in Y$ and $y \notin Y$, then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2]$ and $\bar{\mu}_A(y) = [0,0]$. It follows that

$$\bar{\mu}_A(x \ast y) \geq [0,0] = \min \{[\alpha_1, \alpha_2], [0,0]\} = \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\}. \quad (3.27)$$

Similarly for the case $x \notin Y$ and $y \in Y$, we get $\bar{\mu}_A(x \ast y) \geq \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\}$. Therefore $A$ is an i-v fuzzy BCI-subalgebra of $X$, and the proof is complete. \qed

**Theorem 3.10.** Let $Y$ be a subset of $X$ and let $A$ be an i-v fuzzy set on $X$ which is given in the proof of Theorem 3.9. If $A$ is an i-v fuzzy BCI-subalgebra of $X$, then $Y$ is a BCI-subalgebra of $X$.

**Proof.** Assume that $A$ is an i-v fuzzy BCI-subalgebra of $X$. Let $x,y \in Y$. Then $\bar{\mu}_A(x) = [\alpha_1, \alpha_2] = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x \ast y) \geq \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \min \{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]. \quad (3.28)$$

This implies that $x \ast y \in Y$. Hence $Y$ is a BCI-subalgebra of $X$. \qed

**Theorem 3.11.** If $A$ is an i-v fuzzy BCI-subalgebra of $X$, then the set

$$X_{\bar{\mu}_A} := \{x \in X \mid \bar{\mu}_A(x) = \bar{\mu}_A(0)\} \quad (3.29)$$

is a BCI-subalgebra of $X$.

**Proof.** Let $x,y \in X_{\bar{\mu}_A}$. Then $\bar{\mu}_A(x) = \bar{\mu}_A(0) = \bar{\mu}_A(y)$, and so

$$\bar{\mu}_A(x \ast y) \geq \min \{\bar{\mu}_A(x), \bar{\mu}_A(y)\} = \min \{\bar{\mu}_A(0), \bar{\mu}_A(0)\} = \bar{\mu}_A(0). \quad (3.30)$$

Combining this and Lemma 3.5, we get $\bar{\mu}_A(x \ast y) = \bar{\mu}_A(0)$, that is, $x \ast y \in X_{\bar{\mu}_A}$. Hence $X_{\bar{\mu}_A}$ is a BCI-subalgebra of $X$. \qed

The following is a way to make a new i-v fuzzy BCI-subalgebra from old one.

**Theorem 3.12.** For an i-v fuzzy BCI-subalgebra $A$ of $X$, the i-v fuzzy set $A^*$ in $X$ defined by $\bar{\mu}_{A^*}(x) = \bar{\mu}_A(0 \ast x)$ for all $x \in X$ is an i-v fuzzy BCI-subalgebra of $X$.

**Proof.** Since the equality $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$ holds for all $x,y \in X$, we have

$$\bar{\mu}_{A^*}(x \ast y) = \bar{\mu}_A(0 \ast (x \ast y)) = \bar{\mu}_A((0 \ast x) \ast (0 \ast y))$$

$$\geq \min \{\bar{\mu}_A(0 \ast x), \bar{\mu}_A(0 \ast y)\} \quad (3.31)$$

$$= \min \{\bar{\mu}_{A^*}(x), \bar{\mu}_{A^*}(y)\}$$

for all $x,y \in X$. Therefore $A^*$ is an i-v fuzzy BCI-subalgebra of $X$. \qed
**Definition 3.13** (Biswas [1]). Let $f$ be a mapping from a set $X$ into a set $Y$. Let $B$ be an i-v fuzzy set in $Y$. Then the **inverse image** of $B$, denoted by $f^{-1}(B)$, is the i-v fuzzy set in $X$ with the membership function given by $\bar{\mu}_{f^{-1}(B)}(x) = \bar{\mu}_B(f(x))$ for all $x \in X$.

**Lemma 3.14** (Biswas [1]). Let $f$ be a mapping from a set $X$ into a set $Y$. Let $m = [m^l, m^u]$ and $n = [n^l, n^u]$ be i-v fuzzy sets in $X$ and $Y$, respectively. Then

(i) $f^{-1}(n) = [f^{-1}(n^l), f^{-1}(n^u)]$,

(ii) $f(m) = [f(m^l), f(m^u)]$.

**Theorem 3.15.** Let $f$ be a homomorphism from a BCI-algebra $X$ into a BCI-algebra $Y$. If $B$ is an i-v fuzzy BCI-subalgebra of $Y$, then the inverse image $f^{-1}(B)$ of $B$ is an i-v fuzzy BCI-subalgebra of $X$.

**Proof.** Since $B = [\mu^l_B, \mu^u_B]$ is an i-v fuzzy BCI-subalgebra of $Y$, it follows from Theorem 3.7 that $\mu^l_B$ and $\mu^u_B$ are fuzzy BCI-subalgebras of $Y$. Using Proposition 3.1, we know that $f^{-1}[\mu^l_B]$ and $f^{-1}[\mu^u_B]$ are fuzzy BCI-subalgebras of $X$. Hence, by Lemma 3.14 and Theorem 3.7, we conclude that $f^{-1}(B) = [f^{-1}[\mu^l_B], f^{-1}[\mu^u_B]]$ is an i-v fuzzy BCI-subalgebra of $X$. □

**Definition 3.16** (Biswas [1]). Let $f$ be a mapping from a set $X$ into a set $Y$. Let $A$ be an i-v fuzzy set in $X$. Then the **image** of $A$, denoted by $f[A]$, is the i-v fuzzy set in $Y$ with the membership function defined by

$$
\bar{\mu}_{f[A]}(y) = \begin{cases} 
\sup_{z \in f^{-1}(y)} \bar{\mu}_A(z) & \text{if } f^{-1}(y) \neq \emptyset, \forall y \in Y, \\
[0,0] & \text{otherwise},
\end{cases}
$$

where $f^{-1}(y) = \{x \mid f(x) = y\}$.

**Theorem 3.17.** Let $f$ be a homomorphism from a BCI-algebra $X$ into a BCI-algebra $Y$. If $A$ is an i-v fuzzy BCI-subalgebra of $X$, then the image $f[A]$ of $A$ is an i-v fuzzy BCI-subalgebra of $Y$.

**Proof.** Assume that $A$ is an i-v fuzzy BCI-subalgebra of $X$. Note that $A = [\mu^l_A, \mu^u_A]$ is an i-v fuzzy BCI-subalgebra of $X$ if and only if $\mu^l_A$ and $\mu^u_A$ are fuzzy BCI-subalgebras of $X$. It follows from Proposition 3.2 that the images $f[\mu^l_A]$ and $f[\mu^u_A]$ are fuzzy BCI-subalgebras of $Y$. Combining Theorem 3.7 and Lemma 3.14, we conclude that $f[A] = [f[\mu^l_A], f[\mu^u_A]]$ is an i-v fuzzy BCI-subalgebra of $Y$. □

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