FINITE-RANK INTERMEDIATE HANKEL OPERATORS
ON THE BERGMAN SPACE

TAKAHIKO NAKAZI and TOMOKO OSAWA

(Received 11 January 1998)

2000 Mathematics Subject Classification. Primary 47B35.

1. Introduction. Let $D$ be the open unit disc in $\mathbb{C}$ and let $d\mu$ be the finite positive Borel measure on $D$. Let $L^2 = L^2(D,dr, d\theta/\pi)$ be the Lebesgue space on the open unit disc and let $L^2 = L^2 \cap \mathcal{H}(D)$ be the Bergman space. Let $P$ be the orthogonal projection of $L^2$ onto $L^2$ and let $Q$ be the orthogonal projection onto $L^2 \ominus \{g \in L^2; \hat{g} \in L^2, g(0) = 0\}$. Then $I - P \succeq Q$. The big Hankel operator and the small Hankel operator on $L^2$ are defined as:

$H^\text{big}_f(f) = (I - P)(\phi f)$ and $H^\text{small}_f(f) = Q(\phi f)(f \in L^2)$. In this paper, the finite-rank intermediate Hankel operators between $H^\text{big}_f$ and $H^\text{small}_f$ are studied. We are working on the more general space, that is, the weighted Bergman space.

For arbitrary symbol $\phi$ in $L^\infty$, $H^\text{big}_f(f) = (I - P)(\phi f)$ and $H^\text{small}_f(f) = Q(\phi f)(f \in L^2)$. However, it seems to have not been studied when they are finite-rank operators. When $\phi$ is in $L^\infty$, it is known (see [12, page 155]) that if $H^\text{big}_f$ is a finite-rank operator, then $H^\text{big}_f = 0$ and if $\phi$ is a polynomial, then $H^\text{small}_f$ is a finite-rank operator. In this paper, for arbitrary symbol $\phi$ in $L^\infty$ we show that if $H^\text{big}_f$ is a finite-rank operator, then $H^\text{big}_f = 0$, and we study when $H^\text{small}_f$ is a finite-rank operator. In fact, we study such problems for the intermediate Hankel operators $H^\text{int}_f$ on the weighted Bergman space $L^2(\mu)$. 

Let $L^2 = L^2(\mu) = L^2(D, d\mu)$ and $\mathcal{H}(D)$ be the set of all holomorphic functions on $D$. The weighted Bergman space $L^2 = L^2(\mu)$ is the intersection of $L^2$ and $\mathcal{H}(D)$. In general, $L^2$ is not closed. In [6, Theorem 8], when $(\text{supp } \mu) \cap D$ is a uniqueness set for $\mathcal{H}(D)$, the first author and M. Yamada gave a necessary and sufficient condition for that $L^2$ is closed. Throughout this paper, we assume that $L^2$ is closed. When $d\mu = r dr, d\theta/\pi$, $L^2$ is the usual Bergman space.

For $\mu$ such that $L^2(\mu)$ is closed, when $\mathcal{M}$ is the closed subspace of $L^2(\mu)$ and $z \mathcal{M} \subseteq \mathcal{M}$, $\mathcal{M}$ is called an invariant subspace. Suppose that $\mathcal{M} \supseteq zL^2$. $P^\mathcal{M}$ denotes the orthogonal projection from $L^2$ onto $\mathcal{M}$. For $\phi$ in $L^\infty = L^\infty(\mu) = L^\infty(D, d\mu)$, the intermediate Hankel operator $H^\mathcal{M}_f$ is defined by

$$H^\mathcal{M}_f(f) = (I - P^\mathcal{M})(\phi f) \quad (f \in L^2).$$

When $\mathcal{M} = L^2$, $H^\mathcal{M}_f$ is called a big Hankel operator $H^\text{big}_f$ and when $\mathcal{M} = (zL^2)^{-1}$, $H^\mathcal{M}_f$ is called a small Hankel operator $H^\text{small}_f$. Note that $H^\mathcal{M}_f$ is called a little Hankel operator when $\mathcal{M} = (L^2)^{-1}$.
In [2, 7, 9, 10], intermediate Hankel operators were studied in special weights, \( d\mu(\alpha+1)(1-r^2)\sigma r\,dr\,d\theta/\pi \) for \(-1 < \alpha < \infty\). In particular, Strouse [9] studied finite-rank intermediate Hankel operators.

Let \( d\mu = d\sigma(r)\,d\theta \) be a Borel measure on \( D \), where \( d\sigma(r) \) is a positive measure on \([0, 1]\) with \( d\sigma([0, 1]) = 1/2\pi \) and \( d\theta \) is the Lebesgue measure on \( \partial D \). \( L^2_\sigma(\mu) \) is closed if \( d\sigma([t, 1)) > 0 \) for any \( t > 0 \) (see [6]). For this type measures, it is possible to study more precisely the intermediate Hankel operators. In fact, \( L^2 \) has the following orthogonal decomposition:

\[
L^2 = \sum_{j=-\infty}^{\infty} \mathbb{F}^2 e^{ij\theta},
\]

where \( \mathbb{F}^2 = L^2(d\sigma) = L^2([0, 1), d\sigma) \). Set

\[
H^2 = \sum_{j=0}^{\infty} \mathbb{F}^2 e^{ij\theta},
\]

then \( L^2_\sigma \subset H^2 \subset (2L^2_\sigma)^\perp \) and \( L^2 = H^2 \oplus \mathbb{E} e^{-i\theta}H^2 \). If \( \mathcal{M} = H^2 \), it is easy comparatively to determine finite-rank Hankel operators \( H^u_\phi \) and we can do it completely in Section 5. We can expect that \( H^u_\phi \) is close to \( H^b_\phi \) in case \( \mathcal{M} \equiv H^2 \) (see Section 5) and \( H^u_\phi \) is close to \( H^\text{small}_\phi \) in case \( \mathcal{M} \equiv H^2 \) (see Section 6).

In Section 2, we describe an invariant subspace in \( L^2_\sigma \) whose codimension is of finite. Moreover we show that there does not exist an invariant subspace which contains \( L^2_\sigma \) properly and in which Hankel operators are studied in this paper. In Section 3, we describe finite-rank intermediate Hankel operators for arbitrary measure \( \mu \) such that \( L^2_\sigma(\mu) \) is closed. Moreover, we show that there does not exist any nonzero finite-rank Hankel operators \( H^b_\phi \) and there exists a nonzero finite-rank Hankel operator \( H^\text{small}_\phi \). In fact, we give two necessary and sufficient conditions for that if \( H^u_\phi \) is of finite rank \( \leq \ell \), then \( H^u_\phi = 0 \). In Sections 3, 4, and 5, we use the Fourier coefficients \( \{\mathcal{M}_j\}_{j=-\infty}^{\infty} \) of \( \mathcal{M} \) and so we assume \( d\mu = d\sigma(r)\,d\theta \). Using the Fourier coefficients of \( \phi \) and \( \mathcal{M} \), we give a necessary and sufficient condition for that \( H^u_\phi \) is of finite rank \( \leq \ell \). Assuming that \( \phi \) is a harmonic function, we can get a better necessary and sufficient condition. When \( \mathcal{M} \equiv H^2 \), using the Fourier coefficients \( \{\mathcal{M}_j\}_{j=-\infty}^{\infty} \), we give a necessary condition and a sufficient condition for that if \( H^u_\phi \) is of finite rank \( \leq \ell \), then \( H^u_\phi = 0 \). Two conditions are very similar but are a little different. Applications are given to examples in Section 2.

2. Invariant subspaces. In this section, we assume that \( d\mu = d\sigma(r)\,d\theta \) and \( d\sigma([t, 1)) > 0 \) for any \( t > 0 \), except Propositions 2.1 and 2.2. For our purpose, the invariant subspace \( \mathcal{M} \) must contain \( zL^2_\sigma \) but \( \ker H^u_\phi \) is an invariant subspace in \( L^2_\sigma \). If \( H^u_\phi \) is of finite rank, then the codimension of \( \ker H^u_\phi \) in \( L^2_\sigma \) is finite. In order to study finite-rank intermediate Hankel operators, we need the generalization of a result of Axler and Bourdon [1] which determines finite codimensional invariant subspaces in \( L^2_\sigma \) when \( d\mu = r\,dr\,d\theta/\pi \). In Propositions 2.1 and 2.2, the measure \( \mu \) is an arbitrary finite positive Borel measure such that \( L^2_\mu \) is closed and \( \langle \text{supp} \mu \cap D \rangle \) is a uniqueness set for \( \mathcal{R} \text{ol}(D) \). Since \( H^2 \subset L^\infty \) is an extended weak-* Dirichlet algebra in \( L^\infty \),
Proposition 2.3 is a corollary of [4, Theorem 1]. We will give several examples of invariant subspaces which contain $zL^2_a$.

**PROPOSITION 2.1.** Suppose $\mathcal{M}$ is an invariant subspace in $L^2_a$ and $\ell$ is a positive integer. The codimension of $\mathcal{M}$ in $L^2_a$ is $\ell$, if and only if $\mathcal{M} = qL^2_a$, where $q = \prod_{j=1}^{\ell} (z - a_j)$ and $a_j \in D$ ($1 \leq j \leq \ell$).

**PROOF.** The proof is almost parallel to that in [1, Theorem 1]. We will give a sketch of it. Suppose $\mathcal{M} \supseteq L^2_a$ and $\dim \mathcal{M} = \ell$. Put

$$S_z f = P(z f) \quad (f \in \mathcal{M}) ,$$

where $P$ is an orthogonal projection. Since $\ell < \infty$, there exists an analytic polynomial $b$ such that $b(S_z) = S_b(z) = 0$ and the degree of $b$ is less than or equal to $\ell$. Hence $b.M \subseteq \mathcal{M}$ and so $bL^2_a \subseteq \mathcal{M}$. We show that the zeros of $b$ are only in $D$ and the degree of $b = \ell$. Then $\mathcal{M} = bL^2_a$. It is clear that the degree of $b = \ell$. In this direction, we did not need the condition such that $(\text{supp} \mu) \cap D$ is a uniqueness set.

If $a \notin D$, $(z - a)L^2_a$ is dense in $L^2_a$. Assuming $a \geq 1$ and so $a = 1$ without a loss of generality, if $\varepsilon > 0$, then $(z - 1)L^2_a = (z - 1)(z - (1 + \varepsilon))^{-1}L^2_a$. For any $f \in L^2_a$, it is easy to see that

$$\int_D \left| \frac{z - 1}{z - (1 + \varepsilon)} f - f \right|^2 d\mu \to 0 \quad (\varepsilon \to 0).$$

This implies that $(z - 1)L^2_a$ is dense in $L^2_a$. Thus all zeros of $b$ must be in $D$. The “if” part is clear because any point $a \in D$ gives a bounded evaluation functional. Here we used the condition such that $(\text{supp} \mu) \cap D$ is a uniqueness set (see [6, (1) of Theorem 8]).

\[ \square \]

**PROPOSITION 2.2.** Suppose that $(z - a)^{-1}$ does not belong to $L^2$ for each $a \in D$. If $\mathcal{M}$ is an invariant subspace which contains $L^2_a$ properly, then the codimension of $L^2_a$ in $\mathcal{M}$ is infinite.

**PROOF.** If $\dim \mathcal{M} \cap L^2_a = \ell < \infty$, by the proof of Proposition 2.1, there exists a polynomial $b = \prod_{j=1}^{\ell} (z - a_j)$ such that $b.\mathcal{M} \subseteq L^2_a$ and $a_j \in D$ ($1 \leq j \leq \ell$). Hence there exists a function $\phi$ in $\mathcal{M}$ such that $\phi \notin L^2_a$ and $g = b\phi \in L^2_a$. If $g(a_k) \neq 0$ for some $k$, then $g / (z - a_k) = \phi \prod_{j=1}^{k} (z - a_j)$ cannot belong to $L^2$ because $(z - a_k)^{-1} \notin L^2$. Hence $g(a_j) = 0$ for any $j$. By [6, the proof in (1) of Theorem 8], $g \in bL^2_a$ and so $\phi = g/b$ belongs to $L^2_a$. This contradiction implies that $\dim \mathcal{M} \cap L^2_a = \infty$.

For an invariant subspace $\mathcal{M}$, set

$$\mathcal{M}_j = \left\{ f_j \in L^2_a ; f \in \mathcal{M} , \ f(z) = \sum_{j=-\infty}^{\infty} f_j(r) e^{ij\theta} \right\} .$$

Then $\mathcal{M}_j$ is a subspace in $L^2$, $r.\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ and hence $\dim \mathcal{M}_{j+1} \geq \dim \mathcal{M}_j$. We call $\{\mathcal{M}_j\}_{j=-\infty}^{\infty}$, the Fourier coefficients of $\mathcal{M}$. $\mathcal{M}_j e^{ij\theta}$ may not belong to $\mathcal{M}$. If $\mathcal{M}_j e^{ij\theta}$ belongs to $\mathcal{M}$ for any $j$, then $\mathcal{M}$ has the following decomposition:

$$\mathcal{M} = \sum_{j=-\infty}^{\infty} \phi.\mathcal{M}_j e^{ij\theta} .$$
This decomposition is called the Fourier decomposition of $\mathcal{M}$. In general, $\mathcal{M}$ does not have the Fourier decomposition but we can get an extension $\bar{\mathcal{M}}$ of $\mathcal{M}$ which has the following Fourier decomposition:

$$\bar{\mathcal{M}} = \sum_{j=-\infty}^{\infty} \Phi(\text{closure of } \mathcal{M}_j)e^{ij\theta}. \quad (2.5)$$

**Proposition 2.3.** If $\mathcal{M}$ is an invariant subspace which contains $L^2_{\Delta}$ and $e^{i\theta} \mathcal{M} \subseteq \mathcal{M}$, then $\mathcal{M} = \chi_E \hat{q} H^2 + \chi_F L^2$, where $\chi_E$ is a characteristic function in $L^2$ and $q$ is a unimodular function in $H^2$. Hence $\mathcal{M} \supseteq H^2$. If $\bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M} = \{0\}$, then $\mathcal{M} = q H^2$.

**Proof.** Suppose $S_0 = \mathcal{M} \cap e^{i\theta} \mathcal{M}$, then $\mathcal{M} = (\bigcap_{j=0}^{\infty} \mathcal{M}_k) \supseteq \mathcal{M}_{-\infty}$, where $\mathcal{M}_{-\infty} = \bigcap_{j=0}^{\infty} e^{ij\theta} \mathcal{M}$, and $rS_0 \subseteq S_0$ because $r \mathcal{M}_j \subseteq \mathcal{M}_{j+1}$. It is well known that $\mathcal{M}_{-\infty} = \chi_E L^2$ for a characteristic function $\chi_E$ of some measurable subset in $D$. Put $E = G^c$ then there exists a function $f$ in $S_0$ such that

$$|f| > 0 \quad \text{on } E \quad \text{and} \quad f = 0 \quad \text{on } F. \quad (2.6)$$

Since $f$ is orthogonal to $f e^{ij\theta}$ for all $j \geq 0$, $|f|^2$ belongs to $\mathcal{L}^1 = L^1(\sigma) = L^1([0,1],d\sigma)$ and so $|f|$ belongs to $\mathcal{L}^2$. Hence $\chi_E$ belongs to $\mathcal{L}^2$. Set

$$F(re^{i\theta}) = \begin{cases} \frac{f(re^{i\theta})}{|f(re^{i\theta})|} & \text{if } f \neq 0, \\ 1 & \text{if } f = 0, \end{cases} \quad (2.7)$$

then $F$ is a unimodular function in $L^2$. Since $rS_0 \subseteq S_0$, we can show that $\chi_E F$ belongs to $S_0$ and so $S_0 = \chi_E F \mathcal{L}^2$. Hence $\mathcal{M} \cap \mathcal{M}_{-\infty} = \chi_E \mathcal{H}^2$. Since $1 \in \mathcal{M}$, $\chi_E \hat{F} \in \mathcal{H}^2$ and $q = \hat{F} \in \mathcal{H}^2$. \hfill $\Box$

**Example 2.4.** (i) For $0 < \beta < 1$, put

$$T_\beta = \overline{\text{spans}} \{ z^n \bar{z}^m; \ \beta n \geq m \geq 0 \}. \quad (2.8)$$

Then $T_\beta$ is an invariant subspace and $T_\beta \supseteq L^2_{\Delta}$. Put $T_\beta = L^2_{\Delta}$ for $\beta = 0$ and $T_\beta = \mathcal{H}^2$ for $\beta = 1$. In general, $L^2_{\Delta} \subset T_\beta \subset \mathcal{H}^2$ and $T_\beta (0 \leq \beta < 1)$ has the following Fourier decomposition:

$$T_\beta = \sum_{j=0}^{\infty} \oplus(T_\beta)_j e^{ij\theta}. \quad (2.9)$$

where $(T_\beta)_j = \overline{\text{spans}} \{ r^j p_j(r^2); \ p_j \text{ is a polynomial of degree at most } \beta j/(1-\beta) \}$. Janson and Rochberg [2] studied $H^{\mathbb{R}}_\phi$ when $\mathcal{M} = (T_\beta)^{1/2}$. Then $(T_\beta)^{1/2} = e^{i\theta} \mathcal{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{ \mathcal{L}^2 \ominus (T_\beta)_j \} e^{-ij\theta}.$

(ii) For $k \geq 0$, put $E^k = \overline{\text{spans}} \{ z^m \bar{z}^n; \ m = 0, 1, \ldots, k; \ n = m, m+1, \ldots \}$. $E^k$ is an invariant subspace and $L^2_{\Delta} \subset E^k \subset \mathcal{H}^2$. $E^k$ has the following Fourier decomposition:

$$E^k = \sum_{j=0}^{\infty} \oplus(E^k)_j e^{ij\theta}, \quad (2.10)$$

where $(E^k)_j = \overline{\text{spans}} \{ r^j, \ldots, r^{j+2k} \}$. Strouse [9] studied $H^{\mathbb{R}}_\phi$ when $\mathcal{M} = (E^k)^{1/2}$. Then $(E^k)^{1/2} = e^{i\theta} \mathcal{H}^2 \oplus \sum_{j=0}^{\infty} \oplus \{ \mathcal{L}^2 \ominus (E^k)_j \} e^{-ij\theta}.$
(iii) Fix a polynomial \( p \) of degree \( k \), that is, \( p = \sum_{j=0}^{k} a_j z^j \). Put
\[
Y(p) = \text{span} \{ z^n, z^m \bar{p}; \ n \geq 0, \ m \geq 0 \},
\]
\[
Y^k = \text{span} \{ z^j \bar{z}^\ell; \ \ell \geq 0, \ 0 \leq j \leq k \}.
\] (2.11)
Both \( Y(p) \) and \( Y^k \) are invariant subspaces and \( L^2_a \subseteq Y(p) \subseteq Y^k \), and \( Y^k \) has the following Fourier decomposition:
\[
Y^k = \sum_{j=-k}^{\infty} \Phi(Y^k)_j e^{ij\theta},
\] (2.12)
where \( Y^k_0 = \text{span} \{ 1, r^2, \ldots, r^{2k} \} \) and \( (Y^k)_j = r^j(Y^k_0) \) for \( j \geq 0 \), and \( (Y^k)_{-j} = \text{span} \{ r^{2\ell-j}; j \leq \ell \leq k \} \) for \( 1 \leq j \leq k \). (\( Y(p) \))_j \subseteq (Y^k)_j \) for any \( j \) but \( Y(p) \) does not have a Fourier decomposition. If \( a_j \neq 0 \) for \( 1 \leq j \leq k \), \( (Y(p))_j = (Y^k)_j \) for any \( j \) and so \( \tilde{Y}(p) = Y^k \). Peng, Rochberg, and Wu [7] and Wang and Wu [10] studied \( H^\phi_a / H^5 \) when \( H^\phi_a / H^5 = (\bar{Y}^k)^\perp \). In general, we can define \( Y(g) \) for any function \( g \) in \( L^2 \). Usually, \( Y(g) \) does not have the Fourier decomposition.

(iv) For a unimodular function \( q \) in \( H^2 \), put \( \mathcal{M} = \bar{q}H^2 \). Then \( \mathcal{M} \) is an invariant subspace which contains \( H^2 \). In general, \( \bar{q}H^2 \) may not have the Fourier decomposition but for \( q = e^{i\ell \theta} \), for some \( \ell \geq 0 \),
\[
\mathcal{M} = \sum_{j=-\ell}^{\infty} \oplus L^2 e^{ij\theta}.
\] (2.13)
There are a lot of invariant subspaces between \( H^2 \) and \( e^{-i\ell \theta}H^2 \) even if \( \ell = 1 \).

(v) For arbitrary closed subspaces \( S \) in \( L^2 \), put \( \mathcal{M} = H^2 \oplus Se^{-i\theta} \). Then \( \mathcal{M} \) is an invariant subspace between \( H^2 \) and \( e^{-i\theta}H^2 \).

3. **Kronecker's theorem.** In this section, the measure \( \mu \) is an arbitrary finite positive Borel measure such that \( L^2_a \) is closed. We will write
\[
\mathcal{M}^\infty = \mathcal{M} \cap L^\infty
\] (3.1)
and, for each positive integer \( \ell \),
\[
\mathcal{M}^{\infty,\ell} = \left\{ \phi \in L^\infty; \ \phi(z) = g(z) \prod_{j=1}^{\ell} (z-a_j)^{-1} \ a.e. \ \mu \text{ on } D, g \in \mathcal{M}^\infty \text{ and } a_1, \ldots, a_\ell \in D \right\}.
\] (3.2)
Then \( \mathcal{M}^{\infty} \subseteq \mathcal{M}^{\infty,1} \subseteq \mathcal{M}^{\infty,2} \subseteq \cdots \).

Kronecker (cf. [11, page 210]) described finite-rank Hankel operators on the Hardy space. Theorem 3.1 describes finite-rank intermediate Hankel operators on the (weighted) Bergman space. However the situation is very different from that of Kronecker because \( \mathcal{M}^{\infty} = \mathcal{M}^{\infty,\ell} \) may happen for some \( \ell > 0 \). See Corollaries 3.3 and 3.4.

**Theorem 3.1.** Suppose \( \mathcal{M} \) is an invariant subspace which contains \( zL^2_a \), and \( \phi \) is a function in \( L^\infty \). \( H^\phi_a \) is of finite rank \( \leq \ell \) if and only if \( \phi \) belongs to \( \mathcal{M}^{\infty,\ell} \).

**Proof.** Note that \( \ker H^\phi_a = \{ f \in L^2_a; \ \phi f \in \mathcal{M} \} \). Since \( \mathcal{M} \) is an invariant subspace, \( \ker H^\phi_a \) is also an invariant subspace. Proposition 2.1 implies the theorem. \( \square \)
Theorem 3.2. Suppose $\mathcal{M}$ is an invariant subspace which contains $L^2_\alpha$, and $\phi$ is a function in $L^\infty$. Then the following are equivalent:

1. If $H^\mu_\phi$ is of finite rank, then $H^\mu_\phi = 0$.
2. $\mathcal{M}^\infty = \mathcal{M}^\infty,\ell$ for any $\ell > 0$.
3. If $g \in \mathcal{M}^\infty$, $a \in D$ and $(g(z) - g(a))/(z - a) \in L^\infty$, then $(g(z) - g(a))/(z - a)$ belongs to $\mathcal{M}^\infty$.
4. If $\mathcal{M}'$ is an invariant subspace and $(\mathcal{M}')^\infty \not\supset \mathcal{M}^\infty$, then there does not exist a nonzero polynomial $b$ such that $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$.

Proof. By Theorem 3.1, (1) $\Leftrightarrow$ (2) is clear.

(1) $\Rightarrow$ (3). If there exists $g \in \mathcal{M}^\infty$ such that $(g - g(a))/(z - a) \in L^\infty$ does not belong to $\mathcal{M}^\infty$, put $\phi = (g - g(a))/(z - a)$, then $H^\mu_\phi$ is of rank 1 and $H^\mu_\phi \neq 0$.

(3) $\Rightarrow$ (4). If (4) is not true, there exists $\psi$ such that $\psi \notin \mathcal{M}^\infty$, $\psi \in (\mathcal{M}')^\infty$ and $b\psi \in \mathcal{M}^\infty$ for some polynomial: $b = \prod_{j=1}^{\ell-1} (z - a_j)$ and $a_j \in D (1 \leq j \leq \ell < \infty)$. We may assume that $\phi = \psi \prod_{j=1}^{\ell-1} (z - a_j) \notin \mathcal{M}^\infty$ and $g = (z - a_\ell)\phi \in \mathcal{M}^\infty$. Then

$$
\frac{g - g(a_\ell)}{z - a_\ell} = \phi \in L^\infty, \quad \phi \notin \mathcal{M}^\infty.
$$

(4) $\Rightarrow$ (1). By Theorem 3.1, if $H^\mu_\phi$ is of finite rank $\leq \ell$, then $\phi \in \mathcal{M}^\infty,\ell$. If $\phi \notin \mathcal{M}^\infty$, suppose $\mathcal{M}'$ is an invariant subspace generated by $\phi$ and $\mathcal{M}$, then $(\mathcal{M}')^\infty \not\supset \mathcal{M}^\infty$ but there does not exist a nonzero polynomial $b$ such that $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$. Since $\phi \in \mathcal{M}'$, this contradicts that $\phi \in \mathcal{M}^\infty,\ell$.

Corollary 3.3. Suppose $(\text{supp } \mu) \cap D$ is a uniqueness set for $\Re \sigma(D)$. If $H^\text{big}_\phi$ is of finite rank, then $H^\text{big}_\phi = 0$.

Proof. Theorem 3.2(3) implies the corollary. In fact, if $g \in L^2_\alpha \cap L^\infty$, then $g(z) - g(a) \in (z - a)L^2_\alpha$ by [6, the proof in (1) of Theorem 5.4]. Thus $(g(z) - g(a))/(z - a)$ belongs to $L^2_\alpha \cap L^\infty$.

Corollary 3.4. Suppose $d\mu = r \, dr \, d\theta/\pi$. Let $D_0$ be an open subset of $D$ and $\mathcal{M} = \{f \in L^2; f$ is analytic on $D_0\}$. Then $\mathcal{M}$ is an invariant subspace and if $H^\mu_\phi$ is of finite rank then $H^\mu_\phi = 0$.

Proof. It is easy to see that $\mathcal{M}^\infty$ satisfies Theorem 3.2(3).

Corollary 3.5. Suppose that if $H^\mu_\phi$ is of finite rank then $H^\mu_\phi = 0$. If $\mathcal{M}'$ is an invariant subspace which contains $\mathcal{M}$ properly, then the codimension of $\mathcal{M}$ in $\mathcal{M}'$ is infinite or $(\mathcal{M}')^\infty = \mathcal{M}^\infty$.

Proof. If $\dim \mathcal{M}'/\mathcal{M} < \infty$, as in the proof of Proposition 2.2, then there exists a nonzero polynomial $b$ such that $b\mathcal{M}' \subseteq \mathcal{M}$. Hence $b(\mathcal{M}')^\infty \subseteq \mathcal{M}^\infty$. If $(\mathcal{M}')^\infty \neq \mathcal{M}^\infty$, by Theorem 3.2, this contradicts that if $H^\mu_\phi$ is of finite rank, then $H^\mu_\phi = 0$.

4. General case. In this section, we assume that $d\mu = d\sigma(r) \, d\theta$ and $d\sigma([t, 1)) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{\mathcal{M}_j\}^\infty_{j=-\infty}$ of $\mathcal{M}$. We assume $\mathcal{M} = \hat{\mathcal{M}}$, that is, $\mathcal{M}$ has the Fourier decomposition.
\textbf{Theorem 4.1.} Suppose $M$ is an invariant subspace which contains $zL^2_d$ and $\phi = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta}$ is a function in $L^\infty$. Then $H^u_\phi$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and, for any integer $n$,

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n. \quad (4.1)$$

If $\ell$ is the minimum number of complex numbers $b_1, \ldots, b_\ell$ such that $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for all $n$, then $H^u_\phi$ is of rank $\ell$.

\textbf{Proof.} If $H^u_\phi$ is of rank $\leq \ell$, by Theorem 3.1 there exists a polynomial $b = \sum_{j=0}^{\ell} b_j z^j$ such that $b \phi \in M$. Then

$$\left( \sum_{j=0}^{\ell} b_j r^j e^{ij\theta} \right) \left( \sum_{j=0}^{\ell} b_j r^j e^{ij\theta} \right) = \sum_{n=\infty}^{-\infty} \left( \sum_{j=0}^{\ell} b_{n-j} r^j \right) e^{in\theta}$$

and so $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for any $n$. The converse and the second statement are clear by Theorem 3.2. \hfill $\square$

\textbf{Corollary 4.2.} Let $\phi = \phi_t(r)e^{it\theta}$ for some integer $t$ in Theorem 4.1. Then $H^u_\phi$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and for $t \leq n \leq \ell + t$, $b_n r^{n-t} \phi_t(r) \in M_n$.

\textbf{Proof.} Since $\phi_j(r) = 0$ for $j \neq t$, if $n < t$ or $n > \ell + t$, then $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0$. For $t \leq n \leq \ell + t$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = b_n r^{n-t} \phi_t(r)$, thus the corollary follows. \hfill $\square$

\textbf{Corollary 4.3.} Let $\phi = \sum_{j=1}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j} z^{-j}$ in Theorem 4.1. Then $H^u_\phi$ is of rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$ and for any nonpositive integer $n \sum_{j=0}^{\ell} b_j a_{n-j} r^{2j-n} \in M_n$ and, for $0 < n < \ell$, $\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} \in M_n$.

\textbf{Proof.} If $n \geq \ell$ and $n \neq 0$, then

$$\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = \sum_{j=0}^{\ell} b_j a_{n-j} r^{j+n-j} = \left( \sum_{j=0}^{\ell} b_j a_{n-j} \right) r^n \quad (4.3)$$

and hence $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ because $zL^2_d \subseteq M$. Now Theorem 4.1 implies the corollary.

Theorem 4.1 does not give an exact relation between the rank of $H^u_\phi$ and the number $\ell$ of complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$. However, we can show the following: if $H^u_\phi$ is of rank $\ell$, then there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_\ell = 1$, $\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) \in M_n$ for any $n$ and $b = \sum_{j=0}^{\ell} b_j z^j$ has just $\ell$ zeros in $D$. That is, if $\ell = 1$, then $|b_0| < 1$.

By Theorem 4.1, $H^u_\phi = 0$ if and only if $\phi_n \in M_n$ for any $n$ (i.e., $\phi \in M$). Moreover, $H^u_\phi$ is of rank $\leq 1$ if and only if there exist complex numbers $(b_0, b_1) \neq (0,0)$ such that $b_1 = 1$ and $b_0 \phi_n + b_1 r \phi_{n-1} \in M_n$ for any $n$. \hfill $\square$
5. Big Hankel operator and $\mathcal{M} \subset H^2$. In this section, we assume that $d \mu = d\sigma(r) \, d\theta$ and $d\sigma(t \ldots 1) > 0$ for any $t > 0$. Hence we can define the Fourier coefficients $\{ M_j \}_{j=k}^{\infty}$ of $\mathcal{M}$ and we assume $\mathcal{M} = \mathcal{M}_0$. In this case, $H^2_{\phi^{-}}$ is close to $H^2_{\phi^+}$. Recall examples in Section 2, that is, $T_{ij}, E^k, Y(p)$, and $Y^k$.

**Corollary 5.1.** Suppose $\mathcal{M}$ is a big Hankel operator subspace between $zL^2_\alpha$ and $H^2$, and $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j} \bar{z}^j$. Then $H^2_{\phi^{-}}$ is of finite rank $\leq \ell$ if and only if $a_{-n} = 0$ for $n > \ell$ and there exists complex numbers $b_0, \ldots, b_\ell$ such that $b_{\ell} = 1$ and $\sum_{j=0}^{\infty} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$ for $0 \leq n \leq \ell$ and $\sum_{j=0}^{\infty} b_j a_{n-j} r^{2j-n} = 0$ for $-\ell < n < 0$.

**Proof.** Since $\mathcal{M} \subset H^2$, by Corollary 4.3 $H^2_{\phi^+}$ is of finite rank $\leq \ell$ if and only if there exist complex numbers $b_0, \ldots, b_\ell$ such that $b_{\ell} = 1$ and $\sum_{j=0}^{\infty} b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$ and $\sum_{j=0}^{\infty} b_j a_{n-j} r^{2j-n} \in \mathcal{M}_n$ for $0 \leq n \leq \ell$. If $\sum_{j=0}^{\infty} b_j a_{n-j} r^{2j-n} = 0$ for $n < 0$, then $b_j a_{n-j} = 0$ for $0 \leq j \leq \ell$ and $n < 0$. Hence for each $j$ $(0 \leq j \leq \ell)$, $b_j a_{-t} = 0$ if $t > j$. Thus $a_{-t} = 0$ if $t > \ell$.

**Proposition 5.2.** Suppose $\mathcal{M}$ is an invariant subspace between $zL^2_\alpha$ and $e^{-ik\theta}H^2$ where $k \geq 0$, and $\phi = \sum_{j=0}^{\infty} \phi_j(r) e^{-ij\theta}$ is a function in $L^\infty$. Then $H^2_{\phi^{-}}$ is of finite rank $\leq \ell$ if and only if

$$\phi(z) = \frac{\sum_{j=-k}^{\ell} \psi_j(r) e^{ij\theta}}{\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}},$$

(5.1)

where $\psi_n = \sum_{j=0}^{\ell} b_j r^{j} \phi_{n-j} \in \mathcal{M}_n$, for $-k \leq n \leq \ell$, and $(b_0, \ldots, b_\ell) \in \mathbb{C}^\ell$.

**Proof.** Note that $\mathcal{M} \subset e^{-ik\theta}H^2$ and $\phi_j(r) = 0$ for $j > 0$. If $H^2_{\phi^{-}}$ is of finite rank $\leq \ell$, then, by Theorem 4.1,

$$\left( \sum_{j=0}^{\ell} b_j r^j e^{ij\theta} \right) \left( \sum_{j=0}^{\infty} \phi_{-j}(r) e^{-ij\theta} \right) = \sum_{n=-k}^{\ell} \psi_n(r) e^{in\theta},$$

(5.2)

and $\psi_n = \sum_{j=0}^{\ell} b_j r^j \phi_{n-j} \in \mathcal{M}_n$ for $-k \leq n \leq \ell$. The converse is also a result of Theorem 3.1.

**Corollary 5.3.** Suppose $\mathcal{M}$ is an invariant subspace in Proposition 5.2. If $\phi = \phi_+ + \phi_- = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=0}^{\infty} a_{-j} \bar{z}^j$ and $\phi_+ \in L^\infty$, then $H^2_{\phi^{-}}$ is of finite rank $\leq \ell$ if and only if

$$\phi(z) = \phi_+ + \frac{\sum_{j=-k}^{\ell} \psi_j(r) e^{ij\theta}}{\sum_{j=0}^{\ell} b_j r^j e^{ij\theta}},$$

(5.3)

where $\psi_n = \sum_{j=0}^{\ell} b_j a_{n-j} r^{j+|n-j|} \in \mathcal{M}_n$, for $-k \leq n \leq \ell$, and $(b_0, \ldots, b_\ell) \in \mathbb{C}^\ell$. If $(b_0, \ldots, b_\ell) = (0, \ldots, 0)$, then $\psi_n = 0$ and so $\phi = \phi_+$.

**Theorem 5.4.** Suppose $\mathcal{M}$ is an invariant subspace between $zL^2_\alpha$ and $e^{-ik\theta}H^2$ where $k \geq 0$, and $\phi = \sum_{j=0}^{\infty} \phi_j(r) e^{ij\theta}$ is a function in $L^\infty$.

1. If $\mathcal{M}_j \cap r^{j+1}L^2_\alpha = \{0\}$ for any $j \geq 0$, then there does not exist any finite rank $H^2_{\phi^{-}}$ except $H^2_{\phi} = 0$.

2. If there does not exist any finite rank $H^2_{\phi^{-}}$ except $H^2_{\phi} = 0$, then $\mathcal{M}_{(j-k)} \cap r^{j+1}L^\infty = \{0\}$ for any $j \geq 0$. 
Proof. (1) If $H_\phi^\ell$ is of finite rank $\ell$, by Proposition 5.2, 
$$
\psi_n = \sum_{j=n}^{\ell} b_j r^j \phi_{n-j} \in \mathcal{M}_n, 
$$
for $0 \leq n \leq \ell$ because $\phi_{n-j}(r) = 0$ for $0 \leq j \leq n - 1$. We may assume $b_\ell = 1$. As $n = \ell - 1$, $r^\ell \phi_{-1}(r) \in \mathcal{M}_{\ell-1}$. Since $\mathcal{M}_{\ell-1} \cap r^\ell \mathcal{G}^2 = \{0\}$, $\phi_{-1}(r) = 0$. As $n = \ell - 2,$ 
$$
b_{\ell-1} r^{\ell-1} \phi_{-1}(r) + r^\ell \phi_{-2}(r) \in \mathcal{M}_{\ell-2}. 
$$
Since $\mathcal{M}_{\ell-2} \cap r^\ell \mathcal{G}^2 = \{0\}$ and $\phi_{-1}(r) = 0$, $\phi_{-2}(r) = 0$. we can get $\phi_{-j}(r) = 0$ for $j \leq \ell$. In Proposition 5.2, $\psi_n = 0$ for $0 \leq n \leq \ell$ and so $\phi \equiv 0$.

(2) If $r^{j+1} \mathcal{G} \in \mathcal{M}_{-(k-j)} \cap r^{j+1} \mathcal{G}^\infty$, then put $\phi = ge^{-i(k+1)\theta}$. If $g \neq 0$ then $\phi \not\in \mathcal{M}$ and 
$$
z^{j+1} \phi = r^{j+1} ge^{-i(k-j)\theta} \in \mathcal{M}_{-(k-j)} e^{-i(k-j)\theta}. 
$$
Since $\mathcal{M}$ has the Fourier decomposition, $\mathcal{M} e^{i\theta} \subseteq \mathcal{M}$ and so $z^{j+1} \phi \in \mathcal{M}$. Theorem 3.1 gives a contradiction. \hfill \Box

We will apply results in this section to Example 2.4 in Section 2.

Example 5.5. (i) Suppose $\mathcal{M} = T_\beta$ $(0 \leq \beta < 1)$.

(1) When $\phi = \sum_{j=0}^{\infty} \phi_{-j}(r) e^{-ij\theta}$ is a function in $L^\infty$, there does not exist any finite rank $H_\phi^\infty$ except $H_\phi^\infty = 0$ if and only if $\beta = 0$.

(2) When $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$ is a function in $L^\infty$, there does not exist any finite rank $H_\phi^\infty$ except $H_\phi^\infty = 0$ if and only if $\beta = 0$.

Proof. Recall that $T_\beta = \sum_{j=0}^{\infty} \oplus (T_\beta)_j e^{ij\theta}$ and $(T_\beta)_j = \text{span}\{r^j p_j(r^2); p_j \text{ is a polynomial of degree at most } \beta j/(1-\beta)\}$.

(1) If $\beta = 0$, then $(T_\beta)_j \cap r^{j+1} \mathcal{G}^2 = \{0\}$ for any $j \geq 0$ and if $\beta \neq 0$, then $(T_\beta)_j \cap r^{j+1} \mathcal{G}^\infty \neq \{0\}$ for large $j$. Theorem 5.4 implies (1).

(2) If $\beta \neq 0$, there exists $n$ such that $1-\beta \leq \beta(n-1)$. Hence $(T_\beta)^{n-1} \supset r^{n+1}$. Suppose $\phi = 2$, then $z^n \phi = r^{n+1} e^{i(n-1)\theta}$ and so $z^n \phi \in (T_\beta)^{n-1} e^{i(n-1)\theta} \subseteq T_\beta$. By Theorem 3.1, $H_\phi^\infty$ is of rank $\leq n$ and $H_\phi^\infty \neq 0$. \hfill \Box

(ii) Suppose $\mathcal{M} = E_m$ $(0 < m < \infty)$.

(1) When $\phi = \sum_{j=0}^{\infty} \phi_{-j}(r) e^{-ij\theta}$, there does not exist any finite rank $H_\phi^\infty$ except $H_\phi^\infty = 0$ if and only if $m = 0$.

(2) When $\phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j$ is a function in $L^\infty$, there does not exist any finite rank $H_\phi^\infty$ except $H_\phi^\infty = 0$ if and only if $m = 0$ or $1$.

Proof. We recall that $(E)^m = \sum_{j=0}^{\infty} \oplus (E)^m_j e^{ij\theta}$ and $(E)^m_j = \text{span}\{r^j, \ldots, r^{j+2m}\}$.

(1) If $m = 0$, then $(E)^m_j \cap r^{j+1} \mathcal{G}^2 = \{0\}$ for any $j \geq 0$ and if $m \neq 0$, then $(E)^m_j \cap r^{j+1} \mathcal{G}^\infty \neq \{0\}$ for any $j \geq 0$. Theorem 5.4 implies (1).

(2) If $m = 0$, by (1) there does not exist any finite rank $H_\phi^\infty$ except $H_\phi^\infty = 0$. If $m = 1$, then $(E)^m_j = \text{span}\{r^n, r^{n+2}\}$ for $n \geq 0$. When $H_\phi^M$ is of finite rank $\ell$, by Corollary 5.1, $a_{-n} = 0$ for $n > \ell$ and if $0 \leq n \leq \ell$,

$$
\sum_{j=n}^{\ell} b_j a_{n-j} r^{2j-n} = cr^n + dr^{n+2} 
$$

(5.7)
for complex constants \( c, d \). Hence, for \( 0 \leq n \leq \ell \),

\[
 b_j a_{n-j} = 0 \quad \text{for } n+2 \leq j \leq \ell.
\]  
(5.8)

Since \( b_\ell = 1 \), \( a_{n-\ell} = 0 \) for \( 0 \leq n \leq \ell \) and so \( a_{-j} = 0 \) for \( 0 \leq j \leq \ell \). When \( m \geq 2 \), if \( \phi = z \), then \( z \phi = r^2 \in (E^m)_0 = \text{span}\{1, r^2, \ldots, r^{2m}\} \) and \( z \phi \in E^m \) because \( (E^m)_0 \subset E^m \).

However \( H^u_\phi \neq 0 \).

(iii) Suppose \( \mathcal{M} = Y_k \).

1. When \( \phi = \sum_{j=1}^\infty \phi_j(r) e^{ij \theta} \), there does not exist any finite rank \( H^u_\phi \) except \( H^u_0 = 0 \) if and only if \( k = 0 \).

2. When \( \phi = \phi_+ + \phi_- = \sum_{j=0}^\infty a_j z^j + \sum_{j=1}^\infty a_j \bar{z}^j \) and \( \phi_+ \) are functions in \( L^\infty \), there does not exist any finite rank \( H^u_\phi \) except \( H^u_0 = 0 \) if and only if \( k = 0 \).

**Proof.** Since \( H^u_\phi = H^u_{\phi_+} \), it is sufficient to prove (1). We recall that \( Y_k = \sum_{j=-k}^\infty \phi_j(r) e^{ij \theta} \), where \( \phi_j(r) = \text{span}\{1, r^2, \ldots, r^{2k}\} \) and \( \phi_j(r) = r^j(Y_k)_0 \) for \( j \geq 0 \), and \( \phi_j(r) = \text{span}\{r^{2\ell-j}, j \leq \ell \leq k\} \) for \( 1 \leq j \leq k \). If \( k = 0 \), then \( Y_k = L_2^\infty, k \geq 1 \), \( (Y_k)_- = \text{span}\{r^k\} \). Theorem 5.4(2) implies that there exists a nonzero finite rank \( H^u_\phi \).

6. Small Hankel operator and \( \mathcal{M} \supseteq H^2 \). In this section, we assume that \( d\mu = d\sigma(r) \, d\theta \) and \( d\sigma([t, 1]) > 0 \) for any \( t > 0 \). Hence we can define the Fourier coefficients \( \{M_j\}_{j=-\infty}^\infty \) of \( \mathcal{M} \). In this case, \( H^u_\phi \) is close to \( H^\text{small} \) and far from \( H^\text{big} \). Note that if \( \mathcal{M} \) is an invariant subspace and \( \mathcal{M}' \subseteq e^{it \theta}H^2 \), then \( \mathcal{M} = (\mathcal{M}')^\perp \) is an invariant subspace and \( \mathcal{M} \supseteq e^{it \theta}H^2 \).

**Proposition 6.1.** Suppose \( \mathcal{M} \) is an invariant subspace which contains \( e^{ik \theta}H^2 \) for some nonnegative integer \( k \). If \( \mathcal{M} \neq L^2 \), there exists at least a nonzero finite rank \( H^u_\phi \).

**Proof.** If \( z^n \in \mathcal{M} \) for all \( n \geq 1 \), then \( z^\ell z^n \in \mathcal{M} \) for all \( \ell \geq 1 \) because \( z \mathcal{M} \subseteq \mathcal{M} \). Let \( \mathcal{E} \) be the closed linear span of \( \{z^\ell z^n; n \geq 1, \ell \geq 0\} \), then \( \mathcal{E} \subseteq \mathcal{M} \) and \( g \mathcal{E} \subseteq \mathcal{E} \) for arbitrary polynomial \( g \) of \( z \) and \( z \). It is well known that \( \mathcal{E} = L^2 \). This contradiction implies that there exists at least \( n \) such that \( z^n \notin \mathcal{M} \) and \( n \geq 1 \). If \( \phi = z^n \), then \( z^{n+k} \phi \in \mathcal{M} \). Then \( H^u_\phi \neq 0 \) but \( H^u_\phi \) is of finite rank \( \leq n+k \), by Theorem 3.1.

**Proposition 6.2.** Suppose \( \mathcal{M} \) is an invariant subspace which contains \( e^{ik \theta}H^2 \) for some nonnegative integer \( k \). The following statements are valid.

1. If \( \phi = \sum_{j=-\infty}^\infty \phi_j(r) e^{ij \theta} \) is a function in \( L^\infty \), then there exists a function \( \phi' \) in \( L^2 \) such that \( \phi' = \sum_{j=0}^{k-1} \phi_j(r) e^{ij \theta} + \sum_{j=1}^\infty \phi_j(r) e^{ij \theta} \) and \( H^u_\phi = H^u_{\phi'} \).

2. If \( \phi = \sum_{j=-k}^\infty \phi_j(r) e^{ij \theta} \) is a function in \( L^\infty \), then \( H^u_\phi = 0 \).

3. If \( \phi = \sum_{j=-k}^\infty \phi_j(r) e^{ij \theta} \) is a function in \( L^\infty \), then \( H^u_\phi \) is of rank \( \leq k+\ell < \infty \). Conversely, if one of (1) or (2) is valid, then \( \mathcal{M} \) contains \( e^{it \theta}H^2 \).

**Proof.** Both (1) and (2) are clear because \( \mathcal{M} \supseteq e^{ik \theta}H^2 \). (3) is a result of Theorem 3.1. The converse is also clear.

We will consider Example 2.4 in Section 2.

**Example 6.3.** (ii) Suppose \( \mathcal{M} = (E^k)^\perp (0 \leq k < \infty) \) and \( \phi = \sum_{j=-\infty}^\infty \phi_j(r) e^{ij \theta} \) is a function in \( L^\infty \).
(1) \( H^\mu_\phi = 0 \) if and only if
\[
\int_0^1 \phi_{-j}(r)r^{j+2t}d\sigma = 0 \quad (j \geq 0, \ 0 \leq t \leq k).
\] (6.1)

(2) \( H^\mu_\phi \) is of rank \( \leq 1 \) if and only if there exist complex numbers \((b_0, b_1) \neq (0,0)\) such that
\[
b_0 \int_0^1 \phi_{-j}(r)r^{j+2t}d\sigma = -b_1 \int_0^1 \phi_{-j-1}(r)r^{j+2t+1}d\sigma
\] (6.2)
for \( j \geq 0, \ 0 \leq t \leq k \).

(3) Suppose \( d\sigma = r \, dr/2\pi \). When \( \phi = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=1}^{\infty} a_{-j} \bar{z}^j \), if \( H^\mu_\phi \) is of rank \( \leq 1 \), then \( H^\mu_\phi = 0 \).

**Proof.** From the remark in the last part of Section 4, (1) and (2) follows. (3) By (2), \( H^\mu_\phi \) is of rank \( \leq 1 \) if and only if there exist complex numbers \((b_0, b_1) \neq (0,0)\) such that
\[
b_0 a_{-j} \frac{1}{2j+2t+1} = -b_1 a_{-j-1} \frac{1}{2j+2t+3}
\] (6.3)
for \( j \geq 0, \ 0 \leq t \leq k \). When \( k \neq 0 \), for each \( j \), as \( t = 0 \),
\[
b_0 a_{-j} \frac{1}{2j+1} = -b_1 a_{-j-1} \frac{1}{2j+3},
\]
\[
b_0 a_{-j} \frac{1}{2j+3} = -b_1 a_{-j-1} \frac{1}{2j+5}.
\] (6.4)

This implies that \( a_{-j} = a_{-j-1} = 0 \), for \( j \geq 0 \), and so \( \phi = \sum_{j=1}^{\infty} a_j z^j \). When \( k = 0 \), Corollary 3.3 implies (3).

(iv) Suppose \( \mathcal{M} = q \mathbb{H}^2 \) for some unimodular function \( q \) in \( \mathbb{H}^2 \) and \( \phi \) is a function in \( L^\infty \). \( H^\mu_\phi \) is of finite rank \( \ell \) if and only if
\[
\phi = q \sum_{j=-\ell}^{\infty} \psi_j(r)e^{ij\theta},
\] (6.5)
where \( \psi_{-\ell}(r) \neq 0 \).

**Proof.** If \( \phi = q \sum_{j=-\ell}^{\infty} \psi_j(r)e^{ij\theta} \), then \( z^\ell \phi \in \mathcal{M} \) and so, by Theorem 3.1, \( H^\mu_\phi \) is of finite rank \( \leq \ell \). Since \( \psi_{-\ell}(r) \neq 0 \), \( b\phi \notin \mathcal{M} \) for any polynomial \( b \) of degree \( \leq \ell - 1 \) and so \( H^\mu_\phi \) is of finite rank \( \ell \). The converse is clear.

(v) Suppose \( \mathcal{M} = \mathbb{H}^2 \oplus Se^{-i\theta} \) and \( S \) is a closed subspace in \( L^2 \). Let \( \phi = \sum_{j=-\infty}^{\infty} \phi_j(r)e^{ij\theta} \) be a function in \( L^\infty \). By Theorems 3.1 and 4.1, \( H^\mu_\phi \) is of finite rank \( \leq \ell \) if and only if \( \phi_j(r) = 0 \) for \( j \leq -(\ell + 2) \) and there exist complex numbers \( b_0, \ldots, b_\ell \) such that \( b_\ell = 1 \),
\[
\sum_{j=0}^{\ell} b_j r^j \phi_{n-j}(r) = 0 \quad \text{for} \quad -(\ell + 1) \leq n < -1,
\]
\[
\sum_{j=0}^{\ell} b_j r^j \phi_{-1-j}(r) \in S.
\] (6.6)
7. Restricted shift operator and $\mathcal{M} \subseteq L^2_a$. In this section, we assume $\mu = r \, dr \, d\theta / \pi$ for simplicity. Let $\mathcal{M}$ be an invariant subspace in $L^2_a$ and $\mathcal{K} = L^2_a \ominus \mathcal{M}$. For $\phi$ in $L^\infty_a = L^2_a \cap L^\infty$,  
\begin{equation}
S^X_\phi f = (I - P^X) (\phi f) \quad (f \in \mathcal{K}),
\end{equation}
where $P^X$ is the orthogonal projection from $L^2_a$ to $\mathcal{K}$, $S^X_\phi$ is called a restricted shift operator. For any $\phi$ in $L^\infty_a$, $S^X_\phi$ commutes with $S^X_2$. We do not know whether if the bounded linear operator $T$ on $\mathcal{K}$ commutes with $S^X_2$, then $T = S^X_\phi$ for some $\phi$ in $L^\infty_a$. If $T S^X_2 = S^X_2 T$ and $\phi = T P^X 1$ is bounded, then it is easy to see that $T = S^X_\phi$ (cf. [5, page 784]). In the Hardy space instead of the Bergman space, Sarason [8] showed that this is true without any condition and $\|T\| = \|\phi\|_{\infty}$.

We can define the Hankel operator $H^H_\phi$ as in the introduction. However $H^H_\phi$ is not an intermediate Hankel operator. It is not so difficult to see the following: when $\mathcal{K} = L^2_a \ominus \mathcal{M}$ and $\phi$ in $L^\infty_a$,  
\begin{equation}
\|H^H_\phi\| = \|S^X_\phi\|.
\end{equation}
This is known for the Hardy space. In fact, for $f$ in $L^2_a$,  
\begin{equation}
H^H_\phi f = (I - P^H) \phi f = P^X \phi P^X f
\end{equation}
and so $H^H_\phi f = S^X_\phi P^X f$ for $f$ in $L^2_a$. Hence $H^H_\phi$ is of finite rank $n$ if and only if $S^X_\phi$ is of finite rank $n$. It is easy to see that $S^X_\phi$ is of finite rank $\ell \leq n$ if and only if there exists an analytic polynomial $p$ of degree $\ell \leq n$ such that $p(\phi) \in \mathcal{M}$. When $\phi$ is in $L^\infty_a$, Theorems 3.1 and 4.1 are true for $H^H_\phi$.

Suppose $\phi$ is a function in $L^\infty_a$.

(1) $L^2_a \ni \ker H^H_\phi \ni \mathcal{M}$.

(2) When the common zero set $Z(\mathcal{M})$ of $\mathcal{M}$ in $D$ is empty, if $H^H_\phi$ is of finite rank then $H^H_\phi = 0$. This is a result of (1) and Proposition 2.1.

(3) If $Z(\mathcal{M})$ is not empty, there exists a nonzero finite rank $H^H_\phi$.

Acknowledgement. This research was partially supported by Grant-in-Aid for scientific research, Ministry of Education.

References


Takahiko Nakazi: Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Tomoko Osawa: Mathematical and Scientific Subjects, Asahikawa National College of Technology, Asahikawa 071, Japan
Submit your manuscripts at http://www.hindawi.com