COMPUTATIONS OF NAMBU-POISSON COHOMOLOGIES

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ABSTRACT. We want to associate to an \( n \)-vector on a manifold of dimension \( n \) a cohomology which generalizes the Poisson cohomology of a 2-dimensional Poisson manifold. Two possibilities are given here. One of them, the Nambu-Poisson cohomology, seems to be the most pertinent. We study these two cohomologies locally, in the case of germs of \( n \)-vectors on \( \mathbb{K}^n \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)).

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1. Introduction. A way to study a geometrical object is to associate to it a cohomology. In this paper, we focus on the \( n \)-vectors on an \( n \)-dimensional manifold \( M \).

If \( n = 2 \), the 2-vectors on \( M \) are the Poisson structures thus, we can consider the Poisson cohomology. In dimension 2, this cohomology has three spaces. The first one, \( H^0 \), is the space of functions whose Hamiltonian vector field is zero (Casimir functions). The second one, \( H^1 \), is the quotient of the space of infinitesimal automorphisms (or Poisson vector fields) by the subspace of Hamiltonian vector fields. The last one, \( H^2 \), describes the deformations of the Poisson structure. In a previous paper (see [9]) we have computed the cohomology of germs at 0 of Poisson structures on \( \mathbb{K}^2 \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)).

In order to generalize this cohomology to the \( n \)-dimensional case \( (n \geq 3) \), we can follow the same reasoning. These spaces are not necessarily of finite dimension and it is not always easy to describe them precisely.

Recently, a team of Spanish researchers has defined a cohomology, called Nambu-Poisson cohomology, for the Nambu-Poisson structures (see [6]). In this paper, we adapt their construction to our particular case. We will see that this cohomology generalizes in a certain sense the Poisson cohomology in dimension 2. Then we compute locally this cohomology for germs at 0 of \( n \)-vectors \( \Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n \) on \( \mathbb{K}^n \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \)), with the assumption that \( f \) is a quasihomogeneous polynomial of finite codimension ("most of" the germs of \( n \)-vectors have this form). This computation is based on a preliminary result that we have shown, in the formal case and in the analytical case (so, the \( \mathcal{C}^\infty \) case is not entirely solved). The techniques we use in this paper are quite the same as in [9].

2. Nambu-Poisson cohomology. Let \( M \) be a differentiable manifold of dimension \( n \) \( (n \geq 3) \), admitting a volume form \( \omega \). We denote by \( \mathcal{C}^\infty (M) \) the space of \( \mathcal{C}^\infty \) functions on \( M \), by \( \Omega^k(M) \) \( (k = 0, \ldots, n) \) the \( \mathcal{C}^\infty (M) \)-module of \( k \)-forms on \( M \), and by \( \chi^k(M) \) \( (k = 0, \ldots, n) \) the \( \mathcal{C}^\infty (M) \)-module of \( k \)-vectors on \( M \).
We consider an $n$-vector $\Lambda$ on $M$. Note that $\Lambda$ is a Nambu-Poisson structure on $M$.

Recall that a Nambu-Poisson structure on $M$ of order $r$ is a skew-symmetric $r$-linear map $\{\ldots,\}$

$$\mathfrak{e}^\infty(M) \times \cdots \times \mathfrak{e}^\infty(M) \twoheadrightarrow \mathfrak{e}^\infty(M), \quad (f_1, \ldots, f_r) \mapsto \{f_1, \ldots, f_r\}, \quad (2.1)$$

which satisfies

$$\{f_1, \ldots, f_{r-1}, gh\} = \{f_1, \ldots, f_{r-1}, g\} h + g \{f_1, \ldots, f_{r-1}, h\},$$

$$\{f_1, \ldots, f_{r-1}, \{g_1, \ldots, g_r\}\} = \sum_{i=1}^{r} \{g_1, \ldots, g_{i-1}, f_1, \ldots, f_{r-1}, g_i, g_{i+1}, \ldots, g_r\}, \quad (2.2)$$

for any $f_1, \ldots, f_{r-1}, g, h, g_1, \ldots, g_r$ in $\mathfrak{e}^\infty(M)$. It is clear that we can associate to such a bracket an $r$-vector on $M$. If $r = 2$, we rediscover Poisson structures. Thus, Nambu-Poisson structures can be seen as a kind of generalization of Poisson structures. The notion of Nambu-Poisson structures was introduced in [14] by Takhtajan in order to give a formalism to an idea of Y. Nambu (see [12]).

Here, we suppose that the set $\{x \in M; \Lambda x \neq 0\}$ is dense in $M$. We are going to associate a cohomology to $(M, \Lambda)$.

### 2.1. The choice of the cohomology.

If $M$ is a differentiable manifold of dimension 2, then the Poisson structures on $M$ are the 2-vectors on $M$. If $\Pi$ is a Poisson structure on $M$, then we can associate to $(M, \Pi)$ the complex

$$0 \rightarrow \mathfrak{e}^\infty(M) \xrightarrow{\partial} \chi^1(M) \xrightarrow{\partial} \chi^2(M) \rightarrow 0 \quad (2.3)$$

with $\partial(g) = [g, \Pi] = X_g$ (Hamiltonian of $g$) if $g \in \mathfrak{e}^\infty(M)$ and $\partial(X) = \{X, \Pi\}$ ([ ] indicates Schouten’s bracket) if $X \in \chi^1(M)$. The cohomology of this complex is called the Poisson cohomology of $(M, \Pi)$. This cohomology has been studied for instance in [9, 10, 15].

Now if $M$ is of dimension $n$ with $n \geq 3$, we want to generalize this cohomology. Our first approach was to consider the complex

$$0 \rightarrow (\mathfrak{e}^\infty(M))^{n-1} \xrightarrow{\partial} \chi^1(M) \xrightarrow{\partial} \chi^n(M) \rightarrow 0 \quad (2.4)$$

with $\partial(X) = [X, \Lambda]$ and $\partial(g_1, \ldots, g_{n-1}) = i_{dg_1 \wedge \cdots \wedge dg_{n-1}} \Lambda = X_{g_1 \wedge \cdots \wedge g_{n-1}}$ (Hamiltonian vector field) where we adopt the convention $i_{dg_1 \wedge \cdots \wedge dg_{n-1}} \Lambda = \Lambda(dg_1, \ldots, dg_{n-1}, \cdot)$. We denote by $H^1_\Lambda(M)$, $H^2_\Lambda(M)$, and $H^n_\Lambda(M)$ the three spaces of cohomology of this complex. With this cohomology, we rediscover the interpretation of the first spaces of the Poisson cohomology, that is, $H^1_\Lambda(M)$ describes the infinitesimal deformations of $\Lambda$ and $H^2_\Lambda(M)$ is the quotient of the algebra of vector fields which preserve $\Lambda$ by the ideal of Hamiltonian vector fields.

In [6], the authors associate to any Nambu-Poisson structure on $M$ a cohomology. The second idea is then to adapt their construction to our particular case.

Let $\#_\Lambda$ be the morphism of $\mathfrak{e}^\infty(M)$-modules $\Omega^{n-1}(M) \rightarrow \chi^1(M) : \alpha \mapsto i_\alpha \Lambda$. Note that $\ker \#_\Lambda = \{0\}$ (because the set of regular points of $\Lambda$ is dense). We can define (see [7]) an $\mathbb{R}$-bilinear operator $[[, ]] : \Omega^{n-1}(M) \times \Omega^{n-1}(M) \rightarrow \Omega^{n-1}(M)$ by

$$[[\alpha, \beta]] = \mathcal{L}_{\#_\Lambda \alpha} \beta + (-1)^n (i_{d\alpha} \Lambda) \beta. \quad (2.5)$$
The vector space $\Omega^{n-1}(M)$ equipped with $[,]$ is a Lie algebra (for any Nambu-Poisson structure, it is a Leibniz algebra). Moreover, this bracket verifies that $\#_\Lambda([[,]]) = [\#_\Lambda[,]$] for any $\alpha, \beta$ in $\Omega^{n-1}(M)$. The triple $(\Lambda^{n-1}(T^*(M)),[,],\#_\Lambda)$ is then a Lie algebra and the Nambu-Poisson cohomology of $(M,\Lambda)$ is the Lie algebraic cohomology of $\Lambda^{n-1}(T^*(M))$ (for any Nambu-Poisson structure, it is more elaborate see [6]). More precisely, for every $k \in \{0,\ldots, n\}$, we consider the vector space $C^k(\Omega^{n-1}(M);\mathbb{C}^\infty(M))$ of the skew-symmetric and $\mathbb{C}^\infty(M)$-multilinear maps $\Omega^{n-1}(M) \times \cdots \times \Omega^{n-1}(M) \to \mathbb{C}^\infty(M)$. The cohomology operator $\partial : C^k(\Omega^{n-1}(M);\mathbb{C}^\infty(M)) \to C^{k+1}(\Omega^{n-1}(M);\mathbb{C}^\infty(M))$ is defined by

$$\partial c(\alpha_0,\ldots,\alpha_k) = \sum_{i=0}^k (-1)^i ([\#_\Lambda \alpha_i] \cdot c(\alpha_0,...,\hat{\alpha}_i,...,\alpha_k)$$

$$+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} c([[\alpha_i,\alpha_j]],\alpha_0,...,\hat{\alpha}_i,...,\hat{\alpha}_j,...,\alpha_k)$$

for all $c \in C^k(\Omega^{n-1}(M);\mathbb{C}^\infty(M))$ and $\alpha_0,...,\alpha_k$ in $\Omega^{n-1}(M)$.

The Nambu-Poisson cohomology of $(M,\Lambda)$, denoted by $H^*_\Lambda(M,\Lambda)$, is the cohomology of this complex.

2.2. An equivalent cohomology. So defined, the Nambu-Poisson cohomology is quite difficult to manipulate. We are going to give an equivalent cohomology which is more accessible.

Recall that we assume that $M$ admits a volume form $\omega$.

Let $f \in \mathbb{C}^\infty(M)$, we define the operator

$$d_f : \Omega^k(M) \to \Omega^{k+1}(M), \quad \alpha \mapsto f \, d\alpha - k \, df \wedge \alpha. \tag{2.7}$$

It is easy to prove that $d_f \circ d_f = 0$. We denote by $H^*_f(M)$ the cohomology of this complex. Let $\flat$ be the isomorphism $\chi^1(M) \to \Omega^{-1}(M), X \mapsto i_X \omega$.

**Lemma 2.1.** (1) If $X \in \chi^1(M)$, then $\#_\Lambda(\flat(X)) = (-1)^{n-1} f X$, where $f = i_X \omega$.

(2) If $X$ and $Y$ are in $\chi^1(M)$, then

$$(-1)^{n-1}[\flat(X), \flat(Y)] = f \, \flat([X,Y]) + (X \cdot f) \flat(Y) - (Y \cdot f) \flat(X). \tag{2.8}$$

**Proof.** (1) Obvious.

(2) We have $\#_\Lambda([[\flat(X), \flat(Y)]]) = [[\#_\Lambda(\flat(X)), \#_\Lambda(\flat(Y))]]$ (property of the Lie algebra), which implies that

$$\#_\Lambda([[\flat(X), \flat(Y)]]) = f \, (X \cdot f) \, Y - f \, (Y \cdot f) \, X + f^2 [X,Y]$$

$$= (-1)^{n-1} \#_\Lambda((X \cdot f) \flat(Y) - (Y \cdot f) \flat(X) + f \, \flat([X,Y])). \tag{2.9}$$

The result follows via the injectivity of $\#_\Lambda$. \hfill \Box

**Proposition 2.2.** If we put $f = i_X \omega$, then $H^*_{NP}(M,\Lambda)$ is isomorphic to $H^*_f(M)$.

**Proof.** For every $k$, we consider the application $\varphi : C^k(\Omega^{n-1}(M);\mathbb{C}^\infty(M)) \to \Omega^k(M)$ defined by

$$\varphi(c)(X_1,\ldots,X_k) = c((-1)^{n-1} \flat(X_1),\ldots,(-1)^{n-1} \flat(X_k)). \tag{2.10}$$
where \( c \in C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M)) \) and \( X_1,\ldots,X_k \in \chi^1(M) \). It is easy to see that \( \varphi \) is an isomorphism of vector spaces. We show that it is an isomorphism of complexes.

Let \( c \in C^k(\Omega^{n-1}(M);\mathcal{C}^\infty(M)) \). We put \( \alpha = \varphi(c) \). If \( X_0,\ldots,X_k \) are in \( \chi^1(M) \), then

\[
\varphi(\partial c)(X_0,\ldots,X_k) = (-1)^{(n-1)(k+1)}\partial \big( \varphi(c) \big)(X_0,\ldots,X_k) = A + B,
\]

where

\[
A = (-1)^{i} \sum_{i=0}^{k} (-1)^i \#(\alpha(X_i)) \cdot c \big( \overline{b}(X_0),\ldots,\overline{b}(X_i),\ldots,\overline{b}(X_k) \big),
\]

\[
B = (-1)^{i+j} \sum_{0 \leq i < j \leq k} (-1)^i \big( \alpha(X_i) \cdot f \big) \alpha(X_j,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).
\]

We have \( A = f \sum_{i=0}^{k} (-1)^i X_i \cdot \alpha(X_0,\ldots,\hat{X}_i,\ldots,X_k) \) and

\[
B = f \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha \big( [X_i,X_j] \big) \alpha(X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \big( X_i \cdot f \big) \alpha(X_j,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k)
\]

\[
- \sum_{0 \leq i < j \leq k} (-1)^{i+j} \big( X_j \cdot f \big) \alpha(X_i,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_k).
\]

Consequently, \( \varphi(\partial c) = df \alpha = df \varphi(c) \).

**Remark 2.3.** We claim that this cohomology is a “good” generalization of the Poisson cohomology of a 2-dimensional Poisson manifold. Indeed, if \((M,\Pi)\) is an orientable Poisson manifold of dimension 2, we consider the volume form \( \omega \) on \( M \) and we put

\[
\phi^2 : \chi^2(M) \to \Omega^2(M), \quad \phi^1 : \chi^1(M) \to \Omega^1(M),
\]

defined by

\[
\phi^2(\Gamma) = (i_\Gamma \omega) \omega, \quad \phi^1(X) = -i_X \omega,
\]

for every 2-vector \( \Gamma \) and vector field \( X \).

We also put \( \phi^0 = id : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M) \).

If we denote by \( \partial \) the operator of the Poisson cohomology, and \( f = i_\Pi \omega \), it is quite easy to see that

\[
\phi : (\chi^*(M),\partial) \to (\Omega^*(M), df)
\]

is an isomorphism of complexes.

**Remark 2.4.**

1. The definitions we have given make sense if we work in the holomorphic case or in the formal case.

2. Important: if \( h \) is a function on \( M \) which does not vanish on \( M \), then the cohomologies \( H^*_f(M) \) and \( H^*_{fh}(M) \) are isomorphic.

Indeed, the applications \( (\Omega^k(M),dfh) \to (\Omega^k(M),df) \), \( \alpha \to \alpha/h^k \) give an isomorphism of complexes.
In particular, if $f$ does not vanish on $M$ then $H^f_\ast(M)$ is isomorphic to the de Rham’s cohomology.

**2.3. Other cohomologies.** We can construct other complexes which look like $(\Omega^\ast(M), df)$. More precisely we denote, for $p \in \mathbb{Z}$,

$$d_f^{(p)} : \Omega^k(M) \to \Omega^{k+1}(M), \quad \alpha \mapsto f d\alpha - (k-p) df \wedge \alpha. \quad (2.16)$$

We denote by $H^\ast_{f,p}(M)$ the cohomology of these complexes. We will see in Section 3 some relations between these different cohomologies.

Using the contraction $i_\omega$, it is quite easy to prove the following proposition.

**Proposition 2.5.** The spaces $H^1_\Lambda(M)$ and $H^2_\Lambda(M)$ are isomorphic to $H^{n-1}_{f,n-2}(M)$ and $H^n_{f,n-2}(M)$.

**Remark 2.6.** The two properties of Remark 2.4 are valid for $H^\ast_{f,p}(M)$ with $p \in \mathbb{Z}$.

3. **Computation.** Henceforth, we will work locally. Let $\Lambda$ be a germ of $n$-vectors on $\mathbb{K}^n$ ($\mathbb{K}$ indicates $\mathbb{R}$ or $\mathbb{C}$) with $n \geq 3$. We denote by $\mathcal{F}(\mathbb{K}^n)$ the space of germs at 0 of (holomorphic, analytic, $\mathcal{C}^\infty$, formal) functions (k-forms, vector fields).

We can write $\Lambda$ (with coordinates $(x_1,\ldots,x_n)$) $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$, where $f \in \mathcal{F}(\mathbb{K}^n)$. We assume that the volume form $\omega$ is $dx_1 \wedge \cdots \wedge dx_n$.

We suppose that $f(0) = 0$ (see Remark 2.4) and that $f$ is of finite codimension, which means that $Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$ ($I_f$ is the ideal spanned by $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$) is a finite-dimensional vector space.

**Remark 3.1.** It is important to note that, according to Tougeron’s theorem (cf. [3]), if $f$ is of finite codimension, then the set $f^{-1}(\{0\})$ is, from the topological point of view, the same as the set of zeros of a polynomial.

Therefore, if $g$ is a germ at 0 of functions which satisfies $fg = 0$, then $g = 0$.

Moreover, we suppose that $f$ is a quasihomogeneous polynomial of degree $N$ (for a justification of this additional assumption, see Section 4). We are going to recall the definition of the quasihomogeneity.

3.1. **Quasihomogeneity.** Let $(w_1,\ldots,w_n) \in (\mathbb{N}^\times)^n$. We denote by $W$ the vector field $w_1x_1(\partial/\partial x_1) + \cdots + w_nx_n(\partial/\partial x_n)$ on $\mathbb{K}^n$. We say that a nonzero tensor $T$ is quasihomogeneous with weights $w_1,\ldots,w_n$ and of (quasi)degree $N \in \mathbb{Z}$ if $\mathcal{L}_W T = NT$ ($\mathcal{L}_W$ indicates the Lie derivative operator). Note that $T$ is then polynomial.

If $f$ is a quasihomogeneous polynomial of degree $N$, then $N = k_1w_1 + \cdots + k_nw_n$ with $k_1,\ldots,k_n \in \mathbb{N}$; so, an integer is not necessarily the quasidegree of a polynomial. If $f \in \mathbb{K}[[x_1,\ldots,x_n]]$, we can write $f = \sum_{i=0}^\infty f_i$ with $f_i$ quasihomogeneous of degree $i$ (we adopt the convention that $f_i = 0$ if $i$ is not a quasidegree); $f$ is said to be of order $d$ (ord$(f) = d$) if all of its monomials have a degree $d$ or higher. For more details, see [3].

Since $\mathcal{L}_W$ and the exterior differentiation $d$ commute, if $\alpha$ is a quasihomogeneous $k$-form, then $d\alpha$ is a quasihomogeneous $(k+1)$-form of degree $\deg \alpha$. In particular, it is important to notice that $dx_i$ is a quasihomogeneous 1-form of degree $w_i$ (note that $\partial/\partial x_i$ is a quasihomogeneous vector field of degree $-w_i$). Thus, the volume form
\( \omega = dx_1 \wedge \cdots \wedge dx_n \) is quasihomogeneous of degree \( w_1 + \cdots + w_n \). Note that a quasihomogeneous nonzero \( k \)-form \((k \geq 1)\) has a degree strictly positive.

Note that if \( f \) is a quasihomogeneous polynomial of degree \( N \), then the \( n \)-vector \( \Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n \) is quasihomogeneous of degree \( N - \sum w_i \).

In what follows, the degrees will be quasihomogeneous with respect to \( W = w_1 x_1 (\partial/\partial x_1) + \cdots + w_n x_n (\partial/\partial x_n) \).

We will need the following result.

**Lemma 3.2.** Let \( k_1, \ldots, k_n \in \mathbb{N} \) and put \( p = \sum k_i w_i \). Assume that \( g \in \mathcal{F}(\mathbb{C}^n) \) and \( \alpha \in \Omega^i(\mathbb{C}^n) \) verify \( \text{ord}(j_0^m(g)) = p \) and \( \text{ord}(j_0^m(\alpha)) > p \) \((j_0^m \text{ indicates the } \infty \text{-jet at 0})\). Then

1. there exists \( h \in \mathcal{F}(\mathbb{C}^n) \) such that \( W \cdot h - ph = g \),
2. there exists \( \beta \in \Omega^i(\mathbb{C}^n) \) such that \( \mathcal{L}_W \beta - p \beta = \alpha \).

**Proof.** The first claim is only a generalization of Lemma 3.5 in [9] (it also appears in Lemma 2 in [2]) and it can be proved in the same way. The second claim is a consequence of the first. \( \square \)

Now, we compute the spaces \( H^k_f(\mathbb{C}^n) \) \((i.e., H^k_{NP}(\mathbb{C}^n, \Lambda))\) for \( k = 0, \ldots, n \). We denote by \( Z^k_f(\mathbb{C}^n) \) and \( \Lambda^k_f(\mathbb{C}^n) \) the spaces of \( k \)-cycles and \( k \)-cochains. We also compute some spaces \( H^k_{f,p}(\mathbb{C}^n) \) with particular interest in the spaces \( H^m_{f,n-2}(\mathbb{C}^n) \) \((i.e., H^m_{\Lambda}(\mathbb{C}^n))\) and \( H^{m-1}_{f,n-2}(\mathbb{C}^n) \) \((i.e., H^{m-1}_{\Lambda}(\mathbb{C}^n))\). We denote by \( Z^k_{f,p}(\mathbb{C}^n) \) \((\Lambda^k_{f,p}(\mathbb{C}^n))\) the spaces of \( k \)-cycles \((k \)-cochains\) for the operator \( d_f \).

3.2. Two useful preliminary results. In the computation of these spaces of cohomology, we need the two following propositions. The first is only a corollary of the de Rham’s division lemma (see [4]).

**Proposition 3.3.** Let \( f \in \mathcal{F}(\mathbb{C}^n) \) of finite codimension. If \( \alpha \in \Omega^k(\mathbb{C}^n) \) \((1 \leq k \leq n-1)\) verifies \( df \wedge \alpha = 0 \), then there exists \( \beta \in \Omega^{k-1}(\mathbb{C}^n) \) such that \( \alpha = df \wedge \beta \).

**Proposition 3.4.** Let \( f \in \mathcal{F}(\mathbb{C}^n) \) of finite codimension. Let \( \alpha \) be a \( k \)-form \((2 \leq k \leq n-1)\) which verifies \( d\alpha = 0 \) and \( df \wedge \alpha = 0 \), then there exists \( \gamma \in \Omega^{k-2}(\mathbb{C}^n) \) such that \( \alpha = df \wedge d\gamma \).

**Proof.** We prove this result in the formal case and in the analytical case.

Formal case: let \( \alpha \) be a quasihomogeneous \( k \)-form of degree \( p \) which verifies the hypotheses. Since \( df \wedge \alpha = 0 \), we have \( \alpha = df \wedge \beta_1 \), where \( \beta_1 \) is a quasihomogeneous \((k-1)\)-form of degree \( p-N \). Now, since \( d\alpha = 0 \), we have \( df \wedge d\beta_1 = 0 \) and so \( d\beta_1 = df \wedge \beta_2 \), where \( \beta_2 \) is a quasihomogeneous \((k-1)\)-form of degree \( p-N \) . This way, we can construct a sequence \((\beta_i)\) of quasihomogeneous \((k-1)\)-forms with \( \text{deg} \beta_i = p-n \) which verifies that \( d\beta_i = df \wedge \beta_{i+1} \). Let \( q \in \mathbb{N} \) such that \( p-qN \leq 0 \). Thus, we have \( \beta_q = 0 \) and so \( d\beta_i = 0 \), that is, \( \beta_i = dy_q \), where \( y_q \) is a \((k-2)\)-form. Consequently, \( d\beta_i = df \wedge dy_q \) which implies that \( \beta_i = df \wedge y_q \), where \( y_q \) is a \((k-2)\)-form. In the same way, \( d\beta_{i-1} = df \wedge dy_{q-1} \) so \( \beta_{i-1} = df \wedge y_{q-1} \), where \( y_{q-1} \) is a \((k-2)\)-form. This way, we can show that \( \beta_1 = df \wedge y_1 \), where \( y_1 \) and \( y_2 \) are \((k-2)\)-forms. Therefore, \( \alpha = df \wedge dy_1 \).

Analytical case: in [8], Malgrange gave a result on the relative cohomology of a germ of an analytical function. In particular, he showed that in our case, if \( \beta \) is a germ at
0 of analytical $r$-forms ($r < n - 1$) which verifies $d \beta = df \land \mu$ ($\mu$ is an $r$-form) then there exist two germs of analytical $(r - 1)$-forms $\gamma$ and $\nu$ such that $\beta = d \gamma + df \land \nu$.

Now, we prove our proposition. Let $\alpha$ be a germ of analytical $k$-forms ($2 \leq k \leq n - 1$) which verifies the hypotheses of the proposition. Then there exists a $(k - 1)$-form $\beta$ such that $\alpha = df \land \beta$ (Proposition 3.3). But since $0 = d \alpha = -df \land d\beta$, we have $d\beta = df \land \mu$ and so (see [8]) $\beta = dy + df \land \nu$, where $y$ and $\nu$ are analytical $(k - 2)$-forms. We deduce that $\alpha = df \land dy$, where $y$ is analytic. □

**Remark 3.5.** Important: in fact, some results which appear in [13] lead us to think that this proposition is not true in the real $C^\infty$ case.

The computation of the spaces $H^n_f(\mathbb{K}^n)$, $H^{n-1}_f(\mathbb{K}^n)$ ($p \neq n - 2$), and $H^0_f(\mathbb{K}^n)$ does not use this proposition, so it still holds in the $C^\infty$ case.

The results we find on the other spaces should be the same in the $C^\infty$ case as in the analytical case but another proof need to be found.

### 3.3. Computation of $H^0_{f,p}(\mathbb{K}^n)$

We consider the application $d_f^{(p)} : \Omega^0(\mathbb{K}^n) \to \Omega^1(\mathbb{K}^n)$, $g \mapsto f dg + p df \land g$.

**Theorem 3.6.** (1) If $p > 0$ then $H^0_{f,p}(\mathbb{K}^n) = \{0\}$.

(2) If $p \leq 0$ then $H^0_{f,p}(\mathbb{K}^n) = \mathbb{K} \cdot f^{-p}$.

**Proof.** (1) If $g \in \mathcal{F}(\mathbb{K}^n)$ verifies $d_f^{(p)} g = 0$, then $d(f^p g) = 0$, and so $f^p g$ is constant. But as $f(0) = 0$, $f^p g$ must be 0, that is, $g = 0$ (because $f$ is of finite codimension; see Remark 3.1).

(2) We use an induction to show that for any $k \geq 0$, if $g$ satisfies $f dg = kg df$ then $\gamma = \lambda f^k$, where $\lambda \in \mathbb{K}$.

For $k = 0$ it is obvious.

Now we suppose that the property is true for $k \geq 0$. We show that it is still valid for $k + 1$. Let $g \in \mathcal{F}(\mathbb{K}^n)$ be such that

$$fdg = (k + 1) df.$$  \hspace{1cm} (3.1)

Then $df \land dg = 0$ and so there exists $h \in \mathcal{F}(\mathbb{K}^n)$ such that $dg = h df$ (Proposition 3.3).

Replacing $dg$ by $h df$ in (3.1), we get $f h df = (k + 1) g df$, that is, $g = (1/(k + 1)) fh$.

Now, this former relation gives, on one hand, $f dg = (1/(k + 1))(f^2 dh + fh df)$ and on the other hand, using (3.1), $f dg = f h df$. Consequently, $f dh = kh df$ and so $h = \lambda f^k$ with $\lambda \in \mathbb{K}$.

**3.4. Computation of $H^k_f(\mathbb{K}^n)$ $1 \leq k \leq n - 2$**

**Lemma 3.7.** Let $\alpha \in Z^k_{f,p}(\mathbb{K}^n)$ with $1 \leq k \leq n - 2$. Then $\alpha$ is cohomologous to a closed $k$-form.

**Proof.** We have $fd\alpha - (k - p) df \land \alpha = 0$. If $k = p$ then $\alpha$ is closed. Now we suppose that $k \neq p$. We put $\beta = d \alpha \in \Omega^{k + 1}(\mathbb{K}^n)$. We have

$$0 = df \land (f d\alpha - (k - p) df \land \alpha) = f df \land \alpha,$$  \hspace{1cm} (3.2)

so $df \land \alpha = 0$. 


Now, since \( d\beta = 0 \) and \( df \wedge \beta = 0 \), Proposition 3.4 gives \( \beta = df \wedge dy \) with \( y \in \Omega^{k-1}(\mathbb{K}^n) \). Then, if we consider \( \alpha' = \alpha - (1/(k-p))(fdy - (k-p-1)df \wedge y) \), we have \( d\alpha' = 0 \) and \( fdy - (k-p-1)df \wedge y \in B^{k}_{p}(\mathbb{K}^n) \).

**Theorem 3.8.** If \( k \in \{2, \ldots, n-2\} \) then \( H^k_f(\mathbb{K}^n) = \{0\} \).

**Proof.** Let \( \alpha \in Z^k_f(\mathbb{K}^n) \). Then \( \alpha \in \Omega^k(\mathbb{K}^n) \) and verifies \( f\alpha = -kdf \wedge \alpha = 0 \).

According to Lemma 3.7 we can assume that \( \alpha \) is closed. Now we show that \( \alpha \in B^k_f(\mathbb{K}^n) \). Since \( d\alpha = 0 \) and \( df \wedge \alpha = 0 \), there exists \( \beta \in \Omega^{k-2}(\mathbb{K}^n) \) such that \( \alpha = df \wedge \beta \) (Proposition 3.4). Thus, \( \alpha = df((-1/(k-1))\beta) \).

**Remark 3.9.** It is possible to adapt this proof to show that \( H^k_{f,p}(\mathbb{K}^n) = \{0\} \) if \( k \in \{2, \ldots, n-2\} \) and \( p \neq k, k-1 \).

**Lemma 3.10.** Let \( \alpha \in Z^1_f(\mathbb{K}^n) \). If \( \text{ord}(j^0_\alpha(\beta)) > N \) then \( \alpha \in B^1_f(\mathbb{K}^n) \).

**Proof.** According to Lemma 3.7, we can assume that \( d\alpha = 0 \).

Since \( df \wedge \alpha = 0 \) we have \( \alpha = gdf \) (see Proposition 3.3), where \( g \) is in \( \mathcal{F}(\mathbb{K}^n) \) and verifies \( \text{ord}(j^0_\alpha(g)) > 0 \). We show that \( f \) divides \( g \).

Let \( \hat{g} \in \mathcal{F}(\mathbb{K}^n) \) be such that \( W \cdot \hat{g} = g \) (see Lemma 3.2); note that \( \text{ord}(j^0_\alpha(\hat{g})) > 0 \).

We have \( \mathcal{L}_W(df \wedge d\hat{g}) = Ndf \wedge d\hat{g} + df \wedge dg \), and since \( df \wedge dg = -d\alpha = 0 \), \( df \wedge d\hat{g} \) verifies

\[
\mathcal{L}_W(df \wedge d\hat{g}) = Ndf \wedge d\hat{g},
\]

which means that \( df \wedge d\hat{g} \) is either 0 or quasihomogeneous of degree \( N \).

But since \( \text{ord}(j^0_\alpha(df \wedge d\hat{g})) > N \), \( df \wedge d\hat{g} \) must be 0.

Consequently, there exists \( \nu \in \mathcal{F}(\mathbb{K}^n) \) such that \( \partial \hat{g}/\partial x_i = \nu(\partial f/\partial x_i) \) for any \( i \).

Thus, \( W \cdot \hat{g} = \nu W \cdot f \) and so \( g = \nu f \).

We deduce that \( \alpha = f\beta \) with \( \beta \in \Omega^1(\mathbb{K}^n) \).

Now, we have

\[
0 = d\alpha = df \wedge \beta + f df, \quad 0 = df \wedge \alpha = f df \wedge \beta,
\]

which implies that \( d\beta = 0 \).

Therefore, \( \alpha = f dh = df(h) \) with \( h \in \mathcal{F}(\mathbb{K}^n) \).

**Theorem 3.11.** The space \( H^1_f(\mathbb{K}^n) \) is of dimension 1 and spanned by \( df \).

**Proof.** Let \( \alpha \in Z^1_f(\mathbb{K}^n) \). According to Lemma 3.10, we only have to study the case where \( \alpha \) is quasihomogeneous with \( \text{deg}(\alpha) \leq N \). We have \( f\alpha = -df \wedge \alpha = 0 \), so \( df \wedge d\alpha = 0 \). We deduce that \( d\alpha = df \wedge \beta \), where \( \beta \) is a quasihomogeneous 1-form of degree \( \text{deg}(\alpha) - N \leq 0 \). But since \( dx_i \) is quasihomogeneous of degree \( w_i > 0 \) for any \( i \), every quasihomogeneous nonzero 1-form has a strictly positive degree. We deduce that \( \beta = 0 \) and so \( d\alpha = 0 \). Therefore, \( df \wedge \alpha = 0 \) which implies that \( \alpha = gdf \), where \( g \) is a quasihomogeneous function of degree \( \text{deg}(\alpha) - N \leq 0 \). Consequently, if \( \text{deg}(\alpha) < N \) then \( g = 0 \); otherwise \( g \) is constant. To conclude, note that \( df \) is not a cobord because \( f \) does not divide \( df \).
3.5. Computation of $H^n_{f,p}(\mathbb{K}^n)$. We compute the spaces $H^n_{f,p}(\mathbb{K}^n)$ for $p \neq n-1$. We consider the application

$$d_f^{(n-q)} : \Omega^{n-1}(\mathbb{K}^n) \to \Omega^n(\mathbb{K}^n), \quad \alpha \mapsto f \, d\alpha - (q-1) \, df \wedge \alpha,$$

with $q \neq 1$ (note that if $q = n$ then we obtain the space $H^n_{M,\Lambda}(M,\Lambda)$ and if $q = 2$ then we have $H^2_{\Lambda}(\mathbb{K}^n)$).

We denote $\mathcal{J}^n = \{df \wedge \alpha; \alpha \in \Omega^{n-1}(\mathbb{K}^n)\}$. It is clear that $\mathcal{J}^n \approx I_f$ (recall that $I_f$ is the ideal of $\mathcal{F}(\mathbb{K}^n)$ spanned by $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$) and that $\Omega^n(\mathbb{K}^n)/\mathcal{J}^n \approx Q_f = \mathcal{F}(\mathbb{K}^n)/I_f$.

We put $\sigma = i_0 \omega$ (recall that $W = w_1 x_1 \partial/\partial x_1 + \cdots + w_n x_n \partial/\partial x_n$) and that $\omega$ is the standard volume form on $\mathbb{K}^n$. Note that $\sigma$ is a quasihomogeneous $(n-1)$-form of degree $\sum_i w_i$ and that $dg \wedge \sigma = (W \cdot g) \omega$ if $g \in \mathcal{F}(\mathbb{K}^n)$.

If $\alpha \in \Omega^{n-1}(\mathbb{K}^n)$, we use the notation $\text{div}(\alpha)$ for $d\alpha = \text{div}(\alpha) \omega$; for example, $\text{div}(\sigma) = \sum_i w_i$. Note that if $\alpha$ is quasihomogeneous, then $\text{div}(\alpha)$ is quasihomogeneous of degree $\text{deg} \alpha - \sum_i w_i$.

**Lemma 3.12.** (1) If the $\infty$-jet at 0 of $\gamma$ does not contain a component of degree $qN$ (in particular if $q \leq 0$) then $y \in B^1_{f,n-q}(\mathbb{K}^n) \iff y \in \mathcal{J}^n$.

(2) If $y$ is a quasihomogeneous $n$-form of degree $qN$, then $y \in B^1_{f,n-q}(\mathbb{K}^n) \Rightarrow y \in \mathcal{J}^n$.

**Proof.** If $y = f \, d\alpha - (q-1) \, df \wedge \alpha \in B^n_{f,n-q}(\mathbb{K}^n)$, where $\alpha \in \Omega^{n-1}$ then $y = df \wedge \beta$ with $\beta = -(q-1) \alpha + (\text{div}(\alpha)/N) \sigma$. This shows the second claim and the first part of the first one.

Now we prove the reverse of the first claim.

Formal case: let $y = \sum_{i>0} y^{(i)}$ and $\beta = \sum \beta^{(i-N)}$ (with $y^{(i)}$ of degree $i$, $\gamma^{(qN)} = 0$ and $\beta^{(i-N)}$ of degree $i - N$) such that $y = df \wedge \beta$. If we put $\alpha = (-1/(q-1)) \beta + \sum_i (\text{div}(\beta^{(i-N)}))/(q-1)(i-qN) \sigma$, we have $d_f^{(n-q)}(\alpha) = y$.

Analytical case: if $\beta$ is analytic at 0, the function $\text{div}(\beta)$ is analytic too, and since $\lim_{t \to +\infty} 1/(i-qN) = 0$, the $(n-1)$-form defined above is also analytic at 0.

$\mathcal{C}^\infty$ case: we suppose that $y = df \wedge \beta$. If we denote $\tilde{y} = j_{\infty}^0(y)$, then there exists a formal $(n-1)$-form $\tilde{\alpha}$ such that $\tilde{y} = f \, d\tilde{\alpha} - (q-1) \, df \wedge \tilde{\alpha}$. Let $\alpha$ be a $\mathcal{C}^\infty - (n-1)$-form such that $\tilde{\alpha} = j_{\infty}^0(\alpha)$. This form verifies $f \, d\alpha - (q-1) \, df \wedge \alpha = y + \epsilon$, where $\epsilon$ is flat at 0. Since $B^n_{f,n-q}(\mathbb{K}^n) \subset \mathcal{J}^n$, $\epsilon \in \mathcal{J}^n$ so that $\epsilon = df \wedge \mu$, where $\mu$ is flat at 0. Let $g \in \mathcal{F}(\mathbb{K}^n)$ be such that $W \cdot g - ((q-1)N - \sum w_i)g = \text{div}(\mu)/(q-1)$ (Lemma 3.2). Then the form $\theta = (-1/(q-1)) \mu + g \sigma$ verifies $d_f^{(n-q)}(\theta) = \epsilon$. 

**Remark 3.13.** (1) Lemma 3.12 gives $B^1_{f,n-q}(\mathbb{K}^n) \subset \mathcal{J}^n$. Thus, there is a surjection from $H^n_{f,n-q}(\mathbb{K}^n)$ onto $Q_f$. Therefore, if $f$ is not of finite codimension, then $H^n_{f,n-q}(\mathbb{K}^n)$ is an infinite-dimensional vector space.

(2) According to this lemma, if $y$ is in $\mathcal{J}^n$ then there exits a quasihomogeneous $n$-form $\theta$, of degree $qN$, such that $y + \theta \in B^n_{f,n-q}(\mathbb{K}^n)$. Note that $\theta$ is in $\mathcal{J}^n$.

The first claim of this lemma allows us to state the following theorem.

**Theorem 3.14.** If $q \leq 0$ then $H^n_{f,n-q}(\mathbb{K}^n) \approx Q_f$.

Now we suppose that $q > 1$. 


Lemma 3.15. Let $\alpha \in \Omega^k(\mathbb{K}^n)$ and $p \in \mathbb{Z}$. Then $f d_f^{(p)}(\alpha) = d_f^{(p-1)}(f \alpha)$.

Proof. The proof is obvious. \hfill \Box

Lemma 3.16. (1) Let $q > 2$. If $\alpha \in \Omega^n(\mathbb{K}^n)$ is quasihomogeneous of degree $(q-1)N$ and verifies $f \alpha \in B^n_{f,n-q}(\mathbb{K}^n)$, then $\alpha \in B^n_{f,n-q+1}(\mathbb{K}^n)$.

(2) If $\alpha$ is quasihomogeneous of degree $N$ with $f \alpha \in B^n_{f,n-2}(\mathbb{K}^n)$, then $\alpha = 0$.

Proof. (1) We suppose that $\alpha = g \omega$ with $g \in \mathcal{T}(\mathbb{K}^n)$ quasihomogeneous of degree $(q-1)N - \sum w_i$. We have $f g \omega = f d\beta - (q-1) d f \wedge \beta$, where $\beta$ is a quasihomogeneous $(n-2)$-form of degree $(q-2)N$. Consequently, $\beta = -1/(q-1) d f \wedge \gamma$, where $\gamma$ is a quasihomogeneous $(n-1)$-form of degree $(q-2)N$. Now, a computation shows that $f d\beta - (q-1) d f \wedge \beta = (1/(q-1)) f d f \wedge d y$, that is, $f \alpha = (1/(q-1)) f d f \wedge d y$. Therefore, $\alpha = (1/(q-1)) d f \wedge d y = (1/(q-1)) d f^{(n-2)}((q-2) d y)$.

(2) As in (1) (with $q = 2$), we have $f \alpha = f g \omega = d f^{(n-2)}(f \beta)$ with deg $g = N$ and deg $\beta = N$. We put $\theta = -(q-1)/(q-1) d f \wedge \beta$, where $\gamma$ is a quasihomogeneous $(n-2)$-form of degree 0 which is not possible. So $\theta = 0$, that is, $\beta = -1/(q-1) f d f \wedge \gamma$. We deduce that $f d\beta - d f \wedge \beta = 0$, that is, $\alpha = 0$. \hfill \Box

Let $\mathcal{B}$ be a monomial basis of $Q_f$ (for the existence of such a basis, see [3]). We denote by $r_j$ ($j = 2, \ldots, q-1$) the number of monomials of $\mathcal{B}$ whose degree is $j N - \sum w_i$ (this number does not depend on the choice of $\mathcal{B}$). We also denote by $s$ the dimension of the space of quasihomogeneous polynomials of degree $N - \sum w_i$ and $c$ the codimension of $f$.

Theorem 3.17. Let $\alpha \in \Omega^n(\mathbb{K}^n)$. Then there exist unique polynomials $h_1, \ldots, h_q$ (possibly zero) such that

(a) $h_1$ is quasihomogeneous of degree $N - \sum w_i$,

(b) $h_j$ ($2 \leq j \leq q-1$) is a linear combination of monomials of $\mathcal{B}$ of degree $j N - \sum w_i$,

(c) $h_q$ is a linear combination of monomials of $\mathcal{B}$, and

\[
\alpha = (h_q + fh_{q-1} + \cdots + f^{q-1}h_1) \omega \mod B^n_{f,n-q}(\mathbb{K}^n). \tag{3.6}
\]

In particular, the dimension of $H^n_{f,n-q}(\mathbb{K}^n)$ is $c + r_{q-1} + \cdots + r_2 + s$.

Proof. Existence. We suppose that $\alpha = g \omega$ with $g \in \mathcal{T}(\mathbb{K}^n)$. There exists $h_q$, a linear combination of the monomials of $\mathcal{B}$, such that $g = h_q \mod I_f$. So, according to Lemma 3.12 (see Remark 3.13), $g \omega = h_q \omega + d f \wedge \beta \mod B^n_{f,n-q}(\mathbb{K}^n)$, where $\beta$ is a quasihomogeneous $(n-1)$-form of degree $(q-1)N$.

Consequently, $g \omega = h_q \omega + (1/(q-1)) f d \beta - (1/(q-1)) [f d \beta - (q-1) d f \wedge \beta] \mod B^n_{f,n-q}(\mathbb{K}^n)$, so we can write

\[
g \omega = h_q \omega + f g_{q-1} \omega \mod B^n_{f,n-q}(\mathbb{K}^n), \tag{3.7}
\]

where $g_{q-1}$ is quasihomogeneous of degree $(q-1)N - \sum w_i$. 

In the same way,
\[ g_{q-1} \omega = h_{q-1} \omega + f g_{q-2} \omega \mod B^n_{f,n-q+1}(\mathbb{K}^n), \tag{3.8} \]
where \( h_{q-1} \) is a linear combination of the monomials of \( \mathcal{B} \) of degree \((q - 1)N - \sum w_i\) and \( g_{q-2} \) is quasihomogeneous of degree \((q - 2)N - \sum w_i, \ldots,\)
\[ g_2 \omega = h_2 \omega + f h_1 \omega \mod B^n_{f,n-2}(\mathbb{K}^n), \tag{3.9} \]
where \( h_2 \) is a linear combination of the monomials of \( \mathcal{B} \) of degree \(2N - \sum w_i\) and \( h_1 \) is quasihomogeneous of degree \( N - \sum w_i.\)

Using Lemma 3.15, we get
\[ \alpha = g \omega = (h_q + h_{q-1} + f^2 h_{q-2} + \cdots + f^{q-1} h_1) \omega \mod B^n\left(\lambda^{(n-q)}\right). \tag{3.10} \]

**Unicity.** Let \( g = h_q + f h_{q-1} + \cdots + f^{q-1} h_1 \) with \( h_1, \ldots, h_q \) as in the statement of the theorem. We suppose that \( g \omega \in B^n_{f,n-q}(\mathbb{K}^n). \) Then \( g \omega \in \mathcal{F}^n \), that is, \( g \in \mathcal{L}_f. \) But since \( f h_{q-1} + \cdots + f^{q-1} h_1 \in \mathcal{L}_f \) (because \( f \in \mathcal{L}_f \)) we have \( h_q \in \mathcal{L}_f \) and so \( h_q = 0. \)

Now, according to Lemma 3.16, \( (h_{q-1} + f h_{q-2} + \cdots + f^{q-2} h_1) \omega \) is in \( B^n_{f,n-q+1}(\mathbb{K}^n) \) and so, in the same way, \( h_{q-1} = 0. \)

This way, we get \( h_q = h_{q-1} = \cdots = h_2 = 0 \) and \( f h_1 \omega \in B^n_{f,n-2}(\mathbb{K}^n). \) Lemma 3.16 gives \( h_1 = 0. \)

This theorem allows us to give the dimension of the spaces \( H^n_{A_F}(\mathbb{K}^n, \Lambda) \) and \( H^n_\Lambda(\mathbb{K}^n) \).

**Corollary 3.18.** Let \( \alpha \in \Omega^n(\mathbb{K}^n). \) Then there exist unique polynomials \( h_1, \ldots, h_n \) (Possibly zero) such that
(a) \( h_1 \) is quasihomogeneous of degree \( N - \sum w_i, \)
(b) \( h_j \) (\( 2 \leq j \leq n - 1 \)) is a linear combination of monomials of \( \mathcal{B} \) of degree \( jN - \sum w_i, \)
(c) \( h_n \) is a linear combination of monomials of \( \mathcal{B}, \)
and
\[ \alpha = (h_n + f h_{n-1} + \cdots + f^{n-1} h_1) \omega \mod B^n_f(\mathbb{K}^n). \tag{3.11} \]

In particular, the dimension of \( H^n_{A_F}(\mathbb{K}^n, \Lambda) \) is \( c + r_{n-1} + \cdots + r_2 + s. \)

**Corollary 3.19.** Let \( \alpha \in \Omega^n(\mathbb{K}^n). \) Then there exist unique polynomials \( h_1, h_2 \) (possibly zero) such that
(a) \( h_1 \) is quasihomogeneous of degree \( N - \sum w_i, \)
(b) \( h_2 \) is a linear combination of monomials of \( \mathcal{B}, \)
and
\[ \alpha = (h_2 + f h_1) \omega \mod B^n_{f,n-2}(\mathbb{K}^n). \tag{3.12} \]

In particular, the dimension of \( H^n_\Lambda(\mathbb{K}^n) \) is \( c + s. \)

**Remark 3.20.** If \( q = 1 \), then the space \( H^n_{f,n-1}(\mathbb{K}^n) \) is \( \Omega^n(\mathbb{K}^n) / \Lambda \Omega^n(\mathbb{K}^n) \) which is of infinite dimension.

### 3.6. Computation of \( H^{n-1}_{f,p}(\mathbb{K}^n) \)

We compute the spaces \( H^{n-1}_{f,p}(\mathbb{K}^n) \) with \( p \neq n - 1. \) We consider the piece of complex
\[ \Omega^{n-2}(\mathbb{K}^n) \rightarrow \Omega^{n-1}(\mathbb{K}^n) \rightarrow \Omega^n(\mathbb{K}^n), \tag{3.13} \]
with
\[ d_f^{(n-q)}(\alpha) = f \cdot d\alpha - (q - 2) \cdot d\alpha \wedge \alpha \quad \text{if} \quad \alpha \in \Omega^{n-2}(K^n), \]
\[ d_f^{(n-q)}(\alpha) = f \cdot d\alpha - (q - 1) \cdot d\alpha \wedge \alpha \quad \text{if} \quad \alpha \in \Omega^{n-1}(K^n), \] (3.14)

with \( q \neq 1 \).

Remember that if \( q = n \), we obtain \( H^{n-1}_{\partial \partial'}(K^n, \Lambda) \) and if \( q = 2 \) we have \( H^1(K^n) \).

**Lemma 3.21.** If \( \alpha \in Z^{n-1}_{f,n-q}(K^n) \), then \( \alpha = (\text{div}(\alpha)/(q - 1)N)\sigma + d\alpha \wedge \beta \) with \( \beta \in \Omega^{n-2}(K^n) \) and so, \( d\alpha \) verifies \( \mathcal{L}_W (d\alpha) - (q - 1)N d\alpha = (q - 1)N d\alpha \wedge d\beta \).

**Proof.** It is sufficient to notice that \( d\alpha \wedge (\text{div}(\alpha)/(q - 1)N)\sigma = 0 \) (see Proposition 3.3). For the second claim, we have \( (q - 1)N d\alpha = (W \cdot \text{div}(\alpha) + (\sum \omega_i) \text{div}(\alpha) \omega - (q - 1)N d\alpha \wedge d\beta \). The conclusion follows.

**Lemma 3.22.** If \( \alpha \in Z^{n-1}_{f,n-q}(K^n) \) with \( \text{ord}(j^{(n)}_{f,n-q}(\alpha)) > (q - 1)N \), then \( \alpha \) is cohomologous to a closed \((n-1)\)-form. In particular, if \( q \leq 0 \) then every \((n-1)\)-cocycle for \( d_f^{(n-q)} \) is cohomologous to a closed \((n-1)\)-form.

**Proof.** We have \( \alpha = (\text{div}(\alpha)/(q - 1)N)\sigma + d\alpha \wedge \beta \) (Lemma 3.21) with
\[ \mathcal{L}_W (d\alpha) - (q - 1)N d\alpha = (q - 1)N d\alpha \wedge d\beta. \] (3.15)

Now, let \( \gamma \in \Omega^{n-2}(K^n) \) such that \( \mathcal{L}_W \gamma - (q - 2)N \gamma = (q - 1)N \beta \) (\( \gamma \) exists because \( \text{ord}(j^{(n)}_{f,n-q}(\beta)) > (q - 2)N \), see Lemma 3.2).

We have \( \mathcal{L}_W dy - (q - 2)N dy = (q - 1)N d\beta \). Thus \( d\alpha \wedge d\gamma \) verifies
\[ \mathcal{L}_W (d\alpha \wedge d\gamma) - (q - 1)N d\alpha \wedge d\gamma = (q - 1)N d\alpha \wedge d\beta. \] (3.16)

From (3.15) and (3.16) we get \( d\alpha = d\alpha \wedge d\gamma \).

Indeed, \( \mathcal{L}_W (d\alpha - d\gamma) = (q - 1)N (d\alpha - d\alpha \wedge d\gamma) \) but \( d\alpha - d\alpha \wedge d\gamma \) is not quasi-homogeneous of degree \((q - 1)N \). Now, if we put \( \theta = \alpha - (1/(q - 1))(f \cdot d\gamma - (q - 2)N d\alpha \wedge d\gamma) \), we have \( d\theta = 0 \) and \( \theta \equiv \alpha \mod B^{n-1}_{f,n-q}(K^n) \).

**Lemma 3.22** allows us to state the following theorem.

**Theorem 3.23.** If we suppose that \( q \leq 0 \) then \( H^{n-1}_{f,n-q}(K^n) = \{0\} \).

**Proof.** Let \( \alpha \in Z^{n-1}_{f,n-q}(K^n) \). We can suppose (according to Lemma 3.22) that \( d\alpha = 0 \). Thus we have \( d\alpha \wedge \alpha = 0 \). **Proposition 3.4** gives then, \( \alpha = d\alpha \wedge \gamma \) with \( \gamma \in \Omega^{n-3}(K^n) \). Therefore, \( \alpha = d_f^{(n-q)} (-1/(q - 2)) \cdot d\gamma \).

Now, we assume that \( q > 1 \).

**Lemma 3.24.** If \( \alpha \in Z^{n-1}_{f,n-q}(K^n) \) is a quasi-homogeneous \((n-1)\)-form whose degree is strictly lower than \((q - 1)N \), then \( \alpha \) is cohomologous to a closed \((n-1)\)-form.

**Proof.** According to Lemma 3.21, we have \( \alpha = (\text{div}(\alpha)/(q - 1)N)\sigma + d\alpha \wedge \beta \), and so
\[ d\alpha = \frac{(q - 1)N}{\deg(\alpha) - (q - 1)N} d\alpha \wedge d\beta. \] (3.17)
We deduce that, if we put \( \theta = \alpha - d_f^{[n-q]}((N/(\deg(\alpha) - (q - 1)N))d\beta), \) we have \( d\theta = 0. \)

**Remark 3.25.** A consequence of Lemmas 3.22 and 3.24 is that, if \( q > 1, \) every cocycle \( \alpha \in Z^1_{f,n,q}(\mathbb{K}^n) \) is cohomologous to a cocycle \( \eta + \theta, \) where \( \eta \) is in \( Z^1_{f,n,q}(\mathbb{K}^n) \) and is closed, and \( \theta \) is quasihomogeneous of degree \( (q - 1)N. \)

**Lemma 3.26.** Let \( \alpha = g\sigma, \) where \( g \) is a quasihomogeneous polynomial of degree \( (q-1)N - \sum w_i. \) Then

1. if \( q > 2, \) then \( \alpha \in B^{n-1}_{f,n,q}(\mathbb{K}^n) \) \( \Leftrightarrow g\omega \in B^n_{f,n,q+1}(\mathbb{K}^n), \)
2. if \( q = 2, \) \( \alpha \in B^{n-1}_{f,n-2}(\mathbb{K}^n) \) \( \Leftrightarrow \alpha = 0. \)

**Proof.**

1. (a) We suppose that \( \alpha \in B^{n-1}_{f,n,q}(\mathbb{K}^n), \) that is, \( \alpha = f d\beta - (q - 2)df \wedge \beta \) with \( \beta \in \Omega^{n-2}(\mathbb{K}^n). \) Then \( d\alpha = (q-1)f d\beta \wedge d\beta. \)

   On the other hand, \( d\alpha = (q-1)N\omega \) \( \Leftrightarrow g\omega \in B^n_{f,n,q+1}(\mathbb{K}^n), \)

   (b) Now we suppose that \( g\omega \in B^n_{f,n,q+1}(\mathbb{K}^n), \) that is, \( g\omega = f d\beta - (q - 2)df \wedge \beta, \) where \( \beta \) is a quasihomogeneous \( (n-1) \)-form of degree \( (q - 2)N. \) We put \( y = iw\beta \in \Omega^{n-2}(\mathbb{K}^n). \) We have

   \[
   d_f^{[n-q]}(y) = f df - (q - 2)df \wedge y = f (iw\beta) - (q - 2)df \wedge (iw\beta) \\
   = f (\mathcal{L}_w \beta - iw df) - (q - 2)[ -iw df \wedge \beta + (iw df) \wedge \beta] \\
   = f (q - 2)N\beta - iw df \wedge (q - 2)df \wedge \beta - (q - 2)(W \cdot f)\beta \\
   = -iw df \wedge (q - 2)df \wedge \beta. 
   \]

   Consequently, \( d_f^{[n-q]}(y) = iw(g\omega) = -g\sigma. \)

2. If \( \alpha = f d\beta, \) where \( \beta \) is a quasihomogeneous \( (n-2) \)-form of degree \( \deg \alpha - N = 0, \) then \( \beta = 0 \) and so \( \alpha = 0. \)

We recall that \( \mathcal{B} \) indicates a monomial basis of \( Q_f. \) We adopt the same notations as for Theorem 3.17.

**Theorem 3.27.** We suppose that \( q > 2. \) Let \( \alpha \in Z^1_{f,n,q}(\mathbb{K}^n). \) There exist unique polynomials \( h_1, \ldots, h_{q-1} \) (possibly zero) such that

1. \( h_1 \) is quasihomogeneous of degree \( N - \sum w_i, \)
2. \( h_k \) \( (k \geq 2) \) is a linear combination of monomials of \( \mathcal{B} \) of degree \( kN - \sum w_i, \)

   \[
   \omega = (h_{q-1} + f h_{q-2} + \cdots + f^{q-2} h_1)\sigma \mod B^n_{f,n,q}(\mathbb{K}^n). 
   \]

In particular, the dimension of the space \( H^{n-1}_{f,n,q}(\mathbb{K}^n) \) is \( r_{q-1} + \cdots + r_2 + s. \)

**Proof.** If \( \alpha \in Z^1_{f,n,q}(\mathbb{K}^n), \) then \( \alpha \) is cohomologous to \( \eta + \theta, \) where \( \eta \) is in \( Z^1_{f,n,q}(\mathbb{K}^n) \) and is closed, and \( \theta \) is quasihomogeneous of degree \( (q - 1)N \) (see Remark 3.25).

The same proof as of Theorem 3.23 shows that \( \eta \) is a cobord.

Now, we have to study \( \theta. \) According to Lemma 3.21, we can write \( \theta = (\text{div}(\theta)/(q-1)N)\sigma + d\beta \wedge \beta \) \( (\beta \in \Omega^{n-2}(\mathbb{K}^n)) \) with \( \mathcal{L}_w (d\beta) = (q - 1)N d\beta. \) Since \( \theta \) is quasihomogeneous of degree \( (q - 1)N, \) the former relation gives \( d\beta \wedge d\beta = 0. \)

Consequently, if we put \( y = d\beta, \) Proposition 3.4 gives \( y = d\beta \wedge d\beta. \)

Therefore, \( y = d_f^{[n-q]}(-1)((q - 2))d\beta \) and so \( \theta = (\text{div}(\theta)/(q-1)N)\sigma \mod B^n_{f,n,q}(\mathbb{K}^n). \) The conclusion follows using Lemma 3.26 and Theorem 3.17.
COROLLARY 3.28. We suppose that $q = n$. Let $\alpha \in Z_{f}^{n-1}(\mathbb{K}^{n})$. There exist unique polynomials $h_1, \ldots, h_{n-1}$ (possibly zero) such that

(a) $h_1$ is quasihomogeneous of degree $N - \sum w_i$,

(b) $h_k$ ($k \geq 2$) is a linear combination of monomials of $\mathcal{B}$ of degree $kN - \sum w_i$, and

$$\omega = (h_{n-1} + fh_{n-2} + \cdots + f^{n-2}h_1)\sigma \mod B_{f}^{n-1}(\mathbb{K}^{n}).$$ (3.20)

In particular, the dimension of the space $H_{N_{f}}^{n-1}(\mathbb{K}^{n}, \Lambda)$ is $r_{n-1} + \cdots + r_{2} + s$.

REMARK 3.29. If $q = 2$, the description of the space $H_{f,n-2}^{n-1}(\mathbb{K}^{n})$ (and so $H_{\Lambda}^{1}(\mathbb{K}^{n})$) is more difficult. It is possible to show that this space is not of finite dimension. Indeed, we consider the case $n = 3$ for simplicity (but it is valid for any $n \geq 3$). We put $\alpha = g((\partial f/\partial x)dx + (\partial f/\partial y)dy + (\partial f/\partial z)dz)$, where $g$ is a function which depends only on $z$. We have $d\alpha = 0$ and $df \wedge \alpha = 0$, so $\alpha \in Z_{f,n-1}(\mathbb{K}^{n})$ but $\alpha \notin B_{f,n-2}(\mathbb{K}^{n})$ because $f$ does not divide $\alpha$.

We can yet give more precisions on the space $H_{f,n-2}^{n-1}(\mathbb{K}^{n})$.

THEOREM 3.30. Let $E$ be the space of $(n - 1)$-forms $h\sigma$, where $h$ is a quasihomogeneous polynomial of degree $N - \sum w_i$, and $F$ the quotient of the vector space $\{df \wedge dy; \gamma \in \Omega^{n-3}(\mathbb{K}^{n})\}$ by the subspace $\{df \wedge d(f\beta); \beta \in \Omega^{n-3}(\mathbb{K}^{n})\}$.

Then $H_{f,n-2}^{n-1}(\mathbb{K}^{n}) = E \oplus F$.

PROOF. Let $\alpha \in Z_{f,n-2}^{n-1}(\mathbb{K}^{n})$.

According to Remark 3.25, there exist a closed $(n - 1)$-form $\eta$ with $\eta \in Z_{f,n-2}^{n-1}(\mathbb{K}^{n})$ and a quasihomogeneous $(n - 1)$-form $\theta$ of degree $N$, such that $\alpha$ is cohomologous to $\eta + \theta$.

We have (Lemma 3.21) $\theta = (\text{div}(\theta)/N)\sigma + df \wedge \beta$ with $\beta$ quasihomogeneous of degree 0 which is possible only if $\beta = 0$. So, $\theta = g\sigma$, where $g$ is a quasihomogeneous polynomial of degree $N - \sum w_i$. Lemma 3.26 says that $\theta \in B_{f,n-2}^{n-1}(\mathbb{K}^{n})$ if and only if $\theta = 0$.

Now we study $\eta$. Proposition 3.4 gives $\eta = df \wedge dy$, where $y$ is an $(n - 3)$-form.

If we suppose that $\eta \in B_{f,n-2}^{n-1}(\mathbb{K}^{n})$, then $df \wedge dy = f d\xi$ with $\xi \in \Omega^{n-2}(\mathbb{K}^{n})$, and so $df \wedge d\xi = 0$. Now we apply Proposition 3.4 to $d\xi$ and we obtain $d\xi = df \wedge d\beta$ with $\beta \in \Omega^{n-3}(\mathbb{K}^{n})$. Consequently, $df \wedge dy = f df \wedge d\beta$ which implies that $dy = f d\beta + df \wedge \mu$ with $\mu \in \Omega^{n-3}(\mathbb{K}^{n})$, and so $dy = d(f\beta) + df \wedge \nu$ with $\nu \in \Omega^{n-3}(\mathbb{K}^{n})$.

Therefore, $\eta \in B_{f,n-2}^{n-1}(\mathbb{K}^{n}) \iff \eta = df \wedge d(f\beta)$. \hfill \Box

3.7. Summary. It is time to sum up the results we have found.

The cohomology $H_{f}^{*}(\mathbb{K}^{n})$ (and so the Nambu-Poisson cohomology $H_{\Lambda}^{*}(\mathbb{K}^{n}, \Lambda)$) has been entirely computed (see Theorems 3.6, 3.8, 3.11, and Corollaries 3.18 and 3.28).

The spaces of this cohomology are of finite dimension and only the “extremal” ones (i.e., $H^{0}, H^{1}, H^{n-1}$, and $H^{n}$) are possibly different to $\{0\}$. The spaces $H_{0}^{*}(\mathbb{K}^{n}, \Lambda)$ and $H_{N_{f}}^{*}(\mathbb{K}^{n}, \Lambda)$ are of dimension 1. The dimensions of the spaces $H_{N_{f}}^{n-1}(\mathbb{K}^{n}, \Lambda)$ and $H_{\Lambda}^{n}(\mathbb{K}^{n}, \Lambda)$ depend, on one hand, on the type of the singularity of $\Lambda$ (via the role played by $Q_{f}$), and on the other hand, on the “polynomial nature” of $\Lambda$.

Concerning the cohomology $H_{f,n-2}^{n-1}(\mathbb{K}^{n})$, we have computed $H^{n}$, that is, $H_{\Lambda}^{n}(\mathbb{K}^{n})$ (see Corollary 3.19) and we have given a sketch of description of $H^{n-1}$ (see Theorem 3.30).
We have also computed the spaces $H^0_{f,n-2}(\mathbb{K}^n)$ (see Theorem 3.6) and $H^k_{f,n-2}(\mathbb{K}^n)$ (see Theorem 3.8) for $k + n - 2, n - 1$, but these spaces are not particularly interesting for our problem. The space $H^2_\Lambda(\mathbb{K}^n)$, which describes the infinitesimal deformations of $\Lambda$ is of finite dimension and its dimension has the same property as the dimension of $H^n_{NP}(\mathbb{K}^n, \Lambda)$. On the other hand, the space $H^1_\Lambda(\mathbb{K}^n)$ which is the space of the vector fields preserving $\Lambda$ modulo the Hamiltonian vector fields, is not of finite dimension.

It is interesting to compare the results we have found on these two cohomologies with the ones given in [9] on the computation of the Poisson cohomology in dimension 2.

Finally, if $p \neq 0, n - 2, n - 1$, we have computed the spaces $H^0_{f,p}(\mathbb{K}^n), H^{n-1}_{f,p}(\mathbb{K}^n), H^n_{f,p}(\mathbb{K}^n)$, and $H^k_{f,p}(\mathbb{K}^n)$ with $k \neq p, p + 1$.

If $p = n - 1$, we have computed the spaces $H^0_{f,n-1}(\mathbb{K}^n)$ and $H^k_{f,n-1}(\mathbb{K}^n)$ for $2 \leq k \leq n - 2, k \neq p, p + 1$ (the space $H^n_{f,n-1}(\mathbb{K}^n)$ is of infinite dimension).

4. Examples. In this section, we explicit the cohomology of some particular germs of $n$-vectors.

4.1. Normal forms of $n$-vectors. Let $\Lambda = f(\partial/\partial x_1) \wedge \cdots \wedge \partial/\partial x_n$ be a germ at 0 of $n$-vectors on $\mathbb{K}^n$ ($n \geq 3$) with $f$ of finite codimension (see the beginning of Section 3) and $f(0) = 0$ (if $f(0) \neq 0$, then the local triviality theorem, see [1, 5] or [11], allows us to write, up to a change of coordinates, that $\Lambda = \partial/\partial x_1 \wedge \cdots \wedge \partial/\partial x_n$).

**Proposition 4.1.** If 0 is not a critical point for $f$, then there exist local coordinates $y_1, \ldots, y_n$ such that

$$\Lambda = y_1 \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial y_n}.$$ (4.1)

**Proof.** A similar proposition is shown for instance in [9] in dimension 2. The proof can be generalized to the $n$-dimensional ($n \geq 3$) case.

Now we suppose that 0 is a critical point of $f$. Moreover, we suppose that the germ $f$ is simple, which means that a sufficiently small neighbourhood (with respect to Whitney’s topology; see [3]) of $f$ intersects only a finite number of $R$-orbits (two germs $g$ and $h$ are said to be $R$-equivalent if there exists $\varphi$, a local diffeomorphism at 0, such that $g = h \circ \varphi$). Simple germs are those who present a certain kind of stability under deformation.

The following theorem can be found in [2].

**Theorem 4.2.** Let $f$ be a simple germ at 0 of finite codimension. Suppose that $f$ has at 0 a critical point with critical value 0. Then there exist local coordinates $y_1, \ldots, y_n$ such that the germ $\Lambda = f(\partial/\partial y_1) \wedge \cdots \wedge \partial/\partial y_n$ can be written, up to a multiplicative constant, $g(\partial/\partial y_1) \wedge \cdots \wedge \partial/\partial y_n$, where $g$ is in the following list:

$$A_k : y_1^{k+1} \pm y_2^2 \pm \cdots \pm y_n^2,$$  
$$D_k : y_1^{k+2}y_2 \pm y_2^{k+1} \pm y_3^2 \pm \cdots \pm y_n^2, \quad k \geq 4,$$  
$$E_6 : y_1^3 + y_2^2 \pm y_3^2 \pm \cdots \pm y_n^2,$$  
$$E_7 : y_1^3 + y_1y_2^2 \pm y_3^2 \pm \cdots \pm y_n^2,$$  
$$E_8 : y_1^3 + y_2^2 \pm y_3^2 \pm \cdots \pm y_n^2.$$ (4.2)
Proposition 4.1 and Theorem 4.2 describe most of the germs at 0 of \( n \)-vectors on \( \mathbb{K}^n \) vanishing at 0.

We can notice that the models given in the former list are all quasihomogeneous polynomials; which justifies the assumption we made in Section 3.

4.2. Some examples. (1) The regular case: \( f(x_1, \ldots, x_n) = x_1 \).

It is easy to see that \( Q_f = \{0\} \) and that \( f \) is quasihomogeneous of degree \( N = 1 \), with respect to \( w_1 = \cdots = w_n = 1 \). We have \( N - \sum w_i < 0 \), so \( H_f^0(\mathbb{K}^n) \cong \mathbb{K}, H_f^1(\mathbb{K}^n) = \mathbb{K} \cdot dx_1 \) and \( H_f^2(\mathbb{K}^n) = \{0\} \) for any \( k \geq 2 \).

(2) Nondegenerate singularity: \( f(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2 \) with \( n \geq 3 \).

We have \( N = 2 \) and \( w_1 = \cdots = w_n = 1 \). The space \( Q_f \) is isomorphic to \( \mathbb{K} \) and is spanned by the constant germ 1, which is of degree 0.

We deduce that \( H_f^0(\mathbb{K}^n) \cong \mathbb{K}, H_f^1(\mathbb{K}^n) = \mathbb{K} \cdot (x_1 \, dx_1 + \cdots + x_n \, dx_n) \) and \( H_f^2 = \{0\} \) for \( 2 \leq k \leq n - 2 \).

In order to describe the spaces \( H_f^{n-1}(\mathbb{K}^n) \) and \( H_f^n(\mathbb{K}^n) \), we look for an integer \( k \in \{1, \ldots, n-1\} \) such that \( kN - \sum w_i = \deg 1 \), that is, \( 2k - n = 0 \).

Therefore,

- (a) if \( n \) is even, then \( \{\omega, f^{n/2} \omega\} \) is a basis of \( H_f^n(\mathbb{K}^n) \) and \( H_f^{n-1}(\mathbb{K}^n) \) is spanned by \( \{f^{n/2-1} \sigma\} \),
- (b) if \( n \) is odd, then \( H_f^{n-1}(\mathbb{K}^n) = \{0\} \) and the space \( H_f^n(\mathbb{K}^n) \) is spanned by \( \{\omega\} \).

We recall that \( \omega = dx_1 \wedge \cdots \wedge dx_n \) and

\[
\sigma = i_{\hat{\omega}} \omega = \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \hat{dx}_i \wedge \cdots \wedge dx_n. \tag{4.3}
\]

(3) The case \( A_2 \) with \( n = 3 : f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \).

Here, \( w_1 = 2, w_2 = w_3 = 3 \), and \( n = 6 \). Thus, \( N - \sum w_i = -2, 2N - \sum w_i = 4 \), and \( 3N - \sum w_i = 10 \).

Moreover, \( \mathcal{B} = \{1, x_1\} \) is a monomial basis of \( Q_f \). But as \( \deg 1 = 0 \) and \( \deg x_1 = 3 \), we have

\[
H_f^0(\mathbb{K}^3) \cong \mathbb{K}, \quad H_f^1(\mathbb{K}^3) = \mathbb{K} \cdot (3x_1 \, dx_1 + 2x_2 \, dx_2 + 2x_3 \, dx_3), \quad H_f^3(\mathbb{K}^3) = \{0\}. \tag{4.4}
\]

(4) The case \( D_5 \) with \( n = 4 : f(x_1, x_2, x_3, x_4) = x_1^2 x_2 + x_2^4 + x_3^2 + x_4^2 \).

We have \( w_1 = 3, w_2 = 2, w_3 = w_4 = 4 \), and \( N = 8 \), then \( N - \sum w_i = -5, 2N - \sum w_i = 3, 3N - \sum w_i = 11, \) and \( 4N - \sum w_i = 19 \).

Now, \( \mathcal{B} = \{1, x_1, x_2, x_2^2, x_3^2\} \) is a monomial basis of \( Q_f \). Here, \( \deg 1 = 0, \deg x_1 = 3, \deg x_2 = 2, \deg x_2^2 = 4, \) and \( \deg x_3^2 = 6 \). Thus, the only element of \( \mathcal{B} \) whose degree is of type \( kN - \sum w_i \) is \( x_1 \).

Consequently,

\[
H_f^0(\mathbb{K}^4) \cong \mathbb{K}, \quad H_f^1(\mathbb{K}^4) = \mathbb{K} \cdot (2x_1 x_2 \, dx_1 + (x_1^2 + 4x_2^2) \, dx_2 + 2x_3 \, dx_3 + 2x_4 \, dx_4), \quad H_f^2(\mathbb{K}^4) = \{0\}, \quad H_f^3(\mathbb{K}^4) = \mathbb{K} \cdot (x_1 \, \sigma), \tag{4.5}
\]

and \( \{\omega, x_1 \omega, x_2 \omega, x_2^2 \omega, x_3 \omega, x_4 \omega, x_1 f \omega\} \) is a basis of \( H_f^4(\mathbb{K}^4) \).
Here, we have $W = 3x_1(\partial/\partial x_1) + 2x_2(\partial/\partial x_2) + 4x_3(\partial/\partial x_3) + 4x_4(\partial/\partial x_4)$ and
\[
\sigma = 3x_1 \, dx_2 \wedge dx_3 \wedge dx_4 - 2x_2 \, dx_1 \wedge dx_3 \wedge dx_4 \\
+ 4x_3 \, dx_1 \wedge dx_2 \wedge dx_4 - 4x_4 \, dx_1 \wedge dx_2 \wedge dx_3.
\] (4.6)

References


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