ULTRALOGICS AND PROBABILITY MODELS

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ABSTRACT. We show how nonstandard consequence operators, ultralogics, can generate the general informational content displayed by probability models. In particular, a probability model that predicts that a specific single event will occur and those models that predict that a specific distribution of events will occur.

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1. Introduction. In [3], the theory of nonstandard consequence operators is introduced. Consequence operators, as an informal theory for logical deduction, were introduced by Tarski [6]. There are two such operators investigated, the *finite* and the *general* consequence operator. Let L be any nonempty set that represents a language and $\mathcal{P}$ the set-theoretic power set operator.

DEFINITION 1.1. A mapping $C : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ is a *general* consequence operator (or closure operator) if for each $X, Y \in \mathcal{P}(L)$

- (i) $X \subset C(X) = C(C(X)) \subset L$ and if
- (ii) $X \subset Y$, then $C(X) \subset C(Y)$.

A consequence operator C defined on L is said to be *finite* (*finitary*, or *algebraic*) if it satisfies

- (iii) $C(X) = \cup \{C(A) \mid A \in F(X)\}$, where F is the finite power set operator.

REMARK 1.2. The above axioms (i), (ii), and (iii) are not independent. Indeed, (i) and (iii) imply (ii).

In [3], the language L and the set of all consequence operators defined on L are encoded and embedded into a standard superstructure $\mathcal{M} = \langle \mathcal{N}, \in, = \rangle$. This standard superstructure is further embedded into a nonstandard and elementary extension $^*\mathcal{M} = \langle ^*\mathcal{N}, \in, = \rangle$. For convenience, $^*\mathcal{M}$ is considered to be a $2^{|\mathcal{M}|}$ -saturated enlargement. Then, in the usual constructive manner, $^*\mathcal{M}$ is further embedded into the superstructure, the Grundlegend structure, $\mathcal{U} = \langle Y, \in, = \rangle$ where, usually, the nonstandard analysis occurs. In all that follows in this article, the Grundlegend superstructure $\mathcal{U}$ is altered by adjoining to the construction of $\mathcal{M}$ a set of atoms that corresponds to the real numbers. This yields a $2^{|\mathcal{M}|}$ -saturated enlargement $^*\mathcal{M}_1$ and the corresponding extended Grundlegend structure $\mathcal{U}_1$ [1, 2].

2. The main result. To indicate the intuitive ordering of any sequence of events, the set T of Kleene styled “tick” marks, with a spacing symbol, is used [5, page 202] as they might be metamathematically abbreviated by symbols for the nonzero natural

numbers. Let $G \subset L_1$ be considered as a fixed description for a source that yields, through application of natural laws or processes, the occurrence of an event described by $E \subset L_1$. Further, the statement $E' \subset L_1$ indicates that the event described within the statement E did not occur. Let $L = \{G\} \cup \{E, E'\} \cup T$. As usual, G , E , and E' are assumed to contain associated encoded general information [4]. Note that for subsets of L , bold notation, such as $\mathbf{G}$, denotes the image of G as it is embedded into $\mathcal{M}_1$.

THEOREM 2.1. *For the language L and any $p \in \mathbb{R}$ such that $0 \leq p \leq 1$, where p represents a theory predicted (i.e., a priori) probability that an event will occur, there exists an ultrachoice function *C and an ultralogic P_p with the following properties:*

(1) *When P_p is applied to ${}^*\{\mathbf{G}\} = \{\mathbf{G}\}$ a hyperfinite set of “events” $\{a_1, \dots, a_n, \dots, {}^*a_v\}$ is obtained such that for any “ n ” trials, $\{a_1, \dots, a_n\}$ is a finite identified “event” sequence, where each a_i determines the labeled event $\mathbf{E}$ or labeled non-event $\mathbf{E}'$.*

(2) *The labeled events in (1) are sequentially determined by *C , where C determines a sequence g_{ap} of relative frequencies that converges to p .*

(3) *The sequence of relative frequencies g_{ap} determined by *C gives the appearance of theory dependent random chance.*

PROOF. All of the objects discussed will be members of an informal superstructure at a rather low level and slightly abbreviated definitions, as also discussed in [1, pages 23, 30–31], are utilized. As usual $\mathbb{N}$ is the set of all natural numbers including zero, and $\mathbb{N}^{>0}$ the set of all nonzero natural numbers.

Let $A = \{a \mid (a : \mathbb{N}^{>0} \rightarrow \mathbb{N}) \wedge (\forall n (n \in \mathbb{N}^{>0} \rightarrow (a(1) \leq 1 \wedge 0 \leq a(n+1) - a(n) \leq 1)))\}$. Note that the special sequences in A are nondecreasing and for each $n \in \mathbb{N}^{>0}$, $a(n) \leq n$. Obviously $A \neq \emptyset$, for the basic example to be used below, consider the sequence $a(1) = 0$, $a(2) = 1$, $a(3) = 1$, $a(4) = 2$, $a(5) = 2$, $a(6) = 3$, $a(7) = 3$, $a(8) = 4, \dots$ which is a member of A . Next consider the most basic representation Q for the non-negative rational numbers where we do not consider them as equivalence classes. Thus $Q = \{(n, m) \mid (m \in \mathbb{N}) \wedge (n \in \mathbb{N}^{>0})\}$.

For each member of A , consider the sequence $g_a : \mathbb{N} \rightarrow Q$ defined by $g_a(n) = (n, a(n))$. Let F be the set of all such g_a as $a \in A$. Consider from the above hypotheses, any $p \in \mathbb{R}$ such that $0 \leq p \leq 1$. We show that for any such p there exists an $a \in A$ and a $g_{ap} \in F$ such that $\lim_{n \rightarrow \infty} g_{ap}(n) = p$. For each $n \in \mathbb{N}^{>0}$, consider n subdivisions of $[0, 1]$, and the corresponding intervals $[c_k, c_{k+1})$, where $c_{k+1} - c_k = 1/n$, $0 \leq k < n$, and $c_0 = 0$, $c_n = 1$. If $p = 0$, let $a(n) = 0$ for each $n \in \mathbb{N}^{>0}$. Otherwise, using the customary covering argument relative to such intervals, the number p is a member of one and only one of these intervals, for each $n \in \mathbb{N}^{>0}$. Hence for each such $n > 0$, select the end point c_k of the unique interval $[c_k, c_{k+1})$ that contains p . Notice that for $n = 1$, $c_k = c_0 = 0$. For each such selection, let $a(n) = k$. Using this inductive styled definition for the sequence a , it is immediate, from a simple induction proof, that $a \in A$, $g_{ap} \in F$, and that $\lim_{n \rightarrow \infty} g_{ap}(n) = p$. For example, consider the basic sequence a in paragraph 2 of this proof. Then $g_{ap} = \{(1, 0), (2, 1), (3, 1), (4, 2), (5, 2), (6, 3), (7, 3), (8, 4), \dots\}$ is such a sequence that converges to $1/2$. Let $F_p \subset F$ be the nonempty set of all such g_{ap} . Note that for the set F_p , p is fixed and F_p contains each g_{ap} , as a varies over A , that satisfies the convergence requirement. Thus, for $0 \leq p \leq 1$, A is partitioned into subsets A_p and a single set A' such that each member of A_p determines a $g_{ap} \in F_p$. The elements

of A' are the members of A that are not so characterized by such a p . Let $\mathcal{A}$ denote this set of partitions.

Let $B = \{f \mid \forall n \forall m ((n \in \mathbb{N}^{>0}) \wedge (m \in \mathbb{N}) \wedge (m \leq n)) \rightarrow ((f: ([1, n] \times \{n\}) \times \{m\} \rightarrow \{0, 1\}) \wedge (\forall j ((j \in \mathbb{N}^{>0}) \wedge (1 \leq j \leq n)) \rightarrow (\sum_{i=1}^n f((i, n), n, m) = m)))\}$. The members of B are determined, but not uniquely, by each (n, m) such that $(n \in \mathbb{N}^{>0}) \wedge (m \in \mathbb{N}) \wedge (m \leq n)$. Hence for each such (n, m) , let $f_{nm} \in B$ denote a member of B that satisfies the conditions for a specific (n, m) .

For a given p , by application of the axiom of choice, with respect to $\mathcal{A}$, there is an $a \in A_p$ and a g_{ap} with the properties discussed above. Also there is a sequence $f_{na(n)}$ of partial sequences such that, when $n > 1$, it follows that $(\dagger) f_{na(n)}(j) = f_{(n-1)a(n-1)}(j)$ as $1 \leq j \leq (n-1)$. Relative to the above example, consider the following:

$$\begin{aligned} f_{1a(1)}(1) &= 0, \\ f_{2a(2)}(1) &= 0, \quad f_{2a(2)}(2) = 1, \\ f_{3a(3)}(1) &= 0, \quad f_{3a(3)}(2) = 1, \quad f_{3a(3)}(3) = 0, \\ f_{4a(4)}(1) &= 0, \quad f_{4a(4)}(2) = 1, \quad f_{4a(4)}(3) = 0, \quad f_{4a(4)}(4) = 1, \\ f_{5a(5)}(1) &= 0, \quad f_{5a(5)}(2) = 1, \quad f_{5a(5)}(3) = 0, \quad f_{5a(5)}(4) = 1, \quad f_{5a(5)}(5) = 0, \dots \end{aligned} \tag{2.1}$$

It is obvious how this unique sequence of partial sequences is obtained from any $a \in A$. For each $a \in A$, let $B_a = \{f_{nm} \mid \forall n (n \in \mathbb{N}^{>0} \rightarrow m = a(n))\}$. Let $B_a^\dagger \subset B_a$ such that each $f_{nm} \in B_a^\dagger$ satisfies the partial sequence requirement $(\dagger)$. For each $n \in \mathbb{N}^{>0}$, let $Pf_{na(n)} \in B_a^\dagger$ denote the unique partial sequence of n terms generated by an a and the $(\dagger)$ requirement. In general, as will be demonstrated below, it is the $Pf_{na(n)}$ that yields the set of consequence operators as they are defined on L_1 . Consider an additional map M from the set $PF = \{Pf_{na(n)} \mid a \in A\}$ of these partial sequences into our descriptive language L_1 for the source G and events E, E' as they are now considered as labeled by the tick marks. For each $n \in \mathbb{N}^{>0}$, and $1 \leq j \leq n$, if $Pf_{na(n)}(j) = 0$, then $M(Pf_{na(n)}(j)) = E'$ (i.e., $E' = E$ does not occur); if $Pf_{na(n)}(j) = 1$, then $M(Pf_{na(n)}(j)) = E$ (i.e., E does occur), as $1 \leq j \leq n$, where the partial sequence $j = 1, \dots, n$ models the intuitive concept of an event sequence since each E or E' now contains the appropriate Kleene “tick” symbols or natural number symbols that are an abbreviation for this tick notation.

Consider the set of consequence operators, each defined on L , $H = \{C(X, \{G\}) \mid X \subset L\}$, where if $G \in Y$, then $C(X, \{G\})(Y) = Y \cup X$; if $G \notin Y$, then $C(X, \{G\})(Y) = Y$. Then for each $a \in A_p$, $n \in \mathbb{N}^{>0}$ and the respective $Pf_{na(n)}$, there exists the set of consequence operators $C_{ap} = \{C(\{M(P_{na(n)}(j))\}, \{G\}) \mid 1 \leq j \leq n\} \subset H$. Note that from [3, page 5], H is closed under the finite $\vee$ and the actual consequence operator is $C(\{M(P_{na(n)}(1))\} \cup \dots \cup \{M(P_{na(n)}(n))\}, \{G\})$. Applying a realism relation R (i.e., in general, $R(C(\{G\})) = C(\{G\}) - \{G\}$) to $C(\{M(P_{na(n)}(1))\} \cup \dots \cup \{M(P_{na(n)}(n))\}, \{G\})(\{G\})$ yields the actual labeled or identified event partial sequence $\{M(P_{na(n)}(1)), \dots, M(P_{na(n)}(n))\}$.

Now embed the above intuitive results into the superstructure $\mathcal{M}_1 = \langle \mathcal{R}, \in, = \rangle$ which is further embedded into the nonstandard structure $^*\mathcal{M}_1 = \langle ^*\mathcal{R}, \in, = \rangle$ [1, 2]. Let $p \in \mathbb{R}$ be such that $0 \leq p \leq 1$, where p represents a theory predicted (i.e., a priori) probability that an event will occur. Applying a choice function C to $\mathcal{A}$, there is some $a \in A_p$

such that $g_{ap} \rightarrow p$. Thus $*C$ applied to $*A$ yields $*a \in *A_p$ and $*g_{ap} \in *F_p$. Let $v \in *\mathbb{N}$ be any infinite natural number. The hyperfinite sequence $\{a_1, \dots, a_n, \dots, *a_v\}$ exists and corresponds to $\{a_1, \dots, a_n\}$ for any natural number $n \in \mathbb{N}^{>0}$. Also we know that $\text{st}(*a_\mu) = p$ for any infinite natural number μ . Thus there exists some internal hyperfinite $Pf_{v*a(v)} \in *PF$ with the $*$ -transferred properties mentioned above. Since $*H$ is closed under hyperfinite $\vee$, there is a $P_p \in *H$ such that, after application of the relation $*R$, the result is the hyperfinite sequence

$$S = \{^*M(P_{v*a(v)}(1)), \dots, ^*M(P_{v*a(v)}(j)), \dots, ^*M(P_{v*a(v)}(v))\}. \quad (2.2)$$

Note that if $j \in \mathbb{N}$, then we have that $*E = E$ or $*E' = E'$ as the case may be.

An extended standard mapping that restricts S to internal subsets would restrict S to $\{^*M(P_{v*a(v)}(1)), \dots, ^*M(P_{v*a(v)}(j))\}$, whenever $j \in \mathbb{N}^{>0}$. Such a restriction map models the restriction of S to the natural-world in accordance with the general interpretation given for internal or finite standard objects [2, page 98]. This completes the proof. $\square$

3. Distributions. Prior to considering the statistical notion of a frequency (mass, density) function and the distribution it generates, there is need to consider a finite *Cartesian product* consequence operator. Suppose that we have a finite set of consequence operators $\{C_1, \dots, C_m\}$, where at least one is axiomless, each defined upon its own language L_k . Define the operator ΠC_m as follows: for any $X \subset L_1 \times \dots \times L_m$, using the projections pr_k , consider the Cartesian product $\text{pr}_1(X) \times \dots \times \text{pr}_m(X)$. Then $\Pi C_m(X) = C_1(\text{pr}_1(X)) \times \dots \times C_m(\text{pr}_m(X))$ is a consequence operator on $L_1 \times \dots \times L_m$. If each C_k is a finite consequence operator, then ΠC_m is finite. In all other cases, ΠC_m is a general consequence operator. All of these standard facts also hold within our nonstandard structure under $*$ -transfer.

A distribution's frequency function is always considered to be the probabilistic measure that determines the number of events that occur within a *cell* or "interval" for a specific decomposition of the events into various definable and disjoint cells. There is a specific probability that a specific number of events will be contained in a specific cell and each event must occur in one and only one cell and not occur in any other cell.

For each distribution over a specific set of cells, I_k , there is a specific probability p_k that an event will occur in the cell I_k . Assuming that the distribution does indeed depict physical behavior, we will have a special collection of g_{ap_k} sequences generated. For example, assume that we have three cells and the three probabilities $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 1/4$ occur in I_1, I_2, I_3 , respectively. Assume that the number of events to occur is 6. Then the three partial sequences might appear as follows

$$\begin{aligned} g_{ap_1} &= \{(1, 1), (2, 1), (3, 1), (4, 2), (5, 2), (6, 2)\}, \\ g_{ap_2} &= \{(1, 0), (2, 1), (3, 2), (4, 2), (5, 2), (6, 3)\}, \\ g_{ap_3} &= \{(1, 0), (2, 0), (3, 0), (4, 0), (5, 1), (6, 1)\}. \end{aligned} \quad (3.1)$$

Thus after six events have occurred, 2 events are in the first cell, 3 events are in the second cell, and only 1 event is in the third cell. Of course, as the number of events

continues the first sequence will converge to $1/4$, the second to $1/2$, and the third to $1/4$. Obviously, for any $n \geq 1$, $g_{ap_1}(n) + g_{ap_2}(n) + g_{ap_3}(n) = n$. Clearly, these required g_{ap_i} properties can be formally generated and generalized to any finite number m of cells.

Relative to each factor of the Cartesian product set, all of the standard aspects of [Theorem 2.1](#) will hold. Further, these intuitive results are embedded into the above superstructure and further embedded into our nonstandard structure. Hence, assume that the languages $L_k = L_1$ and that the standard factor consequence operator C_k used to create the product consequence operator is a C_{ap_k} of [Theorem 2.1](#). Under the nonstandard embedding, we would have that for each factor, there is a pure nonstandard consequence operator $P_{p_k} \in {}^*\mathbf{H}_k$. Finally, consider the nonstandard product consequence operator ΠP_{p_m} . For $\ast(\{\mathbf{G}_1\} \times \cdots \times \{\mathbf{G}_m\}) = \{\mathbf{G}_1\} \times \cdots \times \{\mathbf{G}_m\}$, $\mathbf{G}_i = \mathbf{G}$, this nonstandard product consequence operator yields for any fixed event number n , an ordered m -tuple, where one and only one coordinate would have the statement $\mathbf{E}$ and all other coordinates the $\mathbf{E}'$. It would be these m -tuples that guide the proper cell placement for each event and would satisfy the usual requirements of the distribution. Hence, the patterns produced by a specific frequency function for a specific distribution may be rationally assumed to be the result of an application of an ultralogic.

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