ON n-FOLD FUZZY POSITIVE IMPLICATIVE IDEALS OF BCK-ALGEBRAS

YOUNG BAE JUN and KYUNG HO KIM

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ABSTRACT. We consider the fuzzification of the notion of an n-fold positive implicative ideal. We give characterizations of an n-fold fuzzy positive implicative ideal. We establish the extension property for n-fold fuzzy positive implicative ideals, and state a characterization of PIn-Noetherian BCK-algebras. Finally we study the normalization of n-fold fuzzy positive implicative ideals.

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1. Introduction. For the general development of BCK-algebras, the ideal theory plays an important role. In 1999, Huang and Chen [1] introduced the notion of n-fold positive implicative ideals in BCK-algebras. In this paper, we consider the fuzzification of n-fold positive implicative ideals in BCK-algebras. We first define the notion of n-fold fuzzy positive implicative ideals of BCK-algebras, and then discuss the related properties. We give the relation between a fuzzy ideal and an n-fold fuzzy positive implicative ideal. We state a condition for a fuzzy ideal to be an n-fold fuzzy positive implicative ideal. Using level sets, we give a characterization of an n-fold fuzzy positive implicative ideal. We establish the extension property for an n-fold fuzzy positive implicative ideal. Using a family of n-fold fuzzy positive implicative ideals, we make a new n-fold fuzzy positive implicative ideal. We define the notion of PIn-Noetherian BCK-algebras, and give its characterization. Furthermore, we study the normalization of an n-fold fuzzy positive implicative ideal.

2. Preliminaries. By a BCK-algebra we mean an algebra (X; *, 0) of type (2, 0) satisfying the axioms

(I) ((x * y) * (x * z)) * (z * y) = 0,
(II) (x * (x * y)) * y = 0,
(III) x * x = 0,
(IV) 0 * x = 0,
(V) x * y = 0 and y * x = 0 imply x = y,

for all x, y, z ∈ X. We can define a partial ordering ≤ on X by x ≤ y if and only if x * y = 0. A BCK-algebra X is said to be n-fold positive implicative (see Huang and Chen [1]) if there exists a natural number n such that x * y^{n+1} = x * y^n for all x, y ∈ X.

In any BCK-algebra X, the following hold:

(P1) x * 0 = x,
(P2) x * y ≤ x,
(P3) (x * y) * z = (x * z) * y,
(P4) \((x \ast z) \ast (y \ast z) \leq x \ast y\),

(P5) \(x \leq y\) implies \(x \ast z \leq y \ast z\) and \(z \ast y \leq z \ast x\).

Throughout this paper \(X\) will always mean a BCK-algebra unless otherwise specified.

A nonempty subset \(I\) of \(X\) is called an ideal of \(X\) if it satisfies

(I1) \(0 \in I\),

(I2) \(x \ast y \in I\) and \(y \in I\) imply \(x \in I\).

A nonempty subset \(I\) of \(X\) is said to be a positive implicative ideal if it satisfies

(I1) \(0 \in I\),

(I3) \((x \ast y) \ast z \in I\) and \(y \ast z \in I\) imply \(x \ast z \in I\).

**Theorem 2.1** (see [3, Theorem 3]). A nonempty subset \(I\) of \(X\) is a positive implicative ideal of \(X\) if and only if it satisfies

(I1) \(0 \in I\),

(I4) \(((x \ast y) \ast y) \ast z \in I\) and \(z \in I\) imply \(x \ast y \in I\).

We now review some fuzzy logic concepts. A fuzzy set in a set \(X\) is a function \(\mu : X \to [0,1]\). For a fuzzy set \(\mu\) in \(X\) and \(t \in [0,1]\) define \(U(\mu; t)\) to be the set \(U(\mu; t) = \{x \in X \mid \mu(x) \geq t\}\).

A fuzzy set \(\mu\) in \(X\) is said to be a fuzzy ideal of \(X\) if

(F1) \(\mu(0) \geq \mu(x)\) for all \(x \in X\),

(F2) \(\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}\) for all \(x, y \in X\).

Note that every fuzzy ideal \(\mu\) of \(X\) is order reversing, that is, if \(x \leq y\) then \(\mu(x) \geq \mu(y)\).

A fuzzy set \(\mu\) in \(X\) is called a fuzzy positive implicative ideal of \(X\) if it satisfies

(F1) \(\mu(0) \geq \mu(x)\) for all \(x \in X\),

(F3) \(\mu(x \ast z) \geq \min\{\mu((x \ast y) \ast z), \mu(y \ast z)\}\) for all \(x, y, z \in X\).

**Theorem 2.2** (see [2, Proposition 1]). For any fuzzy ideal \(\mu\) of \(X\), we have

\[
\mu(x \ast y) \geq \mu((x \ast y) \ast y) \iff \mu((x \ast z) \ast (y \ast z)) \geq \mu((x \ast y) \ast z) \quad \forall x, y, z \in X.
\] (2.1)

### 3. \(n\)-fold fuzzy positive implicative ideals

For any elements \(x\) and \(y\) of a BCK-algebra, \(x \ast y^n\) denotes

\[
(\cdots ((x \ast y) \ast y) \ast \cdots) \ast y
\] (3.1)

in which \(y\) occurs \(n\) times. Using **Theorem 2.1**, Huang and Chen [1] introduced the concept of an \(n\)-fold positive implicative ideal as follows.

**Definition 3.1.** A subset \(A\) of \(X\) is called an \(n\)-fold positive implicative ideal of \(X\) if

(I1) \(0 \in A\),

(I5) \(x \ast y^n \in A\) whenever \((x \ast y^{n+1}) \ast z \in A\) and \(z \in A\) for every \(x, y, z \in X\).

We try to fuzzify the concept of \(n\)-fold positive implicative ideal.

**Definition 3.2.** Let \(n\) be a positive integer. A fuzzy set \(\mu\) in \(X\) is called an \(n\)-fold fuzzy positive implicative ideal of \(X\) if

(F1) \(\mu(0) \geq \mu(x)\) for all \(x \in X\),
(F4) \( \mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\} \) for all \( x, y, z \in X \).

Notice that the 1-fold fuzzy positive implicative ideal is a fuzzy positive implicative ideal.

**Example 3.3.** Let \( X = \{0, a, b\} \) be a BCK-algebra with the following Cayley table:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a fuzzy set \( \mu : X \rightarrow [0, 1] \) by \( \mu(0) = t_0, \mu(a) = t_1, \) and \( \mu(b) = t_2 \) where \( t_0 > t_1 > t_2 \) in \([0, 1]\). Then \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) for every natural number \( n \).

**Proposition 3.4.** Every \( n \)-fold fuzzy positive implicative ideal is a fuzzy ideal for every natural number \( n \).

**Proof.** Let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \). Then

\[
\mu(x) = \mu(x * 0^n) = \mu((x * 0^{n+1}) * z), \mu(z) \\
= \min\{\mu(x * z), \mu(z)\} \quad \forall x, z \in X.
\]

(3.2)

Hence \( \mu \) is a fuzzy ideal of \( X \). \( \square \)

The following example shows that the converse of Proposition 3.4 may not be true.

**Example 3.5.** Let \( X = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} \) is the set of natural numbers, in which the operation \( * \) is defined by \( x * y = \max\{0, x - y\} \) for all \( x, y \in X \). Then \( X \) is a BCK-algebra [1, Example 1.3]. Let \( \mu \) be a fuzzy set in \( X \) given by \( \mu(0) = t_0 > t_1 = \mu(x) \) for all \( x (\neq 0) \in X \). Then \( \mu \) is a fuzzy ideal of \( X \). But \( \mu \) is not a 2-fold fuzzy positive implicative ideal of \( X \) because \( \mu(5 * 2^2) = \mu(1) = t_1 \) and \( \mu((5 * 2^3) * 0) = \mu(0) = t_0 \), and so

\[
\mu(5 * 2^2) \neq \min\{\mu((5 * 2^3) * 0), \mu(0)\}.
\]

(3.3)

Let \( X \) be an \( n \)-fold positive implicative BCK-algebra and let \( \mu \) be a fuzzy ideal of \( X \). For any \( x, y, z \in X \) we have

\[
\mu(x * y^n) = \mu(x * y^{n+1}) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\}.
\]

(3.4)

Hence \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). Combining this and Proposition 3.4, we have the following theorem.

**Theorem 3.6.** In an \( n \)-fold positive implicative BCK-algebra, the notion of \( n \)-fold fuzzy positive implicative ideals and fuzzy ideals coincide.

**Proposition 3.7.** Let \( \mu \) be a fuzzy ideal of \( X \). Then \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if it satisfies the inequality \( \mu(x * y^n) \geq \mu(x * y^{n+1}) \) for all \( x, y \in X \).
Suppose that (F2) does not hold. Then there exist $a, b \in X$. Then

$$\mu(x \ast y^n) \geq \min \{\mu((x \ast y^{n+1}) \ast 0), \mu(0)\}$$

$$= \min \{\mu(x \ast y^{n+1}), \mu(0)\}$$

$$= \mu(x \ast y^{n+1}).$$

(3.5)

Conversely, let $\mu$ be a fuzzy ideal of $X$ satisfying the inequality

$$\mu(x \ast y^n) \geq \mu(x \ast y^{n+1}) \quad \forall x, y \in X. \quad (3.6)$$

Then

$$\mu(x \ast y^n) \geq \mu(x \ast y^{n+1}) \geq \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\} \quad \forall x, y, z \in X. \quad (3.7)$$

Hence $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$. □

**Corollary 3.8.** Every $n$-fold fuzzy positive implicative ideal $\mu$ of $X$ satisfies the inequality $\mu(x \ast y^n) \geq \mu(x \ast y^{n+k})$ for all $x, y \in X$ and $k \in \mathbb{N}$.

**Proof.** Using Proposition 3.7, the proof is straightforward by induction. □

**Lemma 3.9.** Let $A$ be a nonempty subset of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise}, \end{cases} \quad (3.8)$$

where $t_1 > t_2$ in $[0, 1]$. Then $\mu$ is a fuzzy ideal of $X$ if and only if $A$ is an ideal of $X$.

**Proof.** Let $A$ be an ideal of $X$. Since $0 \in A$, therefore $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. Suppose that (F2) does not hold. Then there exist $a, b \in X$ such that $\mu(a) = t_2$ and $\min\{\mu(a \ast b), \mu(b)\} = t_1$. Thus $\mu(a \ast b) = t_1 = \mu(b)$, and so $a \ast b \in A$ and $b \in A$. It follows from (I2) that $a \in A$ so that $\mu(a) = t_1$. This is a contradiction. Suppose that $\mu$ is a fuzzy ideal of $X$. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) = t_1$ and hence $0 \in A$. Let $x, y \in X$ be such that $x \ast y \in A$ and $y \in A$. Using (F2), we get $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} = t_1$ and so $\mu(x) = t_1$, that is, $x \in A$. Consequently, $A$ is an ideal of $X$. □

**Proposition 3.10.** Let $A$ be a nonempty subset of $X$, $n$ a positive integer, and $\mu$ a fuzzy set in $X$ defined as follows:

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A, \\ t_2 & \text{otherwise}, \end{cases} \quad (3.9)$$

where $t_1 > t_2$ in $[0, 1]$. Then $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if $A$ is an $n$-fold positive implicative ideal of $X$.

**Proof.** Assume that $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$. Then $\mu$ is a fuzzy ideal of $X$. It follows from Lemma 3.9 that $A$ is an ideal of $X$. Let $x, y \in X$ be such that $x \ast y^{n+1} \in A$. Using Proposition 3.7, we get $\mu(x \ast y^n) \geq \mu(x \ast y^{n+1}) = t_1$ and so
\[ \mu(x \ast y^n) = t_1, \]  
that is, \( x \ast y^n \in A \). Hence by [1, Theorem 1.5], we conclude that \( A \) is an \( n \)-fold positive implicative ideal of \( X \). Conversely, suppose that \( A \) is an \( n \)-fold positive implicative ideal of \( X \). Then \( A \) is an ideal of \( X \) (see [1, Proposition 1.2]). It follows from Lemma 3.9 that \( \mu \) is a fuzzy ideal of \( X \). For any \( x, y \in X \), either \( x \ast y^n \in A \) or \( x \ast y^n \notin A \). The former induces \( \mu(x \ast y^n) = t_1 \geq \mu(x \ast y^{n+1}) \). In the latter, we know that \( x \ast y^{n+1} \notin A \) by [1, Theorem 1.5]. Hence \( \mu(x \ast y^n) = t_2 = \mu(x \ast y^{n+1}) \). From Proposition 3.7 it follows that \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). \( \square \)

**Proposition 3.11.** A fuzzy set \( \mu \) in \( X \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if it satisfies

\begin{align*}
& (F1) \quad \mu(0) \geq \mu(x), \\
& (F5) \quad \mu(x \ast z^n) \geq \min\{\mu((x \ast y) \ast z^n), \mu(y \ast z^n)\}, \text{ for all } x, y, z \in X.
\end{align*}

**Proof.** Suppose that \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) and let \( x, y, z \in X \). Then \( \mu \) is a fuzzy ideal of \( X \) (see Proposition 3.4), and so \( \mu \) is order reversing. It follows from (P3), (P4), and (P5) that

\[ \mu((x \ast z^n) \ast (y \ast z^n)) = \mu(((x \ast z^n) \ast (y \ast z^n)) \ast z^n) \geq \mu((x \ast y) \ast z^n). \] (3.10)

Using (F2) and Corollary 3.8, we get

\[ \mu(x \ast z^n) \geq \mu(x \ast z^{2n}) \geq \min\{\mu((x \ast z^n) \ast (y \ast z^n)), \mu(y \ast z^n)\} \]

\[ \geq \min\{\mu((x \ast y) \ast z^n), \mu(y \ast z^n)\}, \] (3.11)

which proves (F5). Conversely, assume that \( \mu \) satisfies conditions (F1) and (F5). Taking \( z = 0 \) in (F5) and using (P1), we conclude that

\[ \mu(x) = \mu(x \ast 0) \geq \min\{\mu((x \ast y) \ast 0^n), \mu(y \ast 0^n)\} = \min\{\mu(x \ast y), \mu(y)\}. \] (3.12)

Hence \( \mu \) is a fuzzy ideal of \( X \). Putting \( z = y \) in (F5) and applying (III), (IV), and (F1), we have

\[ \mu(x \ast y^n) \geq \min\{\mu((x \ast y) \ast y^n), \mu(y \ast y^n)\} \]

\[ = \min\{\mu(x \ast y^{n+1}), \mu(0)\} = \mu(x \ast y^{n+1}). \] (3.13)

By Proposition 3.7, we know that \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). \( \square \)

Now we give a condition for a fuzzy ideal to be an \( n \)-fold fuzzy positive implicative ideal.

**Theorem 3.12.** A fuzzy set \( \mu \) in \( X \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if \( \mu \) is a fuzzy ideal of \( X \) in which the following inequality holds:

\[ (F6) \quad \mu((x \ast z^n) \ast (y \ast z^n)) \geq \mu((x \ast y) \ast z^n) \text{ for all } x, y, z \in X. \]

**Proof.** Assume that \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). By Proposition 3.4, it follows that \( \mu \) is a fuzzy ideal of \( X \). Let \( a = x \ast (y \ast z^n) \) and \( b = x \ast y \). Then

\[ \mu((a \ast b) \ast z^n) = \mu(((x \ast (y \ast z^n)) \ast (x \ast y)) \ast z^n) \]

\[ \geq \mu((y \ast (y \ast z^n)) \ast z^n) = \mu(0), \] (3.14)
and so $\mu((a * b) * z^n) = \mu(0)$. Using (F5) we obtain
\[
\mu((x * z^n) * (y * z^n)) = \mu((x * (y * z^n)) * z^n) = \mu(a * z^n)
\geq \min\{\mu((a * b) * z^n), \mu(b * z^n)\}
= \min\{\mu(0), \mu(b * z^n)\}
= \mu(b * z^n) = \mu((x * y) * z^n),
\]
which is condition (F6). Conversely, let $\mu$ be a fuzzy ideal of $X$ satisfying condition (F6). It is sufficient to show that $\mu$ satisfies condition (F5). For any $x, y, z \in X$ we have
\[
\mu(x * z^n) \geq \min\{\mu((x * z^n) * (y * z^n)), \mu(y * z^n)\}
\geq \min\{\mu((x * y) * z^n), \mu(y * z^n)\},
\]
which is precisely (F5). Hence $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$. \hfill \Box

**Theorem 3.13.** Let $\mu$ be a fuzzy set in $X$ and let $n$ be a positive integer. Then $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$ if and only if the nonempty level set $U(\mu; t)$ of $\mu$ is an $n$-fold positive implicative ideal of $X$ for every $t \in [0, 1]$.

**Proof.** Assume that $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$ and $U(\mu; t) \neq \emptyset$ for every $t \in [0, 1]$. Then there exists $x \in U(\mu; t)$. It follows from (F1) that $\mu(0) \geq \mu(x) \geq t$ so that $0 \in U(\mu; t)$. Let $x, y, z \in X$ be such that $(x * y^{n+1}) * z \in U(\mu; t)$ and $z \in U(\mu; t)$. Then $\mu((x * y^{n+1}) * z) \geq t$ and $\mu(z) \geq t$, which imply from (F4) that
\[
\mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\} \geq t,
\]
so that $x * y^n \in U(\mu; t)$. Therefore $U(\mu; t)$ is an $n$-fold positive implicative ideal of $X$. Conversely, suppose that $U(\mu; t) \neq \emptyset$ is an $n$-fold positive implicative ideal of $X$ for every $t \in [0, 1]$. For any $x \in X$, let $\mu(x) = t$. Then $x \in U(\mu; t)$. Since $0 \in U(\mu; t)$, we get $\mu(0) \geq t = \mu(x)$ and so $\mu(0) \geq \mu(x)$ for all $x \in X$. Now assume that there exist $a, b, c \in X$ such that $\mu(a * b^n) < \min\{\mu((a * b^{n+1}) * c), \mu(c)\}$. Selecting $s_0 = (1/2)(\mu(a * b^n) + \min(\mu((a * b^{n+1}) * c), \mu(c)))$, then
\[
\mu(a * b^n) < s_0 < \min\{\mu((a * b^{n+1}) * c), \mu(c)\}.
\]
It follows that $(a * b^{n+1}) * c \in U(\mu; s_0), c \in U(\mu; s_0)$, and $a * b^n \notin U(\mu; s_0)$. This is a contradiction. Hence $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$. \hfill \Box

**Theorem 3.14.** If $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$, then the set
\[
X_\mu := \{x \in X \mid \mu(x) = \mu(0)\}
\]
is an $n$-fold positive implicative ideal of $X$.

**Proof.** Let $\mu$ be an $n$-fold fuzzy positive implicative ideal of $X$. Clearly $0 \in X_\mu$. Let $x, y, z \in X$ be such that $(x * y^{n+1}) * z \in X_\mu$ and $z \in X_\mu$. Then
\[
\mu(x * y^n) \geq \min\{\mu((x * y^{n+1}) * z), \mu(z)\} = \mu(0).
\]
It follows from (F1) that $\mu(x * y^n) = \mu(0)$ so that $x * y^n \in X_\mu$. Hence $X_\mu$ is an $n$-fold positive implicative ideal of $X$. \hfill \Box
\textbf{Theorem 3.15} (extension property for n-fold fuzzy positive implicative ideals). Let \( \mu \) and \( \nu \) be fuzzy ideals of \( X \) such that \( \mu(0) = \nu(0) \) and \( \mu \equiv \nu \), that is, \( \mu(x) \leq \nu(x) \) for all \( x \in X \). If \( \mu \) is an n-fold fuzzy positive implicative ideal of \( X \), then so is \( \nu \).

\textbf{Proof.} Using Proposition 3.7, it is sufficient to show that \( \nu \) satisfies the inequality \( \nu(x^*y^n) \geq \nu(x^*y^{n+1}) \) for all \( x,y \in X \). Let \( x,y \in X \). Then

\[
\nu(0) = \mu(0) = \mu((x^*y^n)^n) \leq \mu((x^*y^{n+1})^n) = \mu((x^*y^n)\cdot (x^*y^{n+1})) \leq \nu((x^*y^n)\cdot (x^*y^{n+1})).
\]

(3.21)

Since \( \nu \) is a fuzzy ideal, it follows from (F1) and (F2) that

\[
\nu(x^*y^n) \geq \min \{ \nu((x^*y^n)\cdot (x^*y^{n+1})), \nu(x^*y^{n+1}) \}
\]

\[
\geq \min \{ \nu(0), \nu(x^*y^{n+1}) \} = \nu(x^*y^{n+1}).
\]

(3.22)

This completes the proof.

\textbf{4. PI\textsuperscript{n}.Noetherian BCK-algebras}

\textbf{Definition 4.1.} A BCK-algebra \( X \) is said to satisfy the PI\textsuperscript{n}.ascending (resp., PI\textsuperscript{n}.descending) chain condition (briefly, PI\textsuperscript{n}.ACC (resp., PI\textsuperscript{n}.DCC)) if for every ascending (resp., descending) sequence \( A_1 \subseteq A_2 \subseteq \cdots \) (resp., \( A_1 \supseteq A_2 \supseteq \cdots \)) of n-fold positive implicative ideals of \( X \) there exists a natural number \( r \) such that \( A_r = A_k \) for all \( r \geq k \).

If \( X \) satisfies the PI\textsuperscript{n}.ACC, we say that \( X \) is a PI\textsuperscript{n}.Noetherian BCK-algebra.

\textbf{Theorem 4.2.} Let \( \{ A_k \mid k \in \mathbb{N} \} \) be a family of n-fold positive implicative ideals of \( X \) which is nested, that is, \( A_1 \supseteq A_2 \supseteq \cdots \). Let \( \mu \) be a fuzzy set in \( X \) defined by

\[
\mu(x) = \begin{cases} 
\frac{k}{k+1} & \text{if } x \in A_k \setminus A_{k+1}, \; k = 0,1,2,\ldots, \\
1 & \text{if } x \in \cap_{k=0}^\infty A_k,
\end{cases}
\]

(4.1)

for all \( x \in X \), where \( A_0 \) stands for \( X \). Then \( \mu \) is an n-fold fuzzy positive implicative ideal of \( X \).

\textbf{Proof.} Clearly \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Let \( x,y,z \in X \). Suppose that

\[
(x^*y^{n+1})z \in A_k \setminus A_{k+1}, \quad z \in A_r \setminus A_{r+1}
\]

(4.2)

for \( k = 0,1,2,\ldots, r = 0,1,2,\ldots \). Without loss of generality, we may assume that \( k \leq r \). Then obviously \( z \in A_k \). Since \( A_k \) is an n-fold positive implicative ideal, it follows that \( x^*y^n \in A_k \) so that

\[
\mu(x^*y^n) \geq \frac{k}{k+1} = \min \{ \mu((x^*y^{n+1})z), \mu(z) \}.
\]

(4.3)

If \( (x^*y^{n+1})z \in \cap_{k=0}^\infty A_k \) and \( z \in \cap_{k=0}^\infty A_k \), then \( x^*y^n \in \cap_{k=0}^\infty A_k \). Hence

\[
\mu(x^*y^n) = 1 = \min \{ \mu((x^*y^{n+1})z), \mu(z) \}.
\]

(4.4)
If \((x * y^{n+1}) * z \notin \cap_{k=0}^{\infty} A_k\) and \(z \in \cap_{k=0}^{\infty} A_k\), then there exists \(i \in \mathbb{N}\) such that \((x * y^{n+1}) * z \in A_i \setminus A_{i+1}\). It follows that \(x * y^n \in A_i\) so that
\[
\mu(x * y^n) \geq \frac{i}{i+1} = \min \{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.5}
\]

Finally, assume that \((x * y^{n+1}) * z \in \cap_{k=0}^{\infty} A_k\) and \(z \notin \cap_{k=0}^{\infty} A_k\). Then \(z \in A_j \setminus A_{j+1}\) for some \(j \in \mathbb{N}\). Hence \(x * y^n \in A_j\), and thus
\[
\mu(x * y^n) \geq \frac{j}{j+1} = \min \{\mu((x * y^{n+1}) * z), \mu(z)\}. \tag{4.6}
\]

Consequently, \(\mu\) is an \(n\)-fold fuzzy positive implicative ideal of \(X\).

**Theorem 4.2** tells that if every \(n\)-fold fuzzy positive implicative ideal of \(X\) has a finite number of values, then \(X\) satisfies the PI\(^n\)-DCC.

Now we consider the converse of **Theorem 4.2**.

**Theorem 4.3.** Let \(X\) be a BCK-algebra satisfying PI\(^n\)-DCC and let \(\mu\) be an \(n\)-fold fuzzy positive implicative ideal of \(X\). If a sequence of elements of \(\text{Im}(\mu)\) is strictly increasing, then \(\mu\) has a finite number of values.

**Proof.** Let \(\{t_k\}\) be a strictly increasing sequence of elements of \(\text{Im}(\mu)\). Hence \(0 \leq t_1 < t_2 < \cdots < 1\). Then \(U(\mu;r) := \{x \in X \mid \mu(x) \geq t_r\}\) is an \(n\)-fold positive implicative ideal of \(X\) for all \(r = 2, 3, \ldots\). Let \(x \in U(\mu;r)\). Then \(\mu(x) \geq t_r \geq t_{r-1}\), and so \(x \in U(\mu;r-1)\). Hence \(U(\mu;r) \subseteq U(\mu;r-1)\). Since \(t_{r-1} \in \text{Im}(\mu)\), there exists \(x_{r-1} \in X\) such that \(\mu(x_{r-1}) = t_{r-1}\). It follows that \(x_{r-1} \in U(\mu;r-1)\), but \(x_{r-1} \notin U(\mu;r)\). Thus \(U(\mu;r) \subsetneq U(\mu;r-1)\), and so we obtain a strictly descending sequence
\[
U(\mu;1) \supseteq U(\mu;2) \supseteq U(\mu;3) \supseteq \cdots \tag{4.7}
\]
of \(n\)-fold positive implicative ideals of \(X\) which is not terminating. This contradicts the assumption that \(X\) satisfies the PI\(^n\)-DCC. Consequently, \(\mu\) has a finite number of values.

**Theorem 4.4.** The following are equivalent.

(i) \(X\) is a PI\(^n\)-Noetherian BCK-algebra.

(ii) The set of values of any \(n\)-fold fuzzy positive implicative ideal of \(X\) is a well-ordered subset of \([0, 1]\).

**Proof.** (i)\(\Rightarrow\)(ii). Let \(\mu\) be an \(n\)-fold fuzzy positive implicative ideal of \(X\). Assume that the set of values of \(\mu\) is not a well-ordered subset of \([0, 1]\). Then there exists a strictly decreasing sequence \(\{t_k\}\) such that \(\mu(x_k) = t_k\). It follows that
\[
U(\mu;1) \supseteq U(\mu;2) \supseteq U(\mu;3) \supseteq \cdots \tag{4.8}
\]
is a strictly ascending chain of \(n\)-fold positive implicative ideals of \(X\), where \(U(\mu;r) = \{x \in X \mid \mu(x) \geq t_r\}\) for every \(r = 1, 2, \ldots\). This contradicts the assumption that \(X\) is PI\(^n\)-Noetherian.

(ii)\(\Rightarrow\)(i). Assume that condition (i) is satisfied and \(X\) is not PI\(^n\)-Noetherian. Then there exists a strictly ascending chain
\[
A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \tag{4.9}
\]
of \( n \)-fold positive implicative ideals of \( X \). Let \( A = \bigcup_{k \in \mathbb{N}} A_k \). Then \( A \) is an \( n \)-fold positive implicative ideal of \( X \). Define a fuzzy set \( \nu \) in \( X \) by

\[
\nu(x) = \begin{cases} 
0 & \text{if } x \notin A_k, \\
\frac{1}{r} & \text{where } r = \min \{k \in \mathbb{N} \mid x \in A_k\}.
\end{cases} \tag{4.10}
\]

We claim that \( \nu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). Since \( 0 \in A_k \) for all \( k = 1, 2, \ldots \), we have \( \nu(0) = 1 \geq \nu(x) \) for all \( x \in X \). Let \( x, y, z \in X \). If \( (x \ast y^{n+1}) \ast z \in A_k \setminus A_{k-1} \) and \( z \in A_k \setminus A_{k-1} \) for \( k = 2, 3, \ldots \), then \( x \ast y^n \in A_k \). It follows that

\[
\nu(x \ast y^n) \geq \frac{1}{k} - \min \{ \nu((x \ast y^{n+1}) \ast z), \nu(z) \}. \tag{4.11}
\]

Suppose that \( (x \ast y^{n+1}) \ast z \in A_k \) and \( z \in A_k \setminus A_r \) for all \( r < k \). Since \( A_k \) is an \( n \)-fold positive implicative ideal, it follows that \( x \ast y^n \in A_k \). Hence

\[
\nu(x \ast y^n) \geq \frac{1}{k} - \frac{1}{r + 1} \geq \nu(z), \quad \nu(x \ast y^n) \geq \min \{ \nu((x \ast y^{n+1}) \ast z), \nu(z) \}. \tag{4.12}
\]

Similarly for the case \( (x \ast y^{n+1}) \ast z \in A_k \setminus A_r \) and \( z \in A_k \), we have

\[
\nu(x \ast y^n) \geq \min \{ \nu((x \ast y^{n+1}) \ast z), \nu(z) \}. \tag{4.13}
\]

Thus \( \nu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). Since the chain (4.9) is not terminating, \( \nu \) has a strictly descending sequence of values. This contradicts the assumption that the value set of any \( n \)-fold fuzzy positive implicative ideal is well ordered. Therefore \( X \) is PI\(^n\)-Noetherian. This completes the proof.

We note that a set is well ordered if and only if it does not contain any infinite descending sequence.

**Theorem 4.5.** Let \( S = \{t_k \mid k = 1, 2, \ldots \} \cup \{0\} \) where \( \{t_k\} \) is a strictly descending sequence in \((0, 1)\). Then a BCK-algebra \( X \) is PI\(^n\)-Noetherian if and only if for each \( n \)-fold fuzzy positive implicative ideal \( \mu \) of \( X \), \( \text{Im}(\mu) \subseteq S \) implies that there exists a natural number \( k \) such that \( \text{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\} \).

**Proof.** Assume that \( X \) is a PI\(^n\)-Noetherian BCK-algebra and let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \). Then by Theorem 4.4 we know that \( \text{Im}(\mu) \) is a well-ordered subset of \([0, 1]\) and so the condition is necessary.

Conversely, suppose that the condition is satisfied. Assume that \( X \) is not PI\(^n\)-Noetherian. Then there exists a strictly ascending chain of \( n \)-fold positive implicative ideals

\[
A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \tag{4.14}
\]

Define a fuzzy set \( \mu \) in \( X \) by

\[
\mu(x) = \begin{cases} 
t_1 & \text{if } x \in A_1, \\
t_k & \text{if } x \in A_k \setminus A_{k-1}, \quad k = 2, 3, \ldots, \\
0 & \text{if } x \in X \setminus \bigcup_{k=1}^{n} A_k.
\end{cases} \tag{4.15}
\]
Since $0 \in A_1$, we have $\mu(0) = t_1 \geq \mu(x)$ for all $x \in X$. If either $(x \ast y^{n+1}) \ast z$ or $z$ belongs to $X \setminus \bigcup_{k=1}^{\infty} A_k$, then either $\mu((x \ast y^{n+1}) \ast z)$ or $\mu(z)$ is equal to 0 and hence

$$\mu(x \ast y^n) \geq 0 = \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}. \tag{4.16}$$

If $(x \ast y^{n+1}) \ast z \in A_1$ and $z \in A_1$, then $x \ast y^n \in A_1$ and thus

$$\mu(x \ast y^n) = t_1 \geq \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}. \tag{4.17}$$

If $(x \ast y^{n+1}) \ast z \in A_k \setminus A_{k-1}$ and $z \in A_k \setminus A_{k-1}$, then $x \ast y^n \in A_k$. Hence

$$\mu(x \ast y^n) \geq t_k = \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}. \tag{4.18}$$

Assume that $(x \ast y^{n+1}) \ast z \in A_1$ and $z \in A_1 \setminus A_{k-1}$ for $k = 2, 3, \ldots$. Then $x \ast y^n \in A_k$ and therefore

$$\mu(x \ast y^n) \geq t_k = \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}. \tag{4.19}$$

Similarly for $(x \ast y^{n+1}) \ast z \in A_k \setminus A_{k-1}$ and $z \in A_1, k = 2, 3, \ldots$, we obtain

$$\mu(x \ast y^n) \geq t_k = \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\}. \tag{4.20}$$

Consequently, $\mu$ is an $n$-fold fuzzy positive implicative ideal of $X$. This contradicts our assumption. \hfill \Box

5. Normalizations of $n$-fold fuzzy positive implicative ideals

**Definition 5.1.** An $n$-fold fuzzy positive implicative ideal $\mu$ of $X$ is said to be normal if there exists $x \in X$ such that $\mu(x) = 1$.

**Example 5.2.** Let $\{0, a, b\}$ be a BCK-algebra in Example 3.3. Then the fuzzy set $\mu$ in $X$ defined by $\mu(0) = 1$, $\mu(a) = 0.8$, and $\mu(b) = 0.5$ is a normal $n$-fold fuzzy positive implicative ideal of $X$.

Note that if $\mu$ is a normal $n$-fold fuzzy positive implicative ideal of $X$, then clearly $\mu(0) = 1$, and hence $\mu$ is normal if and only if $\mu(0) = 1$.

**Proposition 5.3.** Given an $n$-fold fuzzy positive implicative ideal $\mu$ of $X$ let $\mu^+$ be a fuzzy set in $X$ defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in X$. Then $\mu^+$ is a normal $n$-fold fuzzy positive implicative ideal of $X$ which contains $\mu$.

**Proof.** We have $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu(x)$ for all $x \in X$. For any $x, y, z \in X$, we have

$$\min \{\mu^+((x \ast y^{n+1}) \ast z), \mu^+(z)\}$$

$$= \min \{\mu((x \ast y^{n+1}) \ast z) + 1 - \mu(0), \mu(z) + 1 - \mu(0)\}$$

$$= \min \{\mu((x \ast y^{n+1}) \ast z), \mu(z)\} + 1 - \mu(0)$$

$$\leq \mu(x \ast y^n) + 1 - \mu(0) = \mu^+(x \ast y^n). \tag{5.1}$$

Hence $\mu^+$ is a normal $n$-fold fuzzy positive implicative ideal of $X$, and obviously $\mu \subseteq \mu^+$. \hfill \Box
Noticing that \( \mu \subseteq \mu^+ \), we have the following corollary.

**Corollary 5.4.** If there is \( x \in X \) such that \( \mu^+(x) = 0 \), then \( \mu(x) = 0 \).

Using Proposition 3.10, we know that for any \( n \)-fold positive implicative ideal \( A \) of \( X \), the characteristic function \( \chi_A \) of \( A \) is a normal \( n \)-fold fuzzy positive implicative ideal of \( X \). It is clear that \( \mu \) is a normal \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if \( \mu^+ = \mu \).

**Proposition 5.5.** If \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \), then \( (\mu^+)^+ = \mu^+ \).

**Proof.** The proof is straightforward.

**Corollary 5.6.** If \( \mu \) is a normal \( n \)-fold fuzzy positive implicative ideal of \( X \), then \( (\mu^+)^+ = \mu \).

**Proposition 5.7.** Let \( \mu \) and \( \nu \) be \( n \)-fold fuzzy positive implicative ideals of \( X \). If \( \mu \subseteq \nu \) and \( \mu(0) = \nu(0) \), then \( X_{\mu} \subseteq X_{\nu} \).

**Proof.** If \( x \in X_{\mu} \), then \( \nu(x) \geq \mu(x) = \mu(0) = \nu(0) \) and so \( \nu(x) = \nu(0) \), that is, \( x \in X_{\nu} \). Therefore \( X_{\mu} \subseteq X_{\nu} \).

**Proposition 5.8.** Let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \). If there is an \( n \)-fold fuzzy positive implicative ideal \( \nu \) of \( X \) satisfying \( \nu^+ \subseteq \mu \), then \( \mu \) is normal.

**Proof.** Assume that there is an \( n \)-fold fuzzy positive implicative ideal \( \nu \) of \( X \) such that \( \nu^+ \subseteq \mu \). Then \( 1 = \nu^+(0) \leq \mu(0) \), and so \( \mu(0) = 1 \). Hence \( \mu \) is normal.

Given an \( n \)-fold fuzzy positive implicative ideal, we construct a new normal \( n \)-fold fuzzy positive implicative ideal.

**Theorem 5.9.** Let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \) and let \( f : [0, \mu(0)] \to [0, 1] \) be an increasing function. Let \( \mu_f : X \to [0, 1] \) be a fuzzy set in \( X \) defined by \( \mu_f(x) = f(\mu(x)) \) for all \( x \in X \). Then \( \mu_f \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). In particular, if \( f(\mu(0)) = 1 \) then \( \mu_f \) is normal; and if \( f(t) \geq t \) for all \( t \in [0, \mu(0)] \), then \( \mu \subseteq \mu_f \).

**Proof.** Since \( \mu(0) \geq \mu(x) \) for all \( x \in X \) and since \( f \) is increasing, we have \( \mu_f(0) = f(\mu(0)) \geq f(\mu(x)) = \mu_f(x) \) for all \( x \in X \). For any \( x, y, z \in X \) we get

\[
\min \{ \mu_f((x \ast y^{n+1}) \ast z), \mu_f(z) \} = \min \{ f(\mu((x \ast y^{n+1}) \ast z)), f(\mu(z)) \}
\]

\[
= f(\min \{ \mu((x \ast y^{n+1}) \ast z), \mu(z) \}) \leq f(\mu(x \ast y^n)) = \mu_f(x \ast y^n).
\]

Hence \( \mu_f \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). If \( f(\mu(0)) = 1 \), then clearly \( \mu_f \) is normal. Assume that \( f(t) \geq t \) for all \( t \in [0, \mu(0)] \). Then \( \mu_f(x) = f(\mu(x)) \geq \mu(x) \) for all \( x \in X \), which proves \( \mu \subseteq \mu_f \).

Let \( N(X) \) denote the set of all normal \( n \)-fold fuzzy positive implicative ideals of \( X \).

**Theorem 5.10.** Let \( \mu \in N(X) \) be nonconstant such that it is a maximal element of the poset \( (N(X), \subseteq) \). Then \( \mu \) takes only the values 0 and 1.
**Proof.** Since \( \mu \) is normal, we have \( \mu(0) = 1 \). Let \( x \in X \) be such that \( \mu(x) \neq 1 \). It is sufficient to show that \( \mu(x) = 0 \). If not, then there exists \( a \in X \) such that \( 0 < \mu(a) < 1 \). Define a fuzzy set \( \nu \) in \( X \) by \( \nu(x) = (1/2) \{ \mu(x) + \mu(a) \} \) for all \( x \in X \). Clearly, \( \nu \) is well defined, and we get

\[
\nu(0) = \frac{1}{2} \{ \mu(0) + \mu(a) \} = \frac{1}{2} \{ 1 + \mu(a) \} \geq \frac{1}{2} \{ \mu(x) + \mu(a) \} = \nu(x) \quad \forall x \in X.
\]

Let \( x, y, z \in X \). Then

\[
\nu(x * y^n) = \frac{1}{2} \{ \mu(x * y^n) + \mu(a) \} \geq \frac{1}{2} \{ \min \{ \mu((x * y^{n+1}) * z), \mu(z) \} + \mu(a) \}
\]

\[
= \min \left\{ \frac{1}{2} \{ \mu((x * y^{n+1}) * z) + \mu(a) \}, \frac{1}{2} \{ \mu(z) + \mu(a) \} \right\}
\]

\[
= \min \{ \nu((x * y^{n+1}) * z), \nu(z) \}.
\]

Hence \( \nu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \). By Proposition 5.3, \( \nu^+ \) is a maximal \( n \)-fold fuzzy positive implicative ideal of \( X \), where \( \nu^+ \) is defined by \( \nu^+(x) = \nu(x) + 1 - \nu(0) \) for all \( x \in X \). Note that

\[
\nu^+(a) = \nu(a) + 1 - \nu(0) = \frac{1}{2} \{ \mu(a) + \mu(a) \} + 1 - \frac{1}{2} \{ \mu(0) + \mu(a) \}
\]

\[
= \frac{1}{2} \{ \mu(a) + 1 \} > \mu(a)
\]

and \( \nu^+(a) < 1 = \nu^+(0) \). It follows that \( \nu^+ \) is nonconstant, and \( \mu \) is not a maximal element of \( (\mathcal{N}(X), \subseteq) \). This is a contradiction. \( \square \)

**Definition 5.11.** An \( n \)-fold fuzzy positive implicative ideal \( \mu \) of \( X \) is said to be fuzzy maximal if \( \mu \) is nonconstant and \( \mu^+ \) is a maximal element of the poset \( (\mathcal{N}(X), \subseteq) \).

For any positive implicative ideal \( I \) of \( X \) let \( \mu_I \) be a fuzzy set in \( X \) defined by

\[
\mu_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise}. \end{cases}
\]

**Theorem 5.12.** Let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \). If \( \mu \) is fuzzy maximal, then

(i) \( \mu \) is normal,

(ii) \( \mu \) takes only the values 0 and 1,

(iii) \( \mu^+ \) is a maximal \( n \)-fold positive implicative ideal of \( X \).

**Proof.** Let \( \mu \) be an \( n \)-fold fuzzy positive implicative ideal of \( X \) which is fuzzy maximal. Then \( \mu^+ \) is a nonconstant maximal element of the poset \( (\mathcal{N}(X), \subseteq) \). It follows from Theorem 5.10 that \( \mu^+ \) takes only the values 0 and 1. Note that \( \mu^+(x) = 1 \) if and only if \( \mu(x) = \mu(0) \), and \( \mu^+(x) = 0 \) if and only if \( \mu(x) = \mu(0) - 1 \). By Corollary 5.4, we have \( \mu(x) = 0 \), and so \( \mu(0) = 1 \). Hence \( \mu \) is normal and \( \mu^+ = \mu \). This proves (i) and (ii).

(iii) Observe that \( \mu \subseteq \mu^+ \). If \( \mu(x) = 0 \), then \( \mu \subseteq \mu^+ \). If \( \mu(x) = 1 \), then \( x \in X \mu \) and so \( \mu^+(x) = 1 \). This shows that \( \mu \subseteq \mu^+ \).
(iv) Since \(\mu\) is nonconstant, \(X_\mu\) is a proper \(n\)-fold positive implicative ideal of \(X\). Let \(J\) be an \(n\)-fold positive implicative ideal of \(X\) containing \(X_\mu\). Then \(\mu = \mu_{X_\mu} \subseteq \mu_J\). Since \(\mu\) and \(\mu_J\) are normal \(n\)-fold fuzzy positive implicative ideals of \(X\) and since \(\mu = \mu^+\) is a maximal element of \(N(X)\), we have that either \(\mu = \mu_J\) or \(\mu_J = 1\) where \(1 : X \to [0, 1]\) is a fuzzy set defined by \(1(x) = 1\) for all \(x \in X\). The later case implies that \(J = X\). If \(\mu = \mu_J\), then \(X_\mu = X_{\mu_J} = J\). This shows that \(X_\mu\) is a maximal \(n\)-fold positive implicative ideal of \(X\). This completes the proof.

\[\square\]

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**References**


*Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Jinju 660-701, Korea*  
*E-mail address: ybjun@nongae.gsnu.ac.kr*

*Kyung Ho Kim: Department of Mathematics, Chungju National University, Chungju 380-702, Korea*  
*E-mail address: gkhim@gukwon.chungju.ac.kr*