ON IMAGINABLE $T$-FUZZY SUBALGEBRAS AND IMAGINABLE $T$-FUZZY CLOSED IDEALS IN BCH-ALGEBRAS

YOUNG BAE JUN and SUNG MIN HONG

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ABSTRACT. We inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

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1. Introduction. In 1983, Hu et al. introduced the notion of a BCH-algebra which is a generalization of a BCK/BCI-algebra (see [6, 7]). In [4], Chaudhry et al. stated ideals and filters in BCH-algebras, and studied their properties. For further properties on BCH-algebras, we refer to [2, 3, 5]. In [8], the first author considered the fuzzification of ideals and filters in BCH-algebras, and then described the relation among fuzzy subalgebras, fuzzy closed ideals and fuzzy filters in BCH-algebras. In this paper, we inquire further into the properties on fuzzy closed ideals. We give a characterization of a fuzzy closed ideal using its level set, and establish some conditions for a fuzzy set to be a fuzzy closed ideal. We describe the fuzzy closed ideal generated by a fuzzy set, and give a characterization of a finite-valued fuzzy closed ideal. Using a $t$-norm $T$, we introduce the notion of (imaginable) $T$-fuzzy subalgebras and (imaginable) $T$-fuzzy closed ideals, and obtain some related results. We give relations between an imaginable $T$-fuzzy subalgebra and an imaginable $T$-fuzzy closed ideal. We discuss the direct product and $T$-product of $T$-fuzzy subalgebras. We show that the family of $T$-fuzzy closed ideals is a completely distributive lattice.

2. Preliminaries. By a BCH-algebra we mean an algebra $(X, \ast, 0)$ of type $(2,0)$ satisfying the following axioms:

(H1) $x \ast x = 0$,
(H2) $x \ast y = 0$ and $y \ast x = 0$ imply $x = y$,
(H3) $(x \ast y) \ast z = (x \ast z) \ast y$,

for all $x, y, z \in X$.

In a BCH-algebra $X$, the following statements hold:
(P1) $x \ast 0 = x$. 

(P2) $x \ast 0 = 0$ implies $x = 0$.
(P3) $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$.

A nonempty subset $A$ of a BCH-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in A$ whenever $x, y \in A$. A nonempty subset $A$ of a BCH-algebra $X$ is called a closed ideal of $X$ if

(i) $0 \ast x \in A$ for all $x \in A$,
(ii) $x \ast y \in A$ and $y \in A$ imply that $x \in A$.

In what follows, let $X$ denote a BCH-algebra unless otherwise specified. A fuzzy set in $X$ is a function $\mu : X \rightarrow [0, 1]$. Let $\mu$ be a fuzzy set in $X$. For $\alpha \in [0, 1]$, the set $U(\mu; \alpha) = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level set of $\mu$.

A fuzzy set $\mu$ in $X$ is called a fuzzy subalgebra of $X$ if

$$\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in X.$$  \hspace{1cm} (2.1)

**Definition 2.1** (see [1]). By a $t$-norm $T$ on $[0, 1]$, we mean a function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$, for all $x, y, z \in [0, 1]$.

In what follows, let $T$ denote a $t$-norm on $[0, 1]$ unless otherwise specified. Denote by $\Delta_T$ the set of elements $\alpha \in [0, 1]$ such that $T(\alpha, \alpha) = \alpha$, that is,

$$\Delta_T := \{\alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha\}. \hspace{1cm} (2.2)$$

Note that every $t$-norm $T$ has a useful property:

(P4) $T(\alpha, \beta) \leq \min(\alpha, \beta)$ for all $\alpha, \beta \in [0, 1]$.

3. Fuzzy closed ideals

**Definition 3.1** (see [8]). A fuzzy set $\mu$ in $X$ is called a fuzzy closed ideal of $X$ if

(F1) $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$,
(F2) $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$ for all $x, y \in X$.

**Theorem 3.2.** Let $D$ be a subset of $X$ and let $\mu_D$ be a fuzzy set in $X$ defined by

$$\mu_D(x) = \begin{cases} \alpha_1 & \text{if } x \in D, \\ \alpha_2 & \text{if } x \notin D, \end{cases}$$  \hspace{1cm} (3.1)

for all $x \in X$ and $\alpha_1 > \alpha_2$. Then $\mu_D$ is a fuzzy closed ideal of $X$ if and only if $D$ is a closed ideal of $X$.

**Proof.** Assume that $\mu_D$ is a fuzzy closed ideal of $X$. Let $x \in D$. Then, by (F1), we have $\mu(0 \ast x) \geq \mu(x) = \alpha_1$ and so $\mu(0 \ast x) = \alpha_1$. It follows that $0 \ast x \in D$. Let $x, y \in X$ be such that $x \ast y \in D$ and $y \in D$. Then $\mu_D(x \ast y) = \alpha_1 = \mu_D(y)$, and hence

$$\mu_D(x) \geq \min\{\mu_D(x \ast y), \mu_D(y)\} = \alpha_1.$$  \hspace{1cm} (3.2)

Thus $\mu_D(x) = \alpha_1$, that is, $x \in D$. Therefore $D$ is a closed ideal of $X$. 

Conversely, suppose that \( D \) is a closed ideal of \( X \). Let \( x \in X \). If \( x \in D \), then \( 0 \ast x \in D \) and thus \( \mu_D(0 \ast x) = \alpha_1 = \mu_D(x) \). If \( x \notin D \), then \( \mu_D(x) = \alpha_2 \leq \mu_D(0 \ast x) \). Let \( x, y \in X \). If \( x \ast y \in D \) and \( y \in D \), then \( x \in D \). Hence

\[
\mu_D(x) = \alpha_1 = \min \{ \mu_D(x \ast y), \mu_D(y) \}. \tag{3.3}
\]

If \( x \ast y \notin D \) and \( y \notin D \), then clearly \( \mu_D(x) \geq \min \{ \mu_D(x \ast y), \mu_D(y) \} \). If exactly one of \( x \ast y \) and \( y \) belong to \( D \), then exactly one of \( \mu_D(x \ast y) \) and \( \mu_D(y) \) is equal to \( \alpha_2 \). Therefore, \( \mu_D(x) \geq \alpha_2 = \min \{ \mu_D(x \ast y), \mu_D(y) \} \). Consequently, \( \mu_D \) is a fuzzy closed ideal of \( X \).

Using the notion of level sets, we give a characterization of a fuzzy closed ideal.

**Theorem 3.3.** A fuzzy set \( \mu \) in \( X \) is a fuzzy closed ideal of \( X \) if and only if the nonempty level set \( U(\mu; \alpha) \) of \( \mu \) is a closed ideal of \( X \) for all \( \alpha \in [0, 1] \).

We then call \( U(\mu; \alpha) \) a level closed ideal of \( \mu \).

**Proof.** Assume that \( \mu \) is a fuzzy closed ideal of \( X \) and \( U(\mu; \alpha) \neq \emptyset \) for all \( \alpha \in [0, 1] \). Let \( x \in U(\mu; \alpha) \). Then \( \mu(0 \ast x) \geq \mu(x) \geq \alpha \), and so \( 0 \ast x \in U(\mu; \alpha) \). Let \( x, y \in X \) be such that \( x \ast y \in U(\mu; \alpha) \) and \( y \in U(\mu; \alpha) \). Then

\[
\mu(x) \geq \min \{ \mu(x \ast y), \mu(y) \} \geq \min \{ \alpha, \alpha \} = \alpha, \tag{3.4}
\]

and thus \( x \in U(\mu; \alpha) \). Therefore \( U(\mu; \alpha) \) is a closed ideal of \( X \). Conversely, suppose that \( U(\mu; \alpha) \neq \emptyset \) is a closed ideal of \( X \). If \( \mu(0 \ast a) < \mu(a) \) for some \( a \in X \), then \( \mu(0 \ast a) < \alpha_0 < \mu(a) \) by taking \( \alpha_0 := 1/2 (\mu(0 \ast a) + \mu(a)) \). It follows that \( a \in U(\mu; \alpha_0) \) and \( 0 \ast a \notin U(\mu; \alpha_0) \), which is a contradiction. Hence \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \). Assume that there exist \( x_0, y_0 \in X \) such that

\[
\mu(x_0) < \min \{ \mu(x_0 \ast y_0), \mu(y_0) \}. \tag{3.5}
\]

Taking \( \beta_0 := 1/2 (\mu(x_0) + \min \{ \mu(x_0 \ast y_0), \mu(y_0) \}) \), we get \( \mu(x_0) < \beta_0 < \mu(x_0 \ast y_0) \) and \( \mu(x_0) < \beta_0 < \mu(y_0) \). Thus \( x_0 \ast y_0 \in U(\mu; \beta_0) \) and \( y_0 \in U(\mu; \beta_0) \), but \( x_0 \notin U(\mu; \beta_0) \). This is impossible. Hence \( \mu \) is a fuzzy closed ideal of \( X \). \( \square \)

**Theorem 3.4.** Let \( \mu \) be a fuzzy set in \( X \) and \( \text{Im}(\mu) = \{ \alpha_0, \alpha_1, \ldots, \alpha_n \} \), where \( \alpha_i < \alpha_j \) whenever \( i > j \). Let \( \{ D_k | k = 0, 1, 2, \ldots, n \} \) be a family of closed ideals of \( X \) such that

(i) \( D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n = X \),

(ii) \( \mu(D^+_k) = \alpha_k \), where \( D^+_k = D_k \setminus D_{k-1} \) and \( D_{-1} = \emptyset \) for \( k = 0, 1, \ldots, n \).

Then \( \mu \) is a fuzzy closed ideal of \( X \).

**Proof.** For any \( x \in X \) there exists \( k \in \{ 0, 1, \ldots, n \} \) such that \( x \in D^+_k \). Since \( D_k \) is a closed ideal of \( X \), it follows that \( 0 \ast x \in D_k \). Thus \( \mu(0 \ast x) \geq \alpha_k = \mu(x) \). To prove that \( \mu \) satisfies condition (F2), we discuss the following cases: if \( x \ast y \in D^+_k \) and \( y \in D^+_k \), then \( x \in D_k \) because \( D_k \) is a closed ideal of \( X \). Hence

\[
\mu(x) \geq \alpha_k = \min \{ \mu(x \ast y), \mu(y) \}. \tag{3.6}
\]
If \( x \ast y \notin D^*_{k} \) and \( y \notin D^*_{k} \), then the following four cases arise:

i. \( x \ast y \in X \backslash D_k \) and \( y \in X \backslash D_k \),

ii. \( x \ast y \in D_{k-1} \) and \( y \in D_{k-1} \),

iii. \( x \ast y \in X \backslash D_k \) and \( y \in D_{k-1} \),

iv. \( x \ast y \in D_{k-1} \) and \( y \in X \backslash D_k \).

But, in either case, we know that \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \). If \( x \ast y \in D^*_{k} \) and \( y \notin D^*_{k} \), then either \( y \in D_{k-1} \) or \( y \in X \backslash D_k \). It follows that either \( x \in D_k \) or \( x \in X \backslash D_k \). Thus \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \). Similarly for the case \( x \ast y \notin D^*_{k} \) and \( y \in D^*_{k} \), we have the same result. This completes the proof. \( \square \)

**Theorem 3.5.** Let \( \Lambda \) be a subset of \([0,1]\) and let \( \{D_\lambda \mid \lambda \in \Lambda\} \) be a collection of closed ideals of \( X \) such that

i. \( X = \cup_{\lambda \in \Lambda} D_\lambda \),

ii. \( \alpha \geq \beta \) if and only if \( D_\alpha \subseteq D_\beta \) for all \( \alpha, \beta \in \Lambda \).

Define a fuzzy set \( \mu \) in \( X \) by \( \mu(x) = \sup\{\lambda \in \Lambda \mid x \in D_\lambda\} \) for all \( x \in X \). Then \( \mu \) is a fuzzy closed ideal of \( X \).

**Proof.** Let \( x \in X \). Then there exists \( \alpha_i \in \Lambda \) such that \( x \in D_{\alpha_i} \). It follows that \( 0 \ast x \in D_{\alpha_j} \) for some \( \alpha_j \geq \alpha_i \). Hence

\[
\mu(x) = \sup \{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_i\} \leq \sup \{\alpha_k \in \Lambda \mid \alpha_k \leq \alpha_j\} = \mu(0 \ast x). \quad (3.7)
\]

Let \( x, y \in X \) be such that \( \mu(x \ast y) = m \) and \( \mu(y) = n \), where \( m, n \in [0,1] \). Without loss of generality we may assume that \( m \leq n \). To prove \( \mu \) satisfies condition (F2), we consider the following three cases:

\( (1^*) \lambda \leq m, \quad (2^*) m < \lambda \leq n, \quad (3^*) \lambda > n. \) \quad (3.8)

Case \( (1^*) \) implies that \( x \ast y \in D_\lambda \) and \( y \in D_\lambda \). It follows that \( x \in D_\lambda \) so that

\[
\mu(x) = \sup \{\lambda \in \Lambda \mid x \in D_\lambda\} \geq m = \min\{\mu(x \ast y), \mu(y)\}. \quad (3.9)
\]

For the case \( (2^*) \), we have \( x \ast y \notin D_\lambda \) and \( y \in D_\lambda \). Then either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \), then \( \mu(x) = n \geq \min\{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \), then \( x \in D_\delta - D_\lambda \) for some \( \delta < \lambda \), and so \( \mu(x) > m = \min\{\mu(x \ast y), \mu(y)\} \). Finally, case \( (3^*) \) implies \( x \ast y \notin D_\lambda \) and \( y \notin D_\lambda \). Thus we have that either \( x \in D_\lambda \) or \( x \notin D_\lambda \). If \( x \in D_\lambda \), then obviously \( \mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \). If \( x \notin D_\lambda \), then \( x \in D_\epsilon - D_\lambda \) for some \( \epsilon < \lambda \), and thus \( \mu(x) \geq m = \min\{\mu(x \ast y), \mu(y)\} \). This completes the proof. \( \square \)

Let \( D \) be a subset of \( X \). The least closed ideal of \( X \) containing \( D \) is called the closed ideal generated by \( D \), denoted by \( \langle D \rangle \). Note that if \( C \) and \( D \) are subsets of \( X \) and \( C \subseteq D \), then \( \langle C \rangle \subseteq \langle D \rangle \). Let \( \mu \) be a fuzzy set in \( X \). The least fuzzy closed ideal of \( X \) containing \( \mu \) is called a fuzzy closed ideal of \( X \) generated by \( \mu \), denoted by \( \langle \mu \rangle \).

**Lemma 3.6.** For a fuzzy set \( \mu \) in \( X \), then

\[
\mu(x) = \sup \{\alpha \in [0,1] \mid x \in U(\mu; \alpha)\}, \quad \forall x \in X. \quad (3.10)
\]

**Proof.** Let \( \delta := \sup \{\alpha \in [0,1] \mid x \in U(\mu; \alpha)\} \) and let \( \varepsilon > 0 \) be given. Then \( \delta - \varepsilon < \alpha \) for some \( \alpha \in [0,1] \) such that \( x \in U(\mu; \alpha) \), and so \( \delta - \varepsilon < \mu(x) \). Since \( \varepsilon \) is arbitrary, it
follows that $\mu(x) \geq \delta$. Now let $\mu(x) = \beta$. Then $x \in U(\mu; \beta)$ and hence $\beta \in \{ \alpha \in [0, 1] \mid x \in U(\mu; \alpha) \}$. Therefore

$$\mu(x) = \beta \leq \sup \{ \alpha \in [0, 1] \mid x \in U(\mu; \alpha) \} = \delta,$$  

(3.11)

and consequently $\mu(x) = \delta$, as desired. $\square$

**Theorem 3.7.** Let $\mu$ be a fuzzy set in $X$. Then the fuzzy set $\mu^*$ in $X$ defined by

$$\mu^*(x) = \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}$$  

(3.12)

for all $x \in X$ is the fuzzy closed ideal $(\mu)$ generated by $\mu$.

**Proof.** We first show that $\mu^*$ is a fuzzy closed ideal of $X$. For any $y \in \text{Im}(\mu^*)$, let $y_n = y - 1/n$ for any $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers, and let $x \in U(\mu^*; y)$. Then $\mu^*(x) \geq y$, and so

$$\sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \geq y > y_n,$$  

(3.13)

for all $n \in \mathbb{N}$. Hence there exists $\beta \in [0, 1]$ such that $\beta > y_n$ and $x \in \langle U(\mu; \beta) \rangle$. It follows that $U(\mu; \beta) \subseteq U(\mu; y_n)$ so that $x \in \langle U(\mu; \beta) \rangle \subseteq \langle U(\mu; y_n) \rangle$ for all $n \in \mathbb{N}$. Consequently, $x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$. On the other hand, if $x \in \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, then $y_n \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}$ for any $n \in \mathbb{N}$. Therefore

$$y - \frac{1}{n} = y_n \leq \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} = \mu^*(x),$$  

(3.14)

for all $n \in \mathbb{N}$. Since $n$ is an arbitrary positive integer, it follows that $y \leq \mu^*(x)$ so that $x \in U(\mu^*; y)$. Hence $U(\mu^*; y) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, which is a closed ideal of $X$. Using Theorem 3.3, we know that $\mu^*$ is a fuzzy closed ideal of $X$. We now prove that $\mu^*$ contains $\mu$. For any $x \in X$, let $\beta \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}$. Then $x \in U(\mu; \beta)$ and so $x \in \langle U(\mu; \beta) \rangle$. Thus we get $\beta \in \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}$, and so

$$\{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \subseteq \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \}.$$  

(3.15)

It follows from Lemma 3.6 that

$$\mu(x) = \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} \leq \sup \{ \alpha \in [0, 1] \mid x \in \langle U(\mu; \alpha) \rangle \} = \mu^*(x).$$  

(3.16)

Hence $\mu \subseteq \mu^*$. Finally let $\nu$ be a fuzzy closed ideal of $X$ containing $\mu$ and let $x \in X$. If $\mu^*(x) = 0$, then clearly $\mu^*(x) \leq \nu(x)$. Assume that $\mu^*(x) = \gamma \neq 0$. Then $x \in U(\mu^*; \gamma) = \cap_{n \in \mathbb{N}} \langle U(\mu; y_n) \rangle$, that is, $x \in U(\mu; y_n)$ for all $n \in \mathbb{N}$. It follows that $\nu(x) \geq \mu(x) \geq y_n = y - 1/n$ for all $n \in \mathbb{N}$ so that $\nu(x) \geq \gamma = \mu^*(x)$ since $n$ is arbitrary. This shows that $\mu^* \subseteq \mu$, completing the proof. $\square$

**Definition 3.8.** A fuzzy closed ideal $\mu$ of $X$ is said to be $n$-valued if $\text{Im}(\mu)$ is a finite set of $n$ elements. When no specific $n$ is intended, we call $\mu$ a finite-valued fuzzy closed ideal.
**Theorem 3.9.** Let μ be a fuzzy closed ideal of X. Then μ is finite valued if and only if there exists a finite-valued fuzzy set ν in X which generates μ. In this case, the range sets of μ and ν are identical.

**Proof.** If μ : X → [0, 1] is a finite-valued fuzzy closed ideal of X, then we may choose ν = μ. Conversely, assume that ν : X → [0, 1] is a finite-valued fuzzy set. Let α₁, α₂, ..., αₙ be distinct elements of ν(X) such that α₁ > α₂ > ⋯ > αₙ, and let \( C_i = ν^{-1}(α_i) \) for \( i = 1, 2, ..., n \). Clearly, \( ∪_{i=1}^{j} C_i \subseteq ∪_{i=1}^{k} C_i \) whenever \( j < k \leq n \). Hence if we let \( D_j = (∪_{i=1}^{j} C_i) \), then we have the following chain:

\[
D_1 \subseteq D_2 \subseteq ⋯ \subseteq D_n = X.
\]  

(3.17)

Define a fuzzy set μ : X → [0, 1] as follows:

\[
μ(x) = \begin{cases} 
α_1 & \text{if } x \in D_1, \\
α_j & \text{if } x \in D_j \setminus D_{j-1}.
\end{cases}
\]

(3.18)

We claim that μ is a fuzzy closed ideal of X generated by ν. Clearly μ(0 ⋆ x) ≥ μ(x) for all x ∈ X. Let x, y ∈ X. Then there exist i and j in \{1, 2, ..., n\} such that x ⋆ y ∈ D_i and y ∈ D_j. Without loss of generality, we may assume that i and j are the smallest integers such that i ≥ j, x ⋆ y ∈ D_i, and y ∈ D_j. Since \( D_i \) is a closed ideal of X, it follows from \( D_j \subseteq D_i \) that x ∈ D_i. Hence μ(x) ≥ α_i = min{μ(x ⋆ y), μ(y)}, and so μ is a fuzzy closed ideal of X. If ν(x) = α_j for every x ∈ X, then x ∈ C_j and thus x ∈ D_j. But we have μ(x) ≥ α_j = ν(x). Therefore μ contains ν. Let \( δ : X → [0, 1] \) be a fuzzy closed ideal of X containing ν. Then \( U(ν; α_j) \subseteq U(δ; α_j) \) for every j. Hence \( U(δ; α_j) \), being a closed ideal, contains the closed ideal generated by \( U(ν; α_j) = ∪_{i=1}^{j} C_i \). Consequently, \( D_j \subseteq U(δ; α_j) \). It follows that μ is contained in δ and that μ is generated by ν. Finally, note that |Im(μ)| = n = |Im(ν)|. This completes the proof. □

**Theorem 3.10.** Let \( D_1 \supseteq D_2 \supseteq ⋯ \) be a descending chain of closed ideals of X which terminates at finite step. For a fuzzy closed ideal μ of X, if a sequence of elements of Im(μ) is strictly increasing, then μ is finite valued.

**Proof.** Suppose that μ is infinite valued. Let \{αₙ\} be a strictly increasing sequence of elements of Im(μ). Then 0 ≤ α₁ < α₂ < ⋯ ≤ 1. Note that \( U(μ; α_t) \) is a closed ideal of X for \( t = 1, 2, 3, ⋯ \). Let \( x ∈ U(μ; α_t) \) for \( t = 2, 3, ⋯ \). Then \( μ(x) ≥ α_t > α_{t-1} \), which implies that \( x ∈ U(μ; α_{t-1}) \). Hence \( U(μ; α_t) ⊆ U(μ; α_{t-1}) \) for \( t = 2, 3, ⋯ \). Since \( α_{t-1} ∈ \text{Im}(μ) \), there exists \( x_{t-1} ∈ X \) such that \( μ(x_{t-1}) = α_{t-1} \). It follows that \( x_{t-1} ∈ U(μ; α_{t-1}) \), but \( x_{t-1} ∉ U(μ; α_t) \). Thus \( U(μ; α_t) ∉ U(μ; α_{t-1}) \), and so we obtain a strictly descending chain \( U(μ; α_t) ⊇ U(μ; α_{t-1}) ⊇ ⋯ \) of closed ideals of X which is not terminating. This is impossible and the proof is complete. □

Now we consider the converse of Theorem 3.10.

**Theorem 3.11.** Let μ be a finite-valued fuzzy closed ideal of X. Then every descending chain of closed ideals of X terminates at finite step.
**Proof.** Suppose there exists a strictly descending chain \( D_0 \supseteq D_1 \supseteq D_2 \supseteq \cdots \) of closed ideals of \( X \) which does not terminate at finite step. Define a fuzzy set \( \mu \) in \( X \) by

\[
\mu(x) = \begin{cases} 
    \frac{n}{n+1} & \text{if } x \in D_n \setminus D_{n+1}, \ n = 0, 1, 2, \ldots, \\
    1 & \text{if } x \in \cap_{n=0}^{\infty} D_n,
\end{cases}
\]  

(3.19)

where \( D_0 \) stands for \( X \). Clearly, \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \). Let \( x, y \in X \). Assume that \( x \ast y \in D_n \setminus D_{n+1} \) and \( y \in D_k \setminus D_{k+1} \) for \( n = 0, 1, 2, \ldots \); \( k = 0, 1, 2, \ldots \). Without loss of generality, we may assume that \( n \leq k \). Then clearly \( y \in D_n \), and so \( x \in D_n \) because \( D_n \) is a closed ideal of \( X \). Hence

\[
\mu(x) \geq \frac{n}{n+1} = \min \{ \mu(x \ast y), \mu(y) \}. \tag{3.20}
\]

If \( x \ast y \in \cap_{n=0}^{\infty} D_n \) and \( y \in \cap_{n=0}^{\infty} D_n \), then \( x \in \cap_{n=0}^{\infty} D_n \). Thus \( \mu(x) = 1 = \min \{ \mu(x \ast y), \mu(y) \} \). If \( x \ast y \notin \cap_{n=0}^{\infty} D_n \) and \( y \in \cap_{n=0}^{\infty} D_n \), then there exists a positive integer \( k \) such that \( x \ast y \in D_k \setminus D_{k+1} \). It follows that \( x \in D_k \) so that

\[
\mu(x) \geq \frac{k}{k+1} = \min \{ \mu(x \ast y), \mu(y) \}. \tag{3.21}
\]

Finally suppose that \( x \ast y \in \cap_{n=0}^{\infty} D_n \) and \( y \notin \cap_{n=0}^{\infty} D_n \). Then \( y \in D_r \setminus D_{r+1} \) for some positive integer \( r \). It follows that \( x \in D_r \), and hence

\[
\mu(x) \geq \frac{r}{r+1} = \min \{ \mu(x \ast y), \mu(y) \}. \tag{3.22}
\]

Consequently, we conclude that \( \mu \) is a fuzzy closed ideal of \( X \) and \( \mu \) has an infinite number of different values. This is a contradiction, and the proof is complete. \( \square \)

**Theorem 3.12.** The following are equivalent:

(i) Every ascending chain of closed ideals of \( X \) terminates at finite step.

(ii) The set of values of any fuzzy closed ideal of \( X \) is a well-ordered subset of \( [0,1] \).

**Proof.** (i)\( \Rightarrow \) (ii). Let \( \mu \) be a fuzzy closed ideal of \( X \). Suppose that the set of values of \( \mu \) is not a well-ordered subset of \( [0,1] \). Then there exists a strictly decreasing sequence \( \{ \alpha_n \} \) such that \( \mu(x_n) = \alpha_n \). It follows that

\[
U(\mu; \alpha_1) \subsetneq U(\mu; \alpha_2) \subsetneq U(\mu; \alpha_3) \subsetneq \cdots \tag{3.23}
\]

is a strictly ascending chain of closed ideals of \( X \). This is impossible.

(ii)\( \Rightarrow \) (i). Assume that there exists a strictly ascending chain

\[
D_1 \subsetneq D_2 \subsetneq D_3 \subsetneq \cdots \tag{3.24}
\]

of closed ideals of \( X \). Note that \( D := \cup_{n \in \mathbb{N}} D_n \) is a closed ideal of \( X \). Define a fuzzy set \( \mu \) in \( X \) by

\[
\mu(x) = \begin{cases} 
    0 & \text{if } x \notin D_n, \\
    \frac{1}{k} & \text{where } k = \min \{ n \in \mathbb{N} \mid x \in D_n \}.
\end{cases} \tag{3.25}
\]
We claim that $\mu$ is a fuzzy closed ideal of $X$. Let $x \in X$. If $x \notin D_n$, then obviously $\mu(0 \ast x) \geq 0 = \mu(x)$. If $x \in D_n \setminus D_{n-1}$ for $n = 2,3,\ldots$, then $0 \ast x \in D_n$. Hence $\mu(0 \ast x) \geq 1/n = \mu(x)$. Let $x,y \in X$. If $x \ast y \in D_n \setminus D_{n-1}$ and $y \in D_n \setminus D_{n-1}$ for $n = 2,3,\ldots$, then $x \in D_n$. It follows that

$$\mu(x) \geq \frac{1}{n} = \min\{\mu(x \ast y), \mu(y)\}. \quad (3.26)$$

Suppose that $x \ast y \in D_n$ and $y \in D_n \setminus D_m$ for all $m < n$. Then $x \in D_n$, and so $\mu(x) \geq 1/n \geq 1/m + 1 \geq \mu(y)$. Hence $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$. Similarly for the case $x \ast y \in D_n \setminus D_m$ and $y \in D_n$, we get $\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\}$. Therefore $\mu$ is a fuzzy closed ideal of $X$. Since the chain (3.24) is not terminating, $\mu$ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy closed ideal is well ordered. This completes the proof.

4. $T$-fuzzy subalgebras and $T$-fuzzy closed ideals

DEFINITION 4.1. A fuzzy set $\mu$ in $X$ is said to satisfy imaginable property if $\text{Im}(\mu) \subseteq \Delta_T$.

DEFINITION 4.2. A fuzzy set $\mu$ in $X$ is called a fuzzy subalgebra of $X$ with respect to a $t$-norm $T$ (briefly, $T$-fuzzy subalgebra of $X$) if $\mu(x \ast y) \geq T(\mu(x), \mu(y))$ for all $x,y \in X$. A $T$-fuzzy subalgebra of $X$ is said to be imaginable if it satisfies the imaginable property.

EXAMPLE 4.3. Let $T_m$ be a $t$-norm defined by $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0,1]$ and let $X = \{0,a,b,c,d\}$ be a BCH-algebra with the following Cayley table:

<table>
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<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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<td>d</td>
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<td>b</td>
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<td>d</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set $\mu : X \rightarrow [0,1]$ by

$$\mu(x) = \begin{cases} 0.9 & \text{if } x \in \{0,d\}, \\ 0.09 & \text{otherwise.} \end{cases} \quad (4.1)$$

Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$, which is not imaginable.

(2) Let $\nu$ be a fuzzy set in $X$ defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{0,d\}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.2)$$

Then $\nu$ is an imaginable $T_m$-fuzzy subalgebra of $X$. 
**Proposition 4.4.** Let $A$ be a subalgebra of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$
\mu(x) := \begin{cases} 
\alpha_1 & \text{if } x \in A, \\
\alpha_2 & \text{otherwise}, 
\end{cases}
$$

(4.3)

for all $x \in X$, where $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 > \alpha_2$. Then $\mu$ is a $T_m$-fuzzy subalgebra of $X$. In particular, if $\alpha_1 = 1$ and $\alpha_2 = 0$ then $\mu$ is an imaginable $T_m$-fuzzy subalgebra of $X$, where $T_m$ is the $t$-norm in Example 4.3.

**Proof.** Let $x, y \in X$. If $x \in A$ and $y \in A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_1) = \max (2\alpha_1 - 1, 0)
$$

(4.4)

$$
= \begin{cases} 
2\alpha_1 - 1 & \text{if } \alpha_1 \geq \frac{1}{2} \\
0 & \text{if } \alpha_1 < \frac{1}{2} 
\end{cases}
$$

$$
\leq \alpha_1 = \mu(x \ast y).
$$

If $x \in A$ and $y \notin A$ (or, $x \notin A$ and $y \in A$) then

$$
T_m(\mu(x), \mu(y)) = T_m(\alpha_1, \alpha_2) = \max (\alpha_1 + \alpha_2 - 1, 0)
$$

(4.5)

$$
= \begin{cases} 
\alpha_1 + \alpha_2 - 1 & \text{if } \alpha_1 + \alpha_2 \geq 1 \\
0 & \text{otherwise} 
\end{cases}
$$

$$
\leq \alpha_2 \leq \mu(x \ast y).
$$

If $x, y \notin A$ then

$$
T_m(\mu(x), \mu(y)) = T_m(2\alpha_2 - 1, 0)
$$

(4.6)

$$
= \begin{cases} 
2\alpha_2 - 1 & \text{if } \alpha_2 \geq \frac{1}{2} \\
0 & \text{if } \alpha_2 < \frac{1}{2} 
\end{cases}
$$

$$
\leq \alpha_2 \leq \mu(x \ast y).
$$

Hence $\mu$ is a $T_m$-fuzzy subalgebra of $X$. Assume that $\alpha_1 = 1$ and $\alpha_2 = 0$. Then

$$
T_m(\alpha_1, \alpha_1) = \max (\alpha_1 + \alpha_1 - 1, 0) = 1 = \alpha_1,
$$

$$
T_m(\alpha_2, \alpha_2) = \max (\alpha_2 + \alpha_2 - 1, 0) = 0 = \alpha_2.
$$

(4.7)

Thus $\alpha_1, \alpha_2 \in \Delta_{T_m}$, that is, $\text{Im}(\mu) \subseteq \Delta_{T_m}$ and so $\mu$ is imaginable. This completes the proof.

**Proposition 4.5.** If $\mu$ is an imaginable $T$-fuzzy subalgebra of $X$, then $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$. 

**Proof.** For any \( x \in X \) we have
\[
\mu(0 \ast x) \geq T(\mu(0), \mu(x)) \\
= T(\mu(x \ast x), \mu(x)) \quad \text{[by (H1)]} \\
\geq T(T(\mu(x), \mu(x)), \mu(x)) \quad \text{[by (T2) and (T3)]} \\
= \mu(x), \quad \text{[since } \mu \text{ satisfies the imaginable property]}.
\]
(4.8)
This completes the proof.

**Theorem 4.6.** Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \) and let \( \alpha \in [0, 1] \) be such that \( T(\alpha, \alpha) = \alpha \). Then \( U(\mu; \alpha) \) is either empty or a subalgebra of \( X \), and moreover \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

**Proof.** Let \( x, y \in U(\mu; \alpha) \). Then
\[
\mu(x \ast y) \geq T(\mu(x), \mu(y)) \geq T(\alpha, \alpha) = \alpha,
\]
(4.9)
which implies that \( x \ast y \in U(\mu; \alpha) \). Hence \( U(\mu; \alpha) \) is a subalgebra of \( X \). Since \( x \ast x = 0 \) for all \( x \in X \), we have \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) = \mu(x) \) for all \( x \in X \).

Since \( T(1, 1) = 1 \), we have the following corollary.

**Corollary 4.7.** If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( U(\mu; 1) \) is either empty or a subalgebra of \( X \).

**Theorem 4.8.** Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If there is a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), then \( \mu(0) = 1 \).

**Proof.** Let \( x \in X \). Then \( \mu(0) = \mu(x \ast x) \geq T(\mu(x), \mu(x)) \). Therefore \( \mu(0) \geq T(\mu(x_n), \mu(x_n)) \) for each \( n \in \mathbb{N} \). Since \( 1 \geq \mu(0) \geq \lim_{n \to \infty} T(\mu(x_n), \mu(x_n)) = 1 \), it follows that \( \mu(0) = 1 \), this completes the proof.

Let \( f : X \to Y \) be a mapping of BCH-algebras. For a fuzzy set \( \mu \) in \( Y \), the inverse image of \( \mu \) under \( f \), denoted by \( f^{-1}(\mu) \), is defined by \( f^{-1}(\mu)(x) = \mu(f(x)) \) for all \( x \in X \).

**Theorem 4.9.** Let \( f : X \to Y \) be a homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( Y \), then \( f^{-1}(\mu) \) is a \( T \)-fuzzy subalgebra of \( X \).

**Proof.** For any \( x, y \in X \), we have
\[
f^{-1}(\mu)(x \ast y) = \mu(f(x \ast y)) = \mu(f(x) \ast f(y)) \\
\geq T(\mu(f(x)), \mu(f(y))) \quad \text{(4.10)} \\
= T(f^{-1}(\mu)(x), f^{-1}(\mu)(y)).
\]
This completes the proof.

If \( \mu \) is a fuzzy set in \( X \) and \( f \) is a mapping defined on \( X \). The fuzzy set \( f(\mu) \) in \( f(X) \) defined by \( f(\mu)(y) = \sup\{\mu(x) \mid x \in f^{-1}(y)\} \) for all \( y \in f(X) \) is called the image of \( \mu \) under \( f \). A fuzzy set \( \mu \) in \( X \) is said to have the **sup property** if, for every subset \( T \subseteq X \), there exists \( t_0 \in T \) such that \( \mu(t_0) = \sup\{\mu(t) \mid t \in T\} \).
**Theorem 4.10.** An onto homomorphic image of a fuzzy subalgebra with sup property is a fuzzy subalgebra.

**Proof.** Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras and let \( \mu \) be a fuzzy subalgebra of \( X \) with sup property. Given \( u,v \in Y \), let \( x_0 \in f^{-1}(u) \) and \( y_0 \in f^{-1}(v) \) be such that
\[
\mu(x_0) = \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \quad \mu(y_0) = \sup \{ \mu(t) \mid t \in f^{-1}(v) \},
\]
respectively. Then
\[
f(\mu)(u \ast v) = \sup \{ \mu(z) \mid z \in f^{-1}(u \ast v) \}
\leq \min \{ \mu(x_0), \mu(y_0) \}
= \min \{ \sup \{ \mu(t) \mid t \in f^{-1}(u) \}, \sup \{ \mu(t) \mid t \in f^{-1}(v) \} \}
= \min \{ f(\mu)(u), f(\mu)(v) \}.
\]
Hence \( f(\mu) \) is a fuzzy subalgebra of \( Y \).

**Theorem 4.10** can be strengthened in the following way. To do this we need the following definition.

**Definition 4.11.** A t-norm \( T \) on \([0,1]\) is called a continuous t-norm if \( T \) is a continuous function from \([0,1] \times [0,1]\) to \([0,1]\) with respect to the usual topology.

Note that the function “min” is a continuous t-norm.

**Theorem 4.12.** Let \( T \) be a continuous t-norm and let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. If \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \), then \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

**Proof.** Let \( A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2) \), and \( A_{12} = f^{-1}(y_1 \ast y_2) \), where \( y_1, y_2 \in Y \). Consider the set
\[
A_1 \ast A_2 := \{ x \in X \mid x = a_1 \ast a_2 \text{ for some } a_1 \in A_1, \ a_2 \in A_2 \}.
\]
If \( x \in A_1 \ast A_2 \), then \( x = x_1 \ast x_2 \) for some \( x_1 \in A_1 \) and \( x_2 \in A_2 \) and so
\[
f(x) = f(x_1 \ast x_2) = f(x_1) \ast f(x_2) = y_1 \ast y_2,
\]
that is, \( x \in f^{-1}(y_1 \ast y_2) = A_{12} \). Thus \( A_1 \ast A_2 \subseteq A_{12} \). It follows that
\[
f(\mu)(y_1 \ast y_2) = \sup \{ \mu(x) \mid x \in f^{-1}(y_1 \ast y_2) \} = \sup \{ \mu(x) \mid x \in A_{12} \}
\geq \sup \{ \mu(x) \mid x \in A_1 \ast A_2 \}
\geq \sup \{ \mu(x_1 \ast x_2) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}.
\]
Since \( T \) is continuous, for every \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that if \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - x_1^* \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - x_2^* \leq \delta \) then
\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(x_1^*, x_2^*) \leq \varepsilon.
\]
Choose \( a_1 \in A_1 \) and \( a_2 \in A_2 \) such that \( \sup \{ \mu(x_1) \mid x_1 \in A_1 \} - \mu(a_1) \leq \delta \) and \( \sup \{ \mu(x_2) \mid x_2 \in A_2 \} - \mu(a_2) \leq \delta \). Then

\[
T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) - T(\mu(a_1), \mu(a_2)) \leq \varepsilon. \tag{4.17}
\]

Consequently

\[
f(\mu)(y_1 \ast y_2) \geq \sup \{ T(\mu(x_1), \mu(x_2)) \mid x_1 \in A_1, x_2 \in A_2 \}
\geq T(\sup \{ \mu(x_1) \mid x_1 \in A_1 \}, \sup \{ \mu(x_2) \mid x_2 \in A_2 \}) \tag{4.18}
= T(f(\mu)(y_1), f(\mu)(y_2)),
\]

which shows that \( f(\mu) \) is a \( T \)-fuzzy subalgebra of \( Y \).

\[\square\]

**Lemma 4.13** (see [1]). For all \( \alpha, \beta, y, \delta \in [0,1] \),

\[
T(T(\alpha, \beta), T(y, \delta)) = T(T(\alpha, y), T(\beta, \delta)). \tag{4.19}
\]

**Theorem 4.14.** Let \( X = X_1 \times X_2 \) be the direct product BCH-algebra of BCH-algebras \( X_1 \) and \( X_2 \). If \( \mu_1 \) (resp., \( \mu_2 \)) is a \( T \)-fuzzy subalgebra of \( X_1 \) (resp., \( X_2 \)), then \( \mu = \mu_1 \times \mu_2 \) is a \( T \)-fuzzy subalgebra of \( X \) defined by

\[
\mu(x_1, x_2) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)), \tag{4.20}
\]

for all \( (x_1, x_2) \in X_1 \times X_2 \).

**Proof.** Let \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) be any elements of \( X = X_1 \times X_2 \). Then

\[
\mu(x \ast y) = \mu(\langle x_1, x_2 \rangle \ast \langle y_1, y_2 \rangle) = \mu(x_1 \ast y_1, x_2 \ast y_2)
= T(\mu_1(x_1 \ast y_1), \mu_2(x_2 \ast y_2))
\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\mu_2(x_2), \mu_2(y_2)))
= T(T(\mu_1(x_1), \mu_2(x_2)), T(\mu_1(y_1), \mu_2(y_2)))
= T(\mu(x_1, x_2), \mu(x_1, x_2))
= T(\mu(x_1, x_2), \mu(y)). \tag{4.21}
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

\[\square\]

We will generalize the idea to the product of \( n \) \( T \)-fuzzy subalgebras. We first need to generalize the domain of \( T \) to \( \prod_{i=1}^{n} [0,1] \) as follows:

**Definition 4.15** (see [1]). The function \( T_n : \prod_{i=1}^{n} [0,1] \to [0,1] \) is defined by

\[
T_n(\alpha_1, \alpha_2, \ldots, \alpha_n) = T(\alpha_i, T_{n-1}(\alpha_{i-1}, \alpha_{i-1}, \ldots, \alpha_{i+n})), \tag{4.22}
\]

for all \( 1 \leq i \leq n \), where \( n \geq 2 \), \( T_2 = T \), and \( T_1 = \text{id} \) (identity).

**Lemma 4.16** (see [1]). For every \( \alpha_i, \beta_i \in [0,1] \) where \( 1 \leq i \leq n \) and \( n \geq 2 \),

\[
T_n(T(\alpha_1, \beta_1), T(\alpha_2, \beta_2), \ldots, T(\alpha_n, \beta_n)) = T(T_n(\alpha_1, \alpha_2, \ldots, \alpha_n), T_n(\beta_1, \beta_2, \ldots, \beta_n)). \tag{4.23}
\]
THEOREM 4.17. Let \( \{X_i\}_{i=1}^n \) be the finite collection of BCH-algebras and \( X = \prod_{i=1}^n X_i \) the direct product BCH-algebra of \( \{X_i\} \). Let \( \mu_i \) be a \( T \)-fuzzy subalgebra of \( X_i \), where \( 1 \leq i \leq n \). Then \( \mu = \prod_{i=1}^n \mu_i \) defined by

\[
\mu(x_1, x_2, \ldots, x_n) = \left( \prod_{i=1}^n \mu_i \right)(x_1, x_2, \ldots, x_n)
\]

is a \( T \)-fuzzy subalgebra of the BCH-algebra \( X \).

PROOF. Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be any elements of \( X = \prod_{i=1}^n X_i \). Then

\[
\mu(x \ast y) = \mu(x_1 \ast y_1, x_2 \ast y_2, \ldots, x_n \ast y_n)
\]

\[
= T_n(\mu_1(x_1 \ast y_1), \mu_2(x_2 \ast y_2), \ldots, \mu_n(x_n \ast y_n))
\]

\[
\geq T_n(T(\mu_1(x_1),\mu_1(y_1)), T(\mu_2(x_2),\mu_2(y_2)), \ldots, T(\mu_n(x_n),\mu_n(y_n)))
\]

\[
= T(T_n(\mu_1(x_1),\mu_2(x_2), \ldots, \mu_n(x_n)), T_n(\mu_1(y_1),\mu_2(y_2), \ldots, \mu_n(y_n)))
\]

\[
= T(\mu(x_1, x_2, \ldots, x_n), \mu(y_1, y_2, \ldots, y_n))
\]

Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

DEFINITION 4.18. Let \( \mu \) and \( \nu \) be fuzzy sets in \( X \). Then the \( T \)-product of \( \mu \) and \( \nu \), written \( [\mu \cdot \nu]_T \), is defined by \( [\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x)) \) for all \( x \in X \).

THEOREM 4.19. Let \( \mu \) and \( \nu \) be \( T \)-fuzzy subalgebras of \( X \). If \( T^* \) is a \( t \)-norm which dominates \( T \), that is,

\[
T^*(T(\alpha, \beta), T(\gamma, \delta)) \geq T(T^*(\alpha, \gamma), T^*(\beta, \delta)),
\]

for all \( \alpha, \beta, \gamma, \delta \in [0, 1] \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \( [\mu \cdot \nu]_{T^*} \), is a \( T \)-fuzzy subalgebra of \( X \).

PROOF. For any \( x, y \in X \) we have

\[
[[\mu \cdot \nu]_{T^*}(x \ast y) = T^*(\mu(x \ast y), \nu(x \ast y))
\]

\[
\geq T^*(T(\mu(x), \mu(y)), T(\nu(x), \nu(y)))
\]

\[
\geq T(T^*(\mu(x), \nu(x)), T^*(\mu(y), \nu(y)))
\]

\[
= T([\mu \cdot \nu]_{T^*}(x), [\mu \cdot \nu]_{T^*}(y)).
\]

Hence \( [\mu \cdot \nu]_{T^*} \) is a \( T \)-fuzzy subalgebra of \( X \). \( \square \)

Let \( f : X \to Y \) be an onto homomorphism of BCH-algebras. Let \( T \) and \( T^* \) be \( t \)-norms such that \( T^* \) dominates \( T \). If \( \mu \) and \( \nu \) are \( T \)-fuzzy subalgebras of \( Y \), then the \( T^* \)-product of \( \mu \) and \( \nu \), \( [\mu \cdot \nu]_{T^*} \), is a \( T \)-fuzzy subalgebra of \( Y \). Since every onto homomorphic inverse image of a \( T \)-fuzzy subalgebra is a \( T \)-fuzzy subalgebra, the
inverse images $f^{-1}(\mu)$, $f^{-1}(\nu)$, and $f^{-1}([\mu \cdot \nu]_{T^*})$ are $T$-fuzzy subalgebras of $X$. The next theorem provides that the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and the $T^*$-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

**Theorem 4.20.** Let $f : X \rightarrow Y$ be an onto homomorphism of BCH-algebras. Let $T^*$ be a $t$-norm such that $T^*$ dominates $T$. Let $\mu$ and $\nu$ be $T$-fuzzy subalgebras of $Y$. If $[\mu \cdot \nu]_{T^*}$ is the $T^*$-product of $\mu$ and $\nu$ and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ is the $T^*$-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then

$$f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}.$$  \hspace{1cm} (4.28)

**Proof.** For any $x \in X$ we get

$$f^{-1}([\mu \cdot \nu]_{T^*})(x) = [\mu \cdot \nu]_{T^*}(f(x))$$
$$= T^*(\mu(f(x)), \nu(f(x)))$$
$$= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x))$$
$$= [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}(x),$$ \hspace{1cm} (4.29)

This completes the proof. \hfill \Box

**Definition 4.21.** A fuzzy set $\mu$ in $X$ is called a **fuzzy closed ideal** of $X$ under a $t$-norm $T$ (briefly, $T$-fuzzy closed ideal of $X$) if

- (F1) $\mu(0 \ast x) \geq \mu(x)$ for all $x \in X$,
- (F3) $\mu(x) \geq T(\mu(x \ast y), \mu(y))$ for all $x, y \in X$.

A $T$-fuzzy closed ideal of $X$ is said to be **imaginable** if it satisfies the imaginable property.

**Example 4.22.** Let $T_m$ be a $t$-norm in Example 4.3. Consider a BCH-algebra $X = \{0, a, b, c\}$ with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>c</td>
<td>0</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>c</td>
<td>b</td>
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<tr>
<td>b</td>
<td>b</td>
<td>c</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

(1) Define a fuzzy set $\mu : X \rightarrow [0, 1]$ by $\mu(0) = \mu(c) = 0.8$ and $\mu(a) = \mu(b) = 0.3$. Then $\mu$ is a $T_m$-fuzzy closed ideal of $X$ which is not imaginable.

(2) Let $\nu$ be a fuzzy set in $X$ defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{0, c\}, \\ 0 & \text{otherwise.} \end{cases}$$ \hspace{1cm} (4.30)

Then $\nu$ is an imaginable $T_m$-fuzzy closed ideal of $X$.

**Theorem 4.23.** Every imaginable $T$-fuzzy subalgebra satisfying (F3) is an imaginable $T$-fuzzy closed ideal.

**Proof.** Using Proposition 4.5, it is straightforward. \hfill \Box
Proposition 4.24. If \( \mu \) is an imaginable \( T \)-fuzzy closed ideal of \( X \), then \( \mu(0) \geq \mu(x) \) for all \( x \in X \).

Proof. Using (F1), (F3), and (T2), we have
\[
\mu(0) \geq T(\mu(0 \ast x),\mu(x)) \geq T(\mu(x),\mu(x)) = \mu(x)
\]
for all \( x \in X \), completing the proof.

Theorem 4.25. Every \( T \)-fuzzy closed ideal is a \( T \)-fuzzy subalgebra.

Proof. Let \( \mu \) be a \( T \)-fuzzy closed ideal of \( X \) and let \( x,y \in X \). Then
\[
\mu(x \ast y) \geq T(\mu((x \ast y) \ast x),\mu(x)) \quad \text{[by (F3)]}
\]
\[
= T(\mu(x \ast x) \ast y),\mu(x)) \quad \text{[by (H3)]}
\]
\[
= T(\mu(0 \ast y),\mu(x)) \quad \text{[by (H1)]}
\]
\[
\geq T(\mu(x),\mu(y)) \quad \text{[by (F1), (T2), and (T3)].}
\]
Hence \( \mu \) is a \( T \)-fuzzy subalgebra of \( X \).

The converse of Theorem 4.25 may not be true. For example, the \( T_m \)-fuzzy subalgebra \( \mu \) in Example 4.3(1) is not a \( T_m \)-fuzzy closed ideal of \( X \) since
\[
\mu(a) = 0.09 < 0.9 = T_m(\mu(a \ast d),\mu(d)).
\]
We give a condition for a \( T \)-fuzzy subalgebra to be a \( T \)-fuzzy closed ideal.

Theorem 4.26. Let \( \mu \) be a \( T \)-fuzzy subalgebra of \( X \). If \( \mu \) satisfies the imaginable property and the inequality
\[
\mu(x \ast y) \leq \mu(y \ast x) \quad \forall x,y \in X,
\]
then \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

Proof. Let \( \mu \) be an imaginable \( T \)-fuzzy subalgebra of \( X \) which satisfies the inequality
\[
\mu(x \ast y) \leq \mu(y \ast x) \quad \forall x,y \in X.
\]
It follows from Proposition 4.5 that \( \mu(0 \ast x) \geq \mu(x) \) for all \( x \in X \). Let \( x,y \in X \). Then
\[
\mu(x) = \mu(x \ast 0) \geq \mu(0 \ast x) = \mu((y \ast y) \ast x)
\]
\[
= \mu((y \ast x) \ast y) \geq T(\mu(y \ast x),\mu(y)) \geq T(\mu(x \ast y),\mu(y)).
\]
Hence \( \mu \) is a \( T \)-fuzzy closed ideal of \( X \).

Proposition 4.27. Let \( T_m \) be a \( t \)-norm in Example 4.3. Let \( D \) be a closed ideal of \( X \) and let \( \mu \) be a fuzzy set in \( X \) defined by
\[
\mu(x) = \begin{cases} 
\alpha_1 & \text{if } x \in D, \\
\alpha_2 & \text{otherwise},
\end{cases}
\]
for all \( x \in X \).
Similarly we have $\mu(x) \geq \mu(x \ast y)$ for all $x, y \in X$, completing the proof. \hfill \Box

Similarly, we have $\mu(x) = \mu(x \ast y)$ for all $x, y \in X$, completing the proof. \hfill \Box
**Theorem 4.29.** Every imaginable \( T \)-fuzzy closed ideal is a fuzzy closed ideal.

**Proof.** Let \( \mu \) be an imaginable \( T \)-fuzzy closed ideal of \( X \). Then

\[
\mu(x) \geq T(\mu(x \ast y), \mu(y)) \quad \forall x, y \in X.
\]

(4.42)

Since \( \mu \) is imaginable, we have

\[
\min(\mu(x \ast y), \mu(y)) = T(\min(\mu(x \ast y), \mu(y)), \min(\mu(x \ast y), \mu(y)))
\]

\[
\leq T(\mu(x \ast y), \mu(y))
\]

\[
\leq \min(\mu(x \ast y), \mu(y)).
\]

(4.43)

It follows that \( \mu(x) \geq T(\mu(x \ast y), \mu(y)) = \min(\mu(x \ast y), \mu(y)) \) so that \( \mu \) is a fuzzy closed ideal of \( X \).

Combining Theorems 3.3, 4.29, we have the following corollary.

**Corollary 4.30.** If \( \mu \) is an imaginable \( T \)-fuzzy closed ideal of \( X \), then the nonempty level set of \( \mu \) is a closed ideal of \( X \).

Noticing that the fuzzy set \( \mu \) in Example 4.22(1) is a fuzzy closed ideal of \( X \), we know from Example 4.22(1) that there exists a \( t \)-norm such that the converse of Theorem 4.29 may not be true.

**Proposition 4.31.** Every imaginable \( T \)-fuzzy closed ideal is order reversing.

**Proof.** Let \( \mu \) be an imaginable \( T \)-fuzzy closed ideal of \( X \) and let \( x, y \in X \) be such that \( x \leq y \). Using (P4), (T2), Theorem 4.29, Proposition 4.24, and the definition of a fuzzy closed ideal, we get

\[
\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \geq T(\mu(x \ast y), \mu(y))
\]

\[
= T(\mu(0), \mu(y)) \geq T(\mu(y), \mu(y)) = \mu(y).
\]

(4.44)

This completes the proof.

**Proposition 4.32.** Let \( \mu \) be a \( T \)-fuzzy closed ideal of \( X \), where \( T \) is a diagonal \( t \)-norm on \([0, 1] \), that is, \( T(\alpha, \alpha) = \alpha \) for all \( \alpha \in [0, 1] \). If \( (x \ast a) \ast b = 0 \) for all \( a, b, x \in X \), then \( \mu(x) \geq T(\mu(a), \mu(b)) \).

**Proof.** Let \( a, b, x \in X \) be such that \( (x \ast a) \ast b = 0 \). Then

\[
\mu(x) \geq T(\mu(x \ast a), \mu(a))
\]

\[
\geq T(T(\mu((x \ast a) \ast b), \mu(b)), \mu(a))
\]

\[
= T(T(\mu(0), \mu(b)), \mu(a))
\]

\[
\geq T(T(\mu(b), \mu(b)), \mu(a))
\]

\[
= T(\mu(a), \mu(b)),
\]

(4.45)

completing the proof.
COROLLARY 4.33. Let $\mu$ be a $T$-fuzzy closed ideal of $X$, where $T$ is a diagonal $t$-norm on $[0,1]$. If $(\cdots ((x \ast a_1) \ast a_2) \ast \cdots) \ast a_n = 0$ for all $x, a_1, a_2, \ldots, a_n \in X$, then

$$\mu(x) \geq T_n(\mu(a_1), \mu(a_2), \ldots, \mu(a_n)).$$

(4.46)

PROOF. Using induction on $n$, the proof is straightforward. \qed

THEOREM 4.34. There exists a $t$-norm $T$ such that every closed ideal of $X$ can be realized as a level closed ideal of a $T$-fuzzy closed ideal of $X$.

PROOF. Let $D$ be a closed ideal of $X$ and let $\mu$ be a fuzzy set in $X$ defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in D, \\ 0 & \text{otherwise}, \end{cases}$$

(4.47)

where $\alpha \in (0,1)$ is fixed. It is clear that $U(\mu; \alpha) = D$. We will prove that $\mu$ is a $T_m$-fuzzy closed ideal of $X$, where $T_m$ is a $t$-norm in Example 4.3. If $x \in D$, then $0 \ast x \in D$ and so $\mu(0 \ast x) = \alpha = \mu(x)$. If $x \notin D$, then clearly $\mu(x) = 0 \leq \mu(0 \ast x)$. Let $x, y \in X$. If $x \in D$, then $\mu(x) = \alpha \geq T_m(\mu(x \ast y), \mu(y))$. If $x \notin D$, then $x \ast y \notin D$ or $y \notin D$. It follows that $\mu(x) = 0 = T_m(\mu(x \ast y), \mu(y))$. This completes the proof. \qed

For a family $\{\mu_\alpha \mid \alpha \in \Lambda\}$ of fuzzy sets in $X$, define the join $\vee_{\alpha \in \Lambda} \mu_\alpha$ and the meet $\wedge_{\alpha \in \Lambda} \mu_\alpha$ as follows:

$$(\vee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\}, \quad (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x) = \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\},$$

(4.48)

for all $x \in X$, where $\Lambda$ is any index set.

THEOREM 4.35. The family of $T$-fuzzy closed ideals in $X$ is a completely distributive lattice with respect to meet “$\wedge$” and the join “$\vee$”.

PROOF. Since $[0,1]$ is a completely distributive lattice with respect to the usual ordering in $[0,1]$, it is sufficient to show that $\vee_{\alpha \in \Lambda} \mu_\alpha$ and $\wedge_{\alpha \in \Lambda} \mu_\alpha$ are $T$-fuzzy closed ideals of $X$ for a family of $T$-fuzzy closed ideals $\{\mu_\alpha \mid \alpha \in \Lambda\}$. For any $x \in X$, we have

$$(\vee_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \sup \{\mu_\alpha(0 \ast x) \mid \alpha \in \Lambda\}$$

$$\geq \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\}$$

$$= (\vee_{\alpha \in \Lambda} \mu_\alpha)(x),$$

(4.49)

$$(\wedge_{\alpha \in \Lambda} \mu_\alpha)(0 \ast x) = \inf \{\mu_\alpha(0 \ast x) \mid \alpha \in \Lambda\}$$

$$\geq \inf \{\mu_\alpha(x) \mid \alpha \in \Lambda\}$$

$$= (\wedge_{\alpha \in \Lambda} \mu_\alpha)(x).$$

Let $x, y \in X$. Then

$$(\vee_{\alpha \in \Lambda} \mu_\alpha)(x) = \sup \{\mu_\alpha(x) \mid \alpha \in \Lambda\}$$

$$\geq \sup \{T(\mu_\alpha(x \ast y), \mu_\alpha(y)) \mid \alpha \in \Lambda\}$$

$$\geq T(\sup \{\mu_\alpha(x \ast y) \mid \alpha \in \Lambda\}, \sup \{\mu_\alpha(y) \mid \alpha \in \Lambda\})$$

$$= T((\vee_{\alpha \in \Lambda} \mu_\alpha)(x \ast y), (\vee_{\alpha \in \Lambda} \mu_\alpha)(y)).$$
\[(\land_{\alpha \in \Lambda} \mu_{\alpha})(x) = \inf \{ \mu_{\alpha}(x) \mid \alpha \in \Lambda \}\]
\[\geq \inf \{ T(\mu_{\alpha}(x \ast y), \mu_{\alpha}(y)) \mid \alpha \in \Lambda \}\]
\[\geq T(\inf \{ \mu_{\alpha}(x \ast y) \mid \alpha \in \Lambda \}, \inf \{ \mu_{\alpha}(y) \mid \alpha \in \Lambda \})\]
\[= T((\land_{\alpha \in \Lambda} \mu_{\alpha})(x \ast y), (\land_{\alpha \in \Lambda} \mu_{\alpha})(y)).\]

Hence \[\lor_{\alpha \in \Lambda} \mu_{\alpha}\] and \[\land_{\alpha \in \Lambda} \mu_{\alpha}\] are \(T\)-fuzzy closed ideals of \(X\), completing the proof. \(\square\)

5. Conclusions and future works. We inquired into further properties on fuzzy closed ideals in BCH-algebras, and using a \(t\)-norm \(T\), we introduced the notion of (imaginable) \(T\)-fuzzy subalgebras and (imaginable) \(T\)-fuzzy closed ideals, and obtained some related results. Moreover, we discussed the direct product and \(T\)-product of \(T\)-fuzzy subalgebras. We finally showed that the family of \(T\)-fuzzy closed ideals is a completely distributive lattice. These ideas enable us to define the notion of (imaginable) \(T\)-fuzzy filters in BCH-algebras, and to discuss the direct products and \(T\)-products of \(T\)-fuzzy filters. It also gives us possible problems to discuss relations among \(T\)-fuzzy subalgebras, \(T\)-fuzzy closed ideals and \(T\)-fuzzy filters, and to construct the normalizations. We may also use these ideas to introduce the notion of interval-valued fuzzy subalgebras/closed ideals.

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References


Young Bae Jun: Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: ybjun@nongae.gsnu.ac.kr

Sung Min Hong: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea

E-mail address: smhong@nongae.gsnu.ac.kr