ON THE CONSTRUCTIONS OF TITS AND FAULKNER: AN ISOMORPHISM THEOREM

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(Received 19 December 2000)

ABSTRACT. Classification theory guarantees the existence of an isomorphism between any two E_8 's, at least over an algebraically closed field of characteristic 0. The purpose of this paper is to construct for any Jordan algebra J of degree 3 over a field Φ of characteristic $\neq 2, 3$ an explicit isomorphism between the algebra obtained from J by Faulkner's construction and the algebra obtained from the split octonions and J by Tits construction.

2000 Mathematics Subject Classification. 17B60.

Let $J = J(N, 1)$ be a quadratic Jordan algebra with 1 over a field Φ constructed as in [3] from an admissible nondegenerate cubic form N with base point 1. Let

$$E = \left\{ \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \mid \alpha, \beta \in \Phi, a, b \in J \right\}. \quad (1)$$

Then it is shown in [1] that E is a ternary algebra. Moreover, it is shown in [1] that $S = S(E, R(E)) = E \oplus \tilde{E} \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D$ is a Lie algebra of dimension 248, where $\tilde{E}$ denotes a second copy of E . (Recall from [1] that D is the algebra of derivations of E and $R(E) = \Phi U \oplus D$, the span of $\{R(x, y) \mid x, y \in E\} \cup \{U\}$. For $R = E \oplus \Phi u$, $U(E) = \{A \in \text{Hom}_{\Phi}(R, R) \mid uA \in \Phi u \text{ and } EA \subseteq E\}$ is a Lie algebra with $[AB] = AB - BA$. Here $U \in U(E)$ such that $uU = 2u$ and $xU = x$, for all $x \in E$. So, U is in the center of $U(E)$. Also $R(x, y) \in U(E)$, for $x, y \in E$, is defined by $uR(x, y) = \langle x, y \rangle u$, $zR(x, y) = \langle z, x, y \rangle$, for all $z \in E$, where $\langle, \rangle$ is the bilinear form and $\langle, \cdot, \cdot \rangle$ is the ternary form defining a ternary Φ -module, as in [1].)

On the other hand, let C be the algebra of split octonions and J an arbitrary Jordan algebra of degree 3. Then according to Tits in [4], $L = \text{Der}(C) \oplus C_0 \otimes J_0 \oplus \text{Der}(J)$ is a Lie algebra, where $\text{Der}(C)$ is the algebra of derivations of C and C_0 is the subspace of C whose elements have trace 0.

THEOREM 1. *There is an explicit isomorphism between the Lie algebras constructed by Tits and the Faulkner processes.*

The proof of this theorem is the goal of this paper and the proof completes this paper.

Our attempt is to construct an isomorphism $\Phi : L \rightarrow S$ by breaking Φ into three parts Φ_1 restricted to $\text{Der}(C)$, Φ_2 restricted to $C_0 \otimes J_0$, and Φ_3 restricted to $\text{Der}(J)$.

First we construct a candidate S_1 for the image of Φ_1 in S which in turn plays a significant role throughout the paper. This S_1 is obtained by replacing J by Φ_1 in the

construction of S . Therefore,

$$S_1 = E_1 \oplus \tilde{E}_1 \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D_1. \quad (2)$$

LEMMA 2. *The dimension of $D_1 = 3$.*

The study of the derivation algebra D gives detailed information about S . So, we need to look at the action of $R(x_1, y_1)$, $x_1, y_1 \in E$, on an arbitrary element

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E. \quad (3)$$

Let

$$x_i = \begin{bmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{bmatrix} \in E \quad \text{for } i = 1, 2. \quad (4)$$

Recall from [1] that $R(x, y)$ is the close derivation of E if and only if $\rho(x, y) = \langle x, y \rangle = 0$. Then it is straightforward to check that $R(x_{12}, l_{22})$, $R(x_{21}, l_{11})$, and $R(x_{12}, l_{21})$ are the only distinct derivations of E , where x_{12} is the 2×2 matrix with $(1, 2)$ th element equal to $x \in J_0$ and 0 elsewhere. In case $J = \Phi 1$, similar calculations and tables show that $R(l_{11}, l_{21})$, $R(l_{22}, l_{12})$, and $R(1, 1)$, where 1 is the 2×2 unit matrix in the last term, form a basis of D_1 . Thus, Lemma 2 is proved.

Now it is clear that $\text{Dim}(S_1) = 14$ which is the same as $\text{Dim}(\text{Der}(C))$ as in [4]. So, Φ_1 could be defined conveniently. Moreover, it is necessary to test that the role of S_1 in S is the same as the role of $\text{Der}(C)$ in L . As a matter of fact, this test will provide us with an idea for the definition of Φ_2 .

Now let $\{l_{11}, l_{12}, l_{21}, l_{22}, \tilde{l}_{11}, \tilde{l}_{12}, \tilde{l}_{21}, \tilde{l}_{22}, u, \tilde{u}, U, R(l_{11}, l_{21}), R(l_{22}, l_{12}), R(1, 1)\}$ be a basis of S_1 . There are 7 copies of J_0 in S . Two copies in E , two in $\tilde{E}$ and three in D . Let these copies of J_0 be represented by $x_{12}, x_{21}, \tilde{x}_{12}, \tilde{x}_{21}, R(x_{12}, l_{22}), R(x_{21}, l_{11})$, and $R(x_{12}, l_{21})$.

LEMMA 3. *Let $x \in J_0$, then the Φ -vector space spanned by the seven copies of x is an S_1 -module.*

PROOF. It is enough to show that the Lie product of the seven copies of J_0 with each of the generators of S_1 are elements of J_0 . The computations are lengthy but straightforward. Only two parts are shown below and the rest are similar. The Lie product rule in [1, equation (2.8)] is used.

(1) $[x_{12}, l_{11}] = [x_{12}, l_{12}] = [x_{12}, l_{21}] = [x_{12}, l_{22}] = 0$, $[x_{12}, \tilde{l}_{11}] = 0$, $[x_{12}, \tilde{l}_{12}] = R(x_{21}, l_{11})$, $[x_{12}, \tilde{l}_{21}] = R(x_{12}, l_{21})$, $[x_{12}, \tilde{l}_{22}] = [x_{12}, l_{22}]$, $[x_{12}, u] = 0$, $[x_{12}, \tilde{u}] = \tilde{x}_{12}$, $[x_{12}, U] = x_{12}$, $[x_{12}, R(l_{11}, l_{21})] = 0$, $[x_{12}, R(l_{22}, l_{12})] = -\tilde{x}_{21}$, $[x_{12}, R(1, 1)] = x_{12}$.

(2) $[R(x_{12}, l_{21}), l_{11}] = [R(x_{12}, l_{21}), l_{22}] = 0$, $[R(x_{12}, l_{21}), l_{12}] = 2x_{12}$, $[R(x_{12}, l_{21}), l_{21}] = 2x_{21}$, $[R(x_{12}, l_{21}), \tilde{l}_{11}] = [R(x_{12}, l_{21}), \tilde{l}_{22}] = 0$, $[R(x_{12}, l_{21}), \tilde{l}_{12}] = 2\tilde{x}_{12}$, $[R(x_{12}, l_{21}), \tilde{l}_{21}] = 2\tilde{x}_{21}$, $[R(x_{12}, l_{21}), u] = [R(x_{12}, l_{21}), \tilde{u}] = [R(x_{12}, l_{21}), U] = 0$, $[R(x_{12}, l_{21}), R(l_{11}, l_{21})] = -2R(x_{21}, l_{11})$, $[R(x_{12}, l_{21}), R(l_{22}, l_{12})] = 2R(x_{12}, l_{22})$, $[R(x_{12}, l_{21}), R(1, 1)] = 0$. $\square$

Using the Lie product rule in the Faulkner's construction in [1], we have for $x, y \in J_0$, $[x_{12}, y_{12}] = \langle x_{12}, y_{12} \rangle u = 0$. Similarly, $[x_{21}, y_{21}] = [\tilde{x}_{12}, \tilde{y}_{12}] = [\tilde{x}_{21}, \tilde{y}_{21}] = 0$. Let

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E, \quad \bar{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}. \quad (5)$$

It is easy to check that $z[R(x_{12}, 1_{22}), R(y_{12}, 1_{22})] = z[R(x_{21}, 1_{11}), R(y_{21}, 1_{11})] = \bar{O}$ and $z[R(x_{12}, 1_{21}), R(y_{12}, 1_{21})] = 4\bar{A}[R_x, R_y]$. The use of the ternary product $\langle, \rangle$ as in [1], equations (1.6) and (2.3), may be needed. For future reference, we note that using similar computations implies $[R(x_{12}, 1_{21}), R(y_{12}, 1_{21})] = R(x_{12}, y_{21}) - R(x y_{12}, 1_{21})$.

As in [2, page 105], let $\{e1, e2, f1, f2, f3, g1, g2, g3\}$ be a basis of C . Then C_0 is spanned by $\{e2 - e1, f1, f2, f3, g1, g2, g3\}$. Moreover, $\text{Der}(C)$ is 14-dimensional, and is spanned by $\{D_{f1, f2}, D_{f1, f3}, D_{f2, f3}, D_{g1, g2}, D_{g1, g3}, D_{g2, g3}, D_{e1, gi}, D_{e2, fi}, (i = 1, 2, 3), D_{f1, g1}, (D_{f2, g2} - D_{f3, g3})\}$ from [4, page 77]. [Recall that $D_{x, z} = [R_x, R_y] + [L_x, R_z] + [L_x, L_z]$, where R_x, L_x are the right and left multiplications]. But C_0 is 7-dimensional tells us that L contains 7 copies of J_0 . We name them, for $x \in J_0$, as $(e2 - e1) \otimes x$, $f1 \otimes x$, $f2 \otimes x$, $f3 \otimes x$, $g1 \otimes x$, $g2 \otimes x$, $g3 \otimes x$. Note that $C_0 \otimes J_0$ is a $\text{Der}(C)$ -module, because, for $a \in C_0$, $x \in J_0$, $[a \otimes x, D] = aD \otimes x \in C_0 \otimes J_0$, where $D \in \text{Der}(C)$. Moreover, for $x, y \in J_0$, $[(e2 - e1) \otimes x, (e2 - e1) \otimes y] = [R_x, R_y] \in \text{Der}(J)$, and $[fi \otimes x, fi \otimes y] = [gi \otimes x, gi \otimes y] = 0$, $i = 1, 2, 3$.

Thus, comparing the above two paragraphs, we conclude that $(e2 - e1) \otimes x$ and $R(x_{12}, 1_{21})$ have similar roles in L and S , respectively. Moreover, the annihilator of $(e2 - e1) \otimes x$ is spanned by $\{D_{f1, g1}, (D_{f2, g2} - D_{f3, g3}), D_{f1, g2}, D_{f2, g3}, D_{f3, g1}, D_{f1, g3}, D_{f2, g1}, D_{f3, g2}\}$. The annihilator of $R(x_{12}, 1_{21})$ is spanned by $\{1_{11}, 1_{22}, \tilde{1}_{11}, \tilde{1}_{22}, u, \tilde{u}, U, R(1, 1)\}$ which can be seen in the proof of Lemma 3. Studying the actions of the annihilators of $R(x_{12}, 1_{21})$ and $(e2 - e1) \otimes x$ on the rest of the six copies of J_0 , respectively, we define ϕ_1 and ϕ_2 in the following way: $D_{f1, g2} \phi_1 = -31_{11}$, $D_{f3, g1} \phi_1 = 31_{22}$, $D_{f2, g3} \phi_1 = 3\tilde{u}$, $D_{f1, g3} \phi_1 = 3\tilde{1}_{11}$, $D_{f2, g1} \phi_1 = 3\tilde{1}_{22}$, $D_{f3, g2} \phi_1 = -3u$, $D_{f1, g1} \phi_1 = R(1, 1)$, $(D_{f2, g2} - D_{f3, g3}) \phi_1 = -3U$, $D_{e2, g1} \phi_1 = R(1_{22}, 1_{12})$, $D_{e2, g2} \phi_1 = -1_{12}$, $D_{e2, g3} \phi_1 = \tilde{1}_{12}$, $D_{e1, f1} \phi_1 = R(1_{11}, 1_{21})$, $D_{e1, f2} \phi_1 = \tilde{1}_{21}$, $D_{e1, f3} \phi_1 = 1_{21}$, and $(f1 \otimes x) \phi_2 = R((1/2)x_{21}, 1_{11})$, $(f2 \otimes x) \phi_2 = (1/2)\tilde{x}_{21}$, $(f3 \otimes x) \phi_2 = (1/2)x_{21}$, $(g1 \otimes x) \phi_2 = R((1/2)x_{12}, 1_{22})$, $(g2 \otimes x) \phi_2 = -(1/2)x_{12}$, $(g3 \otimes x) \phi_2 = (1/2)\tilde{x}_{12}$. Recall from [4] that $[a \otimes x, D] = aD \otimes x$, for all $D \in \text{Der}(C)$, $a \in C_0$, $x \in J_0$.

LEMMA 4. *The mapping ϕ_1 is an isomorphism and ϕ_2 is a bijection.*

PROOF. The map ϕ_1 is injective: let $D_1 \phi_1 = D_2 \phi_1$ for $D_1, D_2 \in \text{Der}(C)$ and $D_1 \neq D_2$. We can write $D_1 \phi_1 = (c_1 D_{f1, g2} + c_2 D_{f3, g1} + \dots + c_{14} D_{e1, f3}) \phi_1 = c_1(-31_{11}) + c_2(31_{22}) + \dots + c_{14} 1_{21} = S_1$. Similarly, $D_2 \phi_1 = k_1(-31_{11}) + k_2(31_{22}) + \dots + k_{14} 1_{21} = S_2$, $c_i, k_i \in \Phi$ and ϕ_1 is clearly linear. But $S_1 = S_2$ implies $c_i = k_i$ and hence $D_1 = D_2$.

The map ϕ_1 is surjective: let $S \in S_1$. Then $S = c_1(-31_{11}) + c_2(31_{22}) + \dots + c_{14} 1_{21} = (c_1 D_{f1, g2} + c_2 D_{f3, g1} + \dots + c_{14} D_{e1, f3}) \phi_1 = D \phi_1$ (say). But $D \in \text{Der}(C)$.

The map ϕ_1 preserves Lie product: let $D_1, D_2 \in \text{Der}(C)$. To show that $[D_1 \phi_1, D_2 \phi_1] = [D_1, D_2] \phi_1$. First we notice that for $a \otimes x \in C_0 \otimes J_0$ and $D \in \text{Der}(C)$, $[(a \otimes x) \phi_2, D \phi_1] = [a \otimes x, D] \phi_2 = (aD \otimes x) \phi_2$ from the study of correspondences between $\text{Der}(C)$ and S_1 , and $\text{Der}(C)$ -module in L and S_1 -module in S we discussed before. Using the Jacobi identity, we have, $[[a \otimes x, D_1], D_2] + [[D_1, D_2], a \otimes x] + [[D_2, a \otimes x], D_1] = 0$. Thus we

obtain $a[D_1, D_2] \otimes x = aD_1D_2 \otimes x - aD_2D_1 \otimes x$. Therefore, $[(a \otimes x)\phi_2, [D_1\phi_1, D_2\phi_1]] = [[(a \otimes x)\phi_2, D_1\phi_1], D_2\phi_1] - [[(a \otimes x)\phi_2, D_2\phi_1], D_1\phi_1] = (aD_1D_2 \otimes x - aD_2D_1 \otimes x)\phi_2 = (a[D_1, D_2] \otimes x)\phi_2 = [a \otimes x, [D_1, D_2]]\phi_2$. Thus, $[(a \otimes x)\phi_2, [D_1\phi_1, D_2\phi_1]] = [(a \otimes x)\phi_2, [D_1, D_2]\phi_1]$. But, ϕ_2 is bijective in the same way as ϕ_1 is and by dimension count. Hence, ϕ_1 preserves Lie product and ϕ_1 is an isomorphism. $\square$

LEMMA 5. *The centralizer of S_1 is $\{R \in \text{Der}(E) \mid E_1R = 0\}$.*

PROOF. Let the centralizer of S_1 be denoted by $\text{cent}(S_1)$. Then $\text{cent}(S_1) = \{S \in S \mid [S, S_1] = 0\}$. Let $S = \gamma + \tilde{e} + \alpha u + \beta \tilde{u} + R$ and $S_1 = \gamma_1 + \tilde{e}_1 + \alpha_1 u + \beta_1 \tilde{u} + R_1$. Then, $[S, \gamma_1] = 0$ implies $\langle \gamma, \gamma_1 \rangle u - R(\gamma_1, e) + \beta \gamma_1 - \gamma_1 R = 0$. Hence, $\langle \gamma, \gamma_1 \rangle = 0$, $\gamma_1 R = -R(\gamma_1, e)$, $\beta = 0$. Similarly, $[S, \tilde{e}_1] = 0$ implies $\alpha = 0$, $[S, u] = 0$ implies $e = 0$, $\rho(R) = 0$, $[S, \tilde{u}] = 0$ implies $\tilde{\gamma} = 0$ and finally, $[S, R_1] = 0$ implies $[R, R_1] = 0$. Moreover, $\tilde{\gamma} = 0$ implies $\gamma = 0$, $e = 0$ implies $\tilde{e} = 0$, $\gamma_1 R = 0$ implies $E_1R = 0$. Hence $S = R$ and R is a derivation E . Thus, $\text{cent}(S_1) = \{R \in \text{Der}(E) \mid E_1R = 0 \text{ and } [R, R_1] = 0\}$. If $R_1 = R_1(x_1, \gamma_1)$ for $x_1, \gamma_1 \in E_1$ then $[R, R_1(x_1, \gamma_1)] = -[R_1(x_1, \gamma_1), R] = R_1(x_1R, \gamma_1) + R_1(x_1, \gamma_1R) = 0$ for $E_1R = 0$. Hence, $\text{cent}(S_1) = \{R \in \text{Der}(E) \mid E_1R = 0\}$. $\square$

EXAMPLE 6. The bracket $[R(x_{12}, 1_{21}), R(\gamma_{12}, 1_{21})]$ is an element of $\text{cent}(S_1)$. This is clear from the computations after the proof of Lemma 3. For z and $\tilde{A}$ as defined in these computations, $z[R((1/2)x_{12}, 1_{21}), R((1/2)\gamma_{12}, 1_{21})] = \tilde{A}[R_x, R_y]$, $x, \gamma \in J_0$. But, $[R_x, R_y] \in \text{Der}(J)$, as we have seen before. It is an easy exercise that for $x, \gamma \in J$ there exist $x_0, \gamma_0 \in J_0$ such that $[R_x, R_y] = [R_{x_0}, R_{\gamma_0}]$. Hence, for $x, \gamma \in J_0$, $[R_x, R_y]$ span $\text{Der}(J)$. Write $[R_x, R_y] = D_{x, \gamma}$. Then we define ϕ_3 as follows: $\phi_3 : \text{Der}(J) \rightarrow \text{cent}(S_1)$ by $(D_{x, \gamma})\phi_3 = [R((1/2)x_{12}, 1_{21}), R((1/2)\gamma_{12}, 1_{21})]$. (Recall from [4] that for a central simple Jordan algebra J of degree 3 there is a linear function $T(\cdot)$ such that $x = (1/3)T(x) + x_0$, $x \in J$, $x_0 \in J_0$. $T(1) = 3 \cdot x * \gamma = x \cdot \gamma - (1/3)T(x \cdot \gamma)1$, for $x, \gamma \in J_0$.)

LEMMA 7. *The map ϕ_3 is an isomorphism.*

PROOF. The map ϕ_3 is injective: let $(D_{x, \gamma})\phi_3 = 0$. Then clearly, $D_{x, \gamma} = 0$ from the above discussions.

The map ϕ_3 is surjective: recall that $S = E \oplus \tilde{E} \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D$, where $D = \{\sum_i R(x_i, \gamma_i) \mid \sum_i \langle x_i, \gamma_i \rangle = 0\}$ as in [1, Section 4].

We have $S_1 = E_1 \oplus \tilde{E}_1 \oplus \Phi u \oplus \Phi \tilde{u} \oplus \Phi U \oplus D_1$. Moreover, $J = J_0 \oplus \Phi 1$, as in [4]. These show that we write S as a direct sum of S_1 , the vector space generated by the copies of J_0 and some derivations in D . In fact we will show that $S = S_1 \oplus \text{Im}(\phi_2) \oplus \text{Im}(\phi_3)$. So, it is enough to show that each element of D can be written as a direct sum of elements in these three parts. For each i , let

$$R(x_i, \gamma_i) = R\left(\begin{bmatrix} 1 & x \\ x' & 1 \end{bmatrix}, \begin{bmatrix} 1 & \gamma \\ \gamma' & 1 \end{bmatrix}\right), \quad x, x', \gamma, \gamma' \in J. \quad (6)$$

Using linearity, $R(x_i, \gamma_i) = R(1_{11}, 1_{11}) + R(1_{11}, \gamma_{12}) + R(1_{11}, \gamma'_{21}) + R(1_{11}, 1_{22}) + R(x_{12}, 1_{11}) + R(x_{12}, \gamma_{12}) + R(x_{12}, \gamma'_{21}) + R(x_{12}, 1_{22}) + R(x'_{21}, 1_{11}) + R(x'_{21}, \gamma_{12}) + R(x'_{21}, \gamma'_{21}) + R(x'_{21}, 1_{22}) + R(1_{22}, 1_{11}) + R(1_{22}, \gamma_{12}) + R(1_{22}, \gamma'_{21}) + R(1_{22}, 1_{22})$. Using

$\gamma = (1/3)T(\gamma)1 + \gamma_0$, $\gamma_0 \in J_0$, we get $R(1_{11}, \gamma_{12}) = (1/3)T(\gamma)R(1_{11}, 1_{12}) + R(1_{11}, (\gamma_0)_{12})$. Similarly, $R(1_{12}, \gamma'_{12})$ and $R(x_{12}, \gamma_{12})$ are sum of elements in S_1 and elements in the copies of J_0 . Using a remark after Lemma 3, we have $R((x_0)_{12}, (\gamma_0)_{21}) = [R((x_0)_{12}, 1_{21}), R((\gamma_0)_{12}, 1_{21})] + R((x_0\gamma_0)_{12}, 1_{21})$. Thus, $R(x_{12}, \gamma_{21})$ is a sum of elements in S_1 , in $\text{Im}(\phi_2)$, and in $\text{Im}(\phi_3)$. Moreover, $R(x_{12}, \gamma_{21}) - R(\gamma_{21}, x_{12}) = \langle x_{12}, \gamma_{21} \rangle U$, by [1, equation (2.5)]. Hence, we have shown that $S = S_1 \oplus \text{Im}(\phi_2) \oplus \text{Im}(\phi_3)$. Since ϕ_3 is injective, we obtain $S = S_1 \oplus \text{Im}(\phi_2) \oplus \text{cent}(S_1)$. Thus, $\text{cent}(S_1) = \text{Im}(\phi_3)$ and ϕ_3 is surjective.

The map ϕ_3 preserves Lie product: let $A = [R((1/2)x_{12}, 1_{21}), R((1/2)\gamma_{12}, 1_{21})]$ and $B = [R((1/2)x'_{12}, 1_{21}), R((1/2)\gamma'_{12}, 1_{21})]$ for $x, \gamma, x', \gamma' \in J_0$. Then $z[A, B] = zAB - zBA = \tilde{A}[D_{x, \gamma}, D_{x', \gamma'}]$. By definition, $[D_{x, \gamma}, D_{x', \gamma'}]\phi_3 = [A, B] = [(D_{x, \gamma})\phi_3, (D_{x', \gamma'})\phi_3]$. $\square$

REMARK 8. The Lie product of an element in $\text{cent}(S_1)$ with an element in a copy of J_0 gives an element in a copy of J_0 .

This shows that the behavior of $\text{cent}(S_1)$ in S is the same as that of $\text{Der}(J)$ in L .

To complete the proof of the Theorem the only thing left is to check that ϕ_3 defined on $C_0 \otimes J_0$ preserves Lie product. From [4, page 22], for $x, \gamma \in J_0$ we have $[f1 \otimes x, g2 \otimes \gamma] = (1/12)\langle x, \gamma \rangle D_{f1, g2} + f1 * g2 \otimes x * \gamma - (f1, g2)[R_x, R_\gamma] = (1/12)T(x\gamma)D_{f1, g2} + (f1g2 - (1/2)T(f1g2)) \otimes x * \gamma + (1/2)T(f1g2)[R_x, R_\gamma] = (1/12)T(x\gamma)D_{f1, g2}$. (Note that thus $f1g2 = 0$, $(f1, g2) = (1/2)T(f1, \tilde{g}2) = (1/2)T(f1, -g2) = -(1/2)T(f1g2)$.) Therefore, $[f1 \otimes x, g2 \otimes \gamma]\phi_3 = (1/12)T(x\gamma)(-31_{11})$. On the other hand, for

$$H = \begin{bmatrix} -\frac{1}{4}T(x, \gamma) & 0 \\ 0 & 0 \end{bmatrix}, \quad (7)$$

$[(f1 \otimes x)\phi_3, (g2 \otimes \gamma)\phi_3] = [R((1/2)x_{12}, 1_{11}), -(1/2)\gamma_{12}] = (1/2)\gamma_{12}R((1/2)x_{12}, 1_{11}) = H = (1/12)T(x\gamma)(-31_{11})$, where $T(x, \gamma) = T(x\gamma)$. Similarly, $[f1 \otimes x, g3 \otimes \gamma]\phi_3 = (1/12)T(x\gamma)(3\tilde{1}_{11}) = [(f1 \otimes x)\phi_3, (g3 \otimes \gamma)\phi_3]$. Also $[f2 \otimes x, g1 \otimes \gamma]\phi_3 = (1/12) \times T(x\gamma)(3\tilde{1}_{22}) = [(f2 \otimes x)\phi_3, (g1 \otimes \gamma)\phi_3]$. And $[f2 \otimes x, g3 \otimes \gamma]\phi_3 = (1/12)T(x\gamma)(3\tilde{u}) = [(f2 \otimes x)\phi_3, (g3 \otimes \gamma)\phi_3]$. Similarly, one can continue to check for $[f3 \otimes x, g1 \otimes \gamma]$ and $[f3 \otimes x, g2 \otimes \gamma]$. And $[f1 \otimes x, f2 \otimes \gamma]\phi_3 = -(1/12)T(x\gamma)D_{e2, g3} + g3 \otimes x * \gamma \phi_3 = (1/4)(x \times \gamma)\tilde{1}_{12} = [(f1 \otimes x)\phi_3, (f2 \otimes \gamma)\phi_3]$, where $x \times \gamma = 2x\gamma - T(x\gamma)$. Similar computations for $[f1 \otimes x, f3 \otimes \gamma]$, $[g1 \otimes x, g2 \otimes \gamma]$, and $[g1 \otimes x, g3 \otimes \gamma]$. We also have, $[f2 \otimes x, f3 \otimes \gamma]\phi_3 = -(1/12)T(x\gamma)R(1_{22}, 1_{12}) + R((1/2)(x * \gamma)_{12}, 1_{22})$. For

$$z = \begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \in E, \quad K = \frac{1}{4} \begin{bmatrix} 0 & \alpha x \times \gamma \\ a \times (x \times \gamma) & T(b, x \times \gamma) \end{bmatrix}, \quad (8)$$

$z[(f2 \otimes x, f3 \otimes \gamma)\phi_3] = K = z([(f2 \otimes x)\phi_3, (f3 \otimes \gamma)\phi_3])$. Similarly, $[g2 \otimes x, g3 \otimes \gamma]\phi_3 = ((1/12)T(x\gamma)D_{f1, g1} + (1/2)(e2 - e1) \otimes x * \gamma + (1/2)D_{\gamma, x})\phi_3 = (1/12)T(x\gamma)R(1, 1) - (1/12)T(x\gamma)R(1_{12}, 1_{21}) + (1/4)R(\gamma_{12}, x_{21})$. Then for

$$L = \frac{1}{4} \begin{bmatrix} \alpha T(x\gamma) & aT(x\gamma) - \{ax\gamma\} \\ -bT(x\gamma) + \{b\gamma x\} & -\beta T(x\gamma) \end{bmatrix}, \quad (9)$$

$z([f1 \otimes x, g1 \otimes y]\phi_3) = L = z[(f1 \otimes x)\phi_3, (g1 \otimes y)\phi_3]$. Similar computations are for $[f3 \otimes x, g3 \otimes y]$. We also have, $[f2 \otimes x, g2 \otimes y]\phi_3 = (1/4)R(y_{12}, x_{21}) = [(f2 \otimes x)\phi_3, (g2 \otimes y)\phi_3]$. To check the above two equations, one may need that $(D_{f2, g2})\phi_3 = [D_{e1, f2}, D_{e2, g2}]\phi_3 = [\tilde{I}_{21}, -1_{12}] = R(1_{12}, 1_{21})$. Similarly, $(D_{f3, g3})\phi_3 = R(1_{21}, 1_{12})$. Since $[(e2 - e1) \otimes x, f1 \otimes y]\phi_3 = -(1/6)T(xy)D_{e1, f1} - f1 \otimes x * y\phi_3 = -(1/2)R((xy)_{21}, 1_{11})$. One could use that $D_{e2 - e1, f1} = -2D_{e1, f1}$. For

$$N = \frac{1}{2} \begin{bmatrix} T(a, xy) & b \times xy \\ \beta xy & 0 \end{bmatrix}, \quad (10)$$

$z([(e2 - e1) \otimes x, f1 \otimes y]\phi_3) = N = z([(e2 - e1) \otimes x)\phi_3, (f1 \otimes y)\phi_3]$. Similar computations are for $[(e2 - e1) \otimes x, g1 \otimes y]$. We also have, $[(e2 - e1) \otimes x, f2 \otimes y]\phi_3 = -(1/2)xy\tilde{I}_{21} = [((e2 - e1) \otimes x)\phi_3, (f2 \otimes y)\phi_3]$. Similar computations are for $[(e2 - e1) \otimes x, f3 \otimes y]$, $[(e2 - e1) \otimes x, g2 \otimes y]$, and $[(e2 - e1) \otimes x, g3 \otimes y]$.

Thus, we have shown that ϕ_3 preserves Lie product. Hence ϕ is an isomorphism and the proof of the theorem is completed.

ACKNOWLEDGEMENT. This paper constitutes a part of the author's Master's thesis written under the direction of Professor Michael L. Racine at the University of Ottawa, Ottawa, Ontario, Canada. It is dedicated to the author's parents.

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