ON THE DIOPHANTINE EQUATION $x^3 = dy^2 \pm q^6$

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Abstract. Let $q > 3$ denote an odd prime and $d$ a positive integer without any prime factor $p \equiv 1 \pmod{3}$. In this paper, we have proved that if $(x, q) = 1$, then $x^3 = dy^2 \pm q^6$ has exactly two solutions provided $q \not\equiv \pm 1 \pmod{24}$.

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Cohn [1] and recently Zhang [2, 3] have solved the Diophantine equation

$$x^3 = dy^2 \pm q^6 \quad (1)$$

when $q = 1, 3, 4$, under some conditions on $d$. In this paper, we consider the general case of (1) where $q \neq 3$ is any odd prime by using arguments similar to those used by Cohn [1].

Let $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ be a solution of (1) with $x, y > 0$, then the solution is trivial if $x = 0, \pm q^2$ or $y = \pm 1$. We need the following lemma.

**Lemma 1.** The equation $p^2 = a^4 - 3b^2$, where $p$ denotes an odd prime and $(p, a) = l$, may have a solution in positive integers $a$ and $b$ only if $p \equiv \pm 1 \pmod{24}$.

**Proof.** Suppose $3b^2 = a^4 - p^2$. Then clearly $a$ is odd and $b$ is even. Since $a^4 \equiv 3b^2 \pmod{p}$, and $(p, a) = 1$ therefore the Legendre symbol $(3/p) = 1$ and so $p \equiv \pm 1 \pmod{12}$. Now $(a^2 + p, a^2 - p) = 2$ implies that

$$a^2 \pm p = 3.2c^2,$$

$$a^2 \mp p = 2d^2,$$

where $2cd = b$ and $(c, d) = 1$. Whence

$$a^2 = 3c^2 + d^2. \quad (4)$$

Here $d$ is odd, otherwise we get a contradiction modulo 4. Then considering (3) modulo 8, we get $p \equiv \pm 1 \pmod{8}$. This completes the proof.

Now we consider the upper sign in (1), our main result is laid down in the following.

**Theorem 2.** Let $d$ be a positive integer without prime factor $p \equiv 1 \pmod{3}$ and let $q \neq 3$ be an odd prime. If $q \not\equiv \pm 1 \pmod{24}$ and $(x, q) = 1$, then the Diophantine equation

$$x^3 = dy^2 + q^6 \quad (5)$$
has exactly two solutions given by

\[
x_1 = \frac{3q^4 - 2q^2 - 1}{4}, \quad y = ab, \quad \text{where } a = \frac{3q^4 + 1}{4}, \quad db^2 = \frac{3q^4 - 6q^2 - 1}{4},
\]

\[
x_2 = \frac{q^4 - 2q^2 - 3}{4}, \quad y = 9ab, \quad \text{where } a = \frac{q^4 + 3}{4}, \quad db^2 = \frac{q^4 - 6q^2 - 3}{4}.
\]

**Proof.** If \(d\) has a square factor, then it can be absorbed into \(y^2\), so there is no loss of generality in supposing \(d\) a square free integer. Now

\[
dy^2 = x^3 - q^6 = (x - q^2)(x^2 + q^2x + q^4).
\]

If any prime \(r\) divides both \(d\) and \((x^2 + q^2x + q^4)\), then by hypothesis \(r \equiv 2 \pmod{3}\) or \(r = 3\). But \(r \mid (x^2 + q^2x + q^4)\) implies that \((2x + q^2)^2 + 3q^4 \equiv 0 \pmod{r}\) so the Legendre symbol \((-3/r) = 1\), which is a contradiction, whence \(r = 1\) or 3. Also since \((x,q) = 1\), therefore \((x - q^2, x^2 + q^2x + q^4) = 1\) or 3. So for (7) we have only two possibilities: either

\[
x^2 + q^2x + q^4 = a^2, \quad x - q^2 = db^2,
\]

or

\[
x^2 + q^2x + q^4 = 3a^2, \quad x - q^2 = 3db^2,
\]

where \((q,a) = 1\) and \((q,b) = 1\). Consider the first possibility when \((2x + q^2)^2 + 3q^4 = (2a)^2\) and \(y = ab\). This equation is known to have a finite number of solutions. It can be written as

\[
3q^4 = (2a + 2x + q^2)(2a - (2x + q^2)).
\]

Then for the nontrivial solution of this equation we have only two cases:

**Case 1.**

\[
3q^4 = 2a \pm (2x + q^2), \quad 1 = 2a \mp (2x + q^2),
\]

by subtracting and adding these two equations we get

\[
x = \frac{3q^4 - 2q^2 - 1}{4}, \quad a = \frac{3q^4 + 1}{4}.
\]

Here \(a > 1\), so \(y > 1\), and \(x - q^2 = db^2\) implies that

\[
db^2 = \frac{3q^4 - 6q^2 - 1}{4}.
\]

**Case 2.**

\[
3 = 2a \pm (2x + q^2), \quad q^4 = 2a \mp (2x + q^2).
\]

As in Case 1 we get the nontrivial solution

\[
x = \frac{3q^4 - 2q^2 - 1}{4}, \quad a = \frac{3q^4 + 1}{4}, \quad db^2 = \frac{3q^4 - 6q^2 - 1}{4}.
\]
Now suppose the second possibility. Obviously \( a \) is odd and \( x^2 = 3a^2 \) (mod \( q \)), and since \((q,a) = 1\), so the Legendre symbol \((3/q) = 1\), hence \( q \equiv \pm 1 \) (mod 12). Eliminating \( x \) and dividing by 3, we get

\[
a^2 = q^4 + 3db^2(q^2 + db^2).
\]

Considering (16) modulo 8 we get either \( db^2 \equiv -1 \) (mod 8) or \( db^2 \equiv 0 \) (mod 8).

(1) \( db^2 \equiv -1 \) (mod 8). Then from (16) we get

\[
3d^2b^4 = (2a + 2q^2 + 3db^2)(2a - 2q^2 - 3db^2).
\]

Let \( S \) be a common prime divisor of the two factors in the right-hand side of (17), then \( S \) is odd, \( S \mid 4a \) and \( S \mid 2(2q^2 + 3db^2) \). But \( S^2 \) divides the left-hand side implies that \( S \mid 3db^2 \), so \( S \mid q^2 \). Here \( S = 1 \), otherwise \( x - q^2 = 3db^2 \) implies that \( q \mid x \) which is not true. Thus from (17) we get

\[
2a \pm (2q^2 + 3db^2) = d^2_1b^4_1, \quad 2a \mp (2q^2 + 3db^2) = 3d^2_2b^4_2,
\]

where \( d = d_1d_2 \) and \( b = b_1b_2 \). Whence

\[
\pm 2(2q^2 + 3db^2) = d^2_1b^4_1 - 3d^2_2b^4_2.
\]

Considering this equation modulo 3, we get

\[
4q^2 = d^2_1b^4_1 - 3d^2_2b^4_2 - 6db^2.
\]

Now we prove that \( d_1 = 1 \). Since \( d \) is odd, therefore \( d_1 \) must be odd. Let \( t \) be any odd prime dividing \( d_1 \) then by hypothesis \( t \equiv 2 \) (mod 3) but then from (20) we get

\[
4q^2 \equiv -3d^2_2b^4_2 \pmod{t},
\]

so \((-3/t) = 1\), which is not true. Thus \( d_1 = 1 \) and (20) becomes

\[
q^2 = b^4_1 - 3\left(\frac{b_1^2 + db_2^4}{2}\right)^2,
\]

since \((q,b_1) = 1\), therefore by Lemma 1, \( q \equiv \pm 1 \) (mod 24).

(2) \( db^2 \equiv 0 \) (mod 8). Now we prove that if (16) has a solution, then \( q \equiv \pm 1 \) (mod 24). Since \( d \) is a square free, \( b \) should be even. Suppose \( b = 2m \), then (16) can be written as

\[
12d^2m^4 = (a + q^2 + 6dm^2)(a - q^2 - 6dm^2).
\]

As before we can prove that the common divisor of the two factors in the right-hand side of (23) is 2, so

\[
a \pm (q^2 + 6dm^2) = 2d^2_1m^4_1, \quad a \mp (q^2 + 6dm^2) = 6d^2_2m^4_2,
\]

where \( d = d_1d_2 \) and \( m = m_1m_2 \). It is clear that \( (a,q) = 1 \) implies that \( (m_1,q) = 1 \).
Subtracting the two equations in (24) we get

$$\pm (q^2 + 6dm^2) = d_1^2 m_1^4 - 3d_2^2 m_2^4,$$

(25)

again considering this equation modulo 3, we get $q^2 = d_1^2 m_1^4 - 3d_2^2 m_2^4 - 6dm^2$. As before $d_1$ cannot have any odd prime divisor, so $d_1 = 1$ or 2.

If $d_1 = 1$, then

$$q^2 = 4m_1^4 - 3(m_1^2 + dm_2^2).$$

(26)

Here $m_1$ is odd, otherwise we get a contradiction modulo 8. Since $(m_1, q) = 1$, then from (26) we get

$$2m_1^2 \pm q = 3s^2, \quad 2m_1^2 + q = n^2,$$

(27)

where $sn = m_1^2 + dm_2^2$, so $s$ and $n$ are both odd. Hence $q \equiv \pm 1 \pmod{8}$, combining this result with $q \equiv \pm 1 \pmod{12}$, we get $q \equiv \pm 1 \pmod{24}$.

If $d_1 = 2$, then

$$q^2 = 16b_1^4 - 3(b_1^2 + db_2^2)^2$$

(28)

which is impossible modulo 8. 

Using the same argument as in Theorem 2 we can prove the following theorem.

**Theorem 3.** Let $d$ be a positive integer without prime factor $p \equiv 1 \pmod{3}$ and $q \equiv 3$ an odd prime. If $q \equiv \pm 1 \pmod{24}$ and $(x, q) = 1$, then the Diophantine equation $x^3 = dy^2 - q^6$ has exactly two solutions given by

$$x_1 = \frac{3q^4 + 2q^2 - 1}{4}, \quad y = ab, \quad \text{where} \quad a = \frac{3q^4 + 1}{4}, \quad db^2 = \frac{3q^4 + 6q^2 - 1}{4},$$

$$x_2 = \frac{q^4 + 2q^2 - 3}{4}, \quad y = 9ab, \quad \text{where} \quad a = \frac{q^4 + 3}{4}, \quad db^2 = \frac{q^4 + 6q^2 - 3}{4}.$$  

(29)

Sometimes, combining our results with Cohn’s result [1] we can solve the title equation completely when $d$ has no prime factor $\equiv 1 \pmod{3}$, as we show in the following example.

**Example 4.** Consider the Diophantine equation $x^3 = dy^2 + 5^6$ where $d$ has no prime factor $\equiv 1 \pmod{3}$ and $(5, d) = 1$.

Here $q = 5$, when $(x, 5) = 1$, using Theorem 2 for the positive sign this equation has only two solutions given by $x_1 = 456, \ db^2 = 431$, and $x_2 = 143, \ db^2 = 118$. So $d = 431, 118$. Now let $5 \mid x$, then because $(5, d) = 1$, the equation reduces to the form $x^3 = 5dy^2 + 1$, which by [1, Theorem 1] has no solution in positive integers.

So the equation $x^3 = dy^2 + 5^6$ has a solution only if $d = 431, 118$.

For the negative sign this equation has two solutions when $(x, 5) = 1$ given by

$$x_1 = 481, \quad db^2 = 506, \quad x_2 = 168, \quad db^2 = 193,$$

(30)

that is, when $d = 506, 193$. If $5 \mid x$, then the equation reduces to the form $x^3 = 5dy^2 - 1$, which by [1, Theorem 2] has no solution in positive integers.
ON THE DIOPHANTINE EQUATION \( x^3 = dy^2 \pm q^6 \)

REFERENCES


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