IJMMS 26:6 (2001) 353–357
PII. S0161171201004380
http://ijmms.hindawi.com

SOME ALGEBRAIC UNIVERSAL SEMIGROUP COMPACTIFICATIONS

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(Received 20 November 1999)

ABSTRACT. Universal compactifications of semitopological semigroups with respect to the properties satisfying the varieties of semigroups and groups are studied through two function algebras.

2000 Mathematics Subject Classification. 22A20, 43A60, 54H15.

1. Introduction. In the main approach to semigroup compactification, whose considerations goes back at least to the pioneering paper of de Leeuw and Glicksberg [2], the spectra of some C^*-algebras of functions on a semitopological semigroup are employed to construct certain universal compactifications. For instance, Junghenn in his elaborate study of distal functions characterized the universal group compactification, [7]. The main goal of the present work is to construct two function algebras $\mathcal{F}_V$ and $\mathcal{F}_{\mathcal{V}}$ whose corresponding compactifications are universal with respect to the properties satisfying a variety V of semigroups and a variety $\mathcal{V}$ of groups (the structures of which in terms of subdirect products are given in [1, Section 3.3]).

2. Preliminaries. Here we highlight some required notions and notations from Berglund et al. [1], which is our ground rule. On $\mathcal{C}(S)$, the C^*-algebra of all continuous bounded complex-valued functions on a semitopological semigroup S, the left and right translations L_s and R_s are defined so that, $(L_s f)(t) = f(st) = (R_t f)(s)$. A unital C^*-subalgebra $\mathcal{F}$ of $\mathcal{C}(S)$ which is a left translation invariant (i.e., $L_s f \in \mathcal{F}$ for all $s \in S$ and $f \in \mathcal{F}$) is called m-admissible if the function $s \mapsto (T_\mu f)(s) = \mu(L_s f)$ lies in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $\mu \in S^{\mathcal{F}}$ (equal to the spectrum of $\mathcal{F}$). Then the pair $(\varepsilon, S^{\mathcal{F}})$ is a (semigroup) compactification (called $\mathcal{F}$-compactification) of S, in which $S^{\mathcal{F}}$ is furnished with the Gelfand topology and the multiplication $\mu\nu = \mu \circ T_\nu$, and $\varepsilon : S \to S^{\mathcal{F}}$ is the evaluation mapping.

The m-admissible algebras of the left multiplicatively continuous, (weakly) [strongly] almost periodic, and strongly minimal distal functions, are denoted by $\mathcal{LMC}$, $(\mathcal{WAP})$ $[\mathcal{SAP}]\mathcal{AP}$, and $\mathcal{GP}$, respectively; see [1, 7].

3. The main results. From now on, F_A is a fixed free semigroup on the countable alphabet A, on which the identities of V are expressed, and $p(=\prod_{i=1}^m a_i) = q(=\prod_{j=1}^n b_j)$ is a fixed arbitrary formal identity of V; see [6].

DEFINITION 3.1. We define $\mathcal{F}_V(S)$ as the set of those $f \in \mathcal{LMC}(S)$ such that the identities

$$\lim_{\alpha}\left(\left(\prod_{i=2}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)f\right)(\theta_\alpha(a_1))=\lim_{\alpha}\left(\left(\prod_{j=2}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)f\right)(\theta_\alpha(b_1)), \tag{3.1}$$

$$\lim_{\alpha}\left(\left(\prod_{i=2}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)\left(\lim_{\alpha}R_{s_\alpha}f\right)\right)(\theta_\alpha(a_1))=\lim_{\alpha}\left(\left(\prod_{j=2}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)\left(\lim_{\alpha}R_{s_\alpha}f\right)\right)(\theta_\alpha(b_1)), \tag{3.2}$$

$$\left(\prod_{i=1}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)f=\left(\prod_{j=1}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)f, \tag{3.3}$$

$$\left(\prod_{i=1}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)\left(\lim_{\alpha}R_{s_\alpha}f\right)=\left(\prod_{j=1}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)\left(\lim_{\alpha}R_{s_\alpha}f\right), \tag{3.4}$$

are valid for all identities $\prod_{i=1}^{m}a_i=\prod_{j=1}^{n}b_j$ of V, and all nets θ_α (of mappings from A into S), for which the involved pointwise limits exist. (Note that, the notation $\prod$ is being used for the iterated multiplications in different semigroups, and one should realize that $\prod_{i=2}^{m}\lim_\alpha R_{\theta_\alpha(a_i)}$ plays the role of identity operator whenever $m=1$; similarly for $n=1$.)

The next lemma presents $\mathcal{F}_V$ in terms of $S^{\mathcal{LMC}}$.

Lemma 3.2. *A function $f\in\mathcal{LMC}$ belongs to $\mathcal{F}_V$ if and only if for every homomorphism $\Phi:F_A\to S^{\mathcal{LMC}}$, each identity $p=q$ of V, and every $\mu\in S^{\mathcal{LMC}}$, $\Phi(p)(f)=\Phi(q)(f)$, $(\Phi(p)\mu)(f)=(\Phi(q)\mu)(f)$, $T_{\Phi(p)}f=T_{\Phi(q)}f$, and $T_{(\Phi(p)\mu)}f=T_{(\Phi(q)\mu)}f$.*

Proof. For the necessity, it is enough to show that $\Phi(p)(g)=\Phi(q)(g)$ and $T_{\Phi(p)}g=T_{\Phi(q)}g$, for all $g\in X_f\cup\{f\}$, where X_f is the pointwise closure of R_Sf in $\mathcal{C}(S)$. There exists a net $\theta_\alpha:A\to S$ such that, for each i and j, $\lim_\alpha\varepsilon(\theta_\alpha(a_i))=\Phi(a_i)$ and $\lim_\alpha\varepsilon(\theta_\alpha(b_j))=\Phi(b_j)$. Now using (3.1) and (3.2), we have

$$\begin{aligned}\Phi(p)(g)&=\Phi(a_1)\left(\prod_{i=2}^{m}T_{\Phi(a_i)}\right)g\\&=\lim_{\alpha}\left(\left(\prod_{i=2}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)g\right)(\theta_\alpha(a_1))\\&=\lim_{\alpha}\left(\left(\prod_{j=2}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)g\right)(\theta_\alpha(b_1))\\&=\Phi(q)(g).\end{aligned} \tag{3.5}$$

A similar argument, using (3.3) and (3.4), shows that $T_{\Phi(p)}g=T_{\Phi(q)}g$. Conversely, for any net θ_α, of mappings from A into S, consider the mapping $\theta:A\to S^{\mathcal{LMC}}$ defined by $\theta(a)=\lim_\alpha\varepsilon(\theta_\alpha(a))$. Using the crucial property of free semigroups, θ can be extended to a homomorphism $\Phi:F_A\to S^{\mathcal{LMC}}$. Therefore, for every $g\in X_f\cup\{f\}$,

$$\left(\prod_{i=1}^{m}\lim_{\alpha}R_{\theta_\alpha(a_i)}\right)g=\left(\prod_{i=1}^{m}T_{\Phi(a_i)}\right)g=T_{\Phi(p)}g=T_{\Phi(q)}g=\left(\prod_{j=1}^{n}\lim_{\alpha}R_{\theta_\alpha(b_j)}\right)g; \tag{3.6}$$

which is equivalent to (3.3) and (3.4), taken together. A similar argument can be used to obtain the other required identity, which is equivalent to (3.1) and (3.2) taken together. □

Now we have the next result which describes the main property of F_V.

Theorem 3.3. *$\mathcal{F}_V$ is m-admissible and $\mathcal{F}_V$-compactification is universal with respect to the property satisfying V.*

Proof. Using Lemma 3.2, the m-admissibility of $\mathcal{F}_V$ may be readily verified. Again Lemma 3.2 implies that, for each homomorphism $\Phi : F_A \to S^{\mathcal{F}_V}$ and $f \in \mathcal{F}_V$, $\Phi(p)(f) = \Phi(q)(f)$; which means $S^{\mathcal{F}_V} \in V$. It is enough to show that for every other compactification (ψ, X) of S, with $X \in V$, $\psi^*(\mathcal{C}(X)) \subseteq \mathcal{F}_V(S)$. Let $\pi : S^{\mathcal{LMC}} \to X$ be the homomorphism for which $\pi \circ \varepsilon = \psi$. This implies that $\psi^*(\mathcal{C}(X)) \subseteq \mathcal{LMC}(S)$. Furthermore, if $h \in \mathcal{C}(X)$, then for each homomorphism $\Phi : F_A \to S^{\mathcal{LMC}}$ and each $\mu \in S^{\mathcal{LMC}}$,

$$(\Phi(p)\mu)(\psi^*(h)) = h((\pi \circ \Phi)(p)\pi(\mu)) = h((\pi \circ \Phi)(q)\pi(\mu)) = (\Phi(q)\mu)(\psi^*(h)). \tag{3.7}$$

Similar arguments show that

$$\begin{aligned} \Phi(p)(\psi^*(h)) &= \Phi(q)(\psi^*(h)), \\ T_{\Phi(p)}\psi^*(h) &= T_{\Phi(q)}\psi^*(h), \\ T_{(\Phi(p)\mu)}\psi^*(h) &= T_{(\Phi(q)\mu)}\psi^*(h). \end{aligned} \tag{3.8}$$

Now Lemma 3.2 implies that $\psi^*(h) \in \mathcal{F}_V(S)$, as claimed. □

Let $\mathcal{F}_{V_c}$ consist of those $f \in \mathcal{C}(S)$ such that $f(\Phi(p)) = f(\Phi(q))$, $f(\Phi(p)s) = f(\Phi(q)s)$, $f(s\Phi(p)) = f(s\Phi(q))$, and $f(s\Phi(p)t) = f(s\Phi(q)t)$, for all $s, t \in S$, for every homomorphism $\Phi : F_A \to S$, and all identities $p = q$ of V. Trivially, $\mathcal{F}_V \subseteq \mathcal{F}_{V_c}$ (with the equality holding in the compact setting). Due to joint continuity of the multiplication of $S^{\mathcal{AP}}$, we get the next simplification of $\mathcal{F}_V \cap \mathcal{AP}$.

Proposition 3.4. $\mathcal{F}_V \cap \mathcal{AP} = \mathcal{F}_{V_c} \cap \mathcal{AP}$.

Proof. According to Lemma 3.2 it is enough to show that if $f \in \mathcal{F}_{V_c} \cap \mathcal{AP}$, then for each homomorphism $\Phi : F_A \to S^{\mathcal{AP}}$, and all $\mu \in S^{\mathcal{AP}}$, $\Phi(p)(f) = \Phi(q)(f)$, $(\Phi(p)\mu)(f) = (\Phi(q)\mu)(f)$, $T_{\Phi(p)}f = T_{\Phi(q)}f$, and $T_{(\Phi(p)\mu)}f = T_{(\Phi(q)\mu)}f$. Imitating the methods of the proof of Lemma 3.2, let $\Phi_\alpha : F_A \to S$ be that net (of homomorphisms) for which $\lim_\alpha \varepsilon(\Phi_\alpha(a_i)) = \Phi(a_i)$ and $\lim_\alpha \varepsilon(\Phi_\alpha(b_j)) = \Phi(b_j)$; and also t_α be a net in S such that $\lim_\alpha \varepsilon(t_\alpha) = \mu$. Now, for all $s \in S$;

$$\begin{aligned} (T_{\Phi(p)\mu}f)(s) &= \left(\left(\prod_{i=1}^{m}\Phi(a_i)\right)\mu\right)(L_s f) \\ &= \left(\left(\prod_{i=1}^{m}\lim_\alpha \varepsilon(\Phi_\alpha(a_i))\right)\left(\lim_\alpha \varepsilon(t_\alpha)\right)\right)(L_s f) \\ &= \lim_\alpha \varepsilon(\Phi_\alpha(p)t_\alpha)(L_s f) = \lim_\alpha f(s\Phi_\alpha(p)t_\alpha) \\ &= \lim_\alpha f(s\Phi_\alpha(q)t_\alpha) = (T_{\Phi(q)\mu}f)(s). \end{aligned} \tag{3.9}$$

Similar arguments may apply to the other required equalities. □

Remarks. (a) The present results are a generalization of what we have described in [4], for the familiar varieties of abelian semigroups, AB, bands, BD, and semilattices, SL.

By Definition 3.1, $\mathcal{F}_{AB}$ consists of those $f \in \mathcal{LMC}$ such that the identities

$$\begin{aligned}
&\lim_\alpha \Big(\lim_\alpha R_{s_\alpha} f\Big)(t_\alpha) = \lim_\alpha \Big(\lim_\alpha R_{t_\alpha} f\Big)(s_\alpha),\\
&\lim_\alpha \Big(\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{u_\alpha} f\Big)\Big)(t_\alpha) = \lim_\alpha \Big(\lim_\alpha R_{t_\alpha}\Big(\lim_\alpha R_{u_\alpha} f\Big)\Big)(s_\alpha),\\
&\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{t_\alpha} f\Big) = \lim_\alpha R_{t_\alpha}\Big(\lim_\alpha R_{s_\alpha} f\Big),\\
&\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{t_\alpha}\Big(\lim_\alpha R_{u_\alpha} f\Big)\Big) = \lim_\alpha R_{t_\alpha}\Big(\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{u_\alpha} f\Big)\Big)
\end{aligned} \tag{3.10}$$

hold for all nets s_α, t_α, and u_α in S, for which the limits exist. Since $S^{\mathcal{F}_{AB}}$ is semitopological, one realizes that the latter seemingly complicated limit process is preparatory, and can be condensed so that it presents $\mathcal{F}_{AB}$ in the simple form

$$\{f \in \mathcal{WAP} : f(st) = f(ts), \text{ and } f(stu) = f(sut)\,\forall s,t,u \in S\}. \tag{3.11}$$

In other words, $\mathcal{F}_{AB} = \mathcal{F}_{AB_c} \cap \mathcal{WAP}$.

Again, Definition 3.1 implies that $\mathcal{F}_{BD}$ is the set of all $f \in \mathcal{LMC}$, for which

$$\begin{aligned}
&\lim_\alpha \Big(\lim_\alpha R_{s_\alpha} f\Big)(s_\alpha) = \lim_\alpha f(s_\alpha),\\
&\lim_\alpha \Big(\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{t_\alpha} f\Big)\Big)(s_\alpha) = \lim_\alpha \Big(\lim_\alpha R_{t_\alpha} f\Big)(s_\alpha),\\
&\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{s_\alpha} f\Big) = \lim_\alpha R_{s_\alpha} f,\\
&\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{t_\alpha} f\Big)\Big) = \lim_\alpha R_{s_\alpha}\Big(\lim_\alpha R_{t_\alpha} f\Big);
\end{aligned} \tag{3.12}$$

for all nets s_α and t_α in S, for which the limits exist. In contrast to the situation for $\mathcal{F}_{AB}$, in general $\mathcal{F}_{BD} \nsubseteq \mathcal{WAP}$ (e.g., a direct verification shows that the characteristic function of even numbers on $\mathbb{N}$, with its maximum multiplication, lies in $\mathcal{F}_{BD}$ but not in $\mathcal{WAP}$). Furthermore, as it is seen by easy examples (e.g., consider $[0,1]\times\mathbb{N}$ under its rectangular multiplication) $\mathcal{F}_{BD} \neq \mathcal{F}_{BD_c}$, in general. It would be desirable to investigate the equality of $\mathcal{F}_{BD} = \mathcal{F}_{BD_c} \cap \mathcal{LMC}$, the left and right introversion of $\mathcal{F}_{BD}$, the relation between $\mathcal{F}_{BD}$ and the space similarly defined in terms of the left translations, and to characterize the topological center of $S^{\mathcal{F}_{BD}}$. We believe that there are close connections among these problems.

By the joint continuity theorem of Lawson [8], $\mathcal{F}_{SL} \subseteq \mathcal{AP}$ and so $\mathcal{F}_{SL} = \mathcal{F}_{SL_c} \cap \mathcal{AP}$.

It would be desirable to examine $\mathcal{F}_V$ for the variety of simple semigroups.

(b) It might be readily verified that the conditions (3.3) and (3.4) in the definition of $\mathcal{F}_V$, taken together, are equivalent to the fact that the enveloping semigroup $\sum(S, X_f \cup \{f\})$, of the natural flow $(S, X_f \cup \{f\})$, lies in V; and so the latter is satisfied whenever $f \in \mathcal{F}_V$. A natural question that arises is whether the converse is also true. As a helpful answer, one can verify that; a function $f \in \mathcal{GP}$ lies in $\mathcal{F}_V$ if and only if $\sum(S, X_f)$ lies in V; that is, $\mathcal{F}_V \cap \mathcal{GP} = \{f \in \mathcal{GP} : \sum(S, X_f) \in V\}$.

Now, for a variety $\mathcal{V}$ of groups, defined by the set of laws Ω (see [9]), we define $\mathcal{F}_{\mathcal{V}}(S)$ as the set of all $f \in \mathcal{GP}(S)$ such that $(\prod_{i=1}^m (\lim_\alpha R_{s_{i_\alpha}})^{l_i})(\lim_\alpha R_{s_\alpha} f) = \lim_\alpha R_{s_\alpha} f$, for all laws $\prod_{i=1}^m x_i{}^{l_i}$ in Ω, and all nets s_{i_α} and s_α in S for which the required limits exists.

It is easy to verify that, a function $f \in \mathcal{GP}$ lies in $\mathcal{F}_{\mathcal{V}}$ if and only if $T_{\omega\mu}f = T_{\mu}f$ for all $\mu \in S^{\mathcal{GP}}$ and all values in $S^{\mathcal{GP}}$ of the laws $\omega \in \Omega$; which is also equivalent to the fact that $\sum(S, X_f) \in \mathcal{V}$. Using these, one may obtain the next result, which is the group version of Theorem 3.3.

Theorem 3.5. *$\mathcal{F}_{\mathcal{V}}$ is m-admissible and $\mathcal{F}_{\mathcal{V}}$-compactification is universal with respect to the property satisfying $\mathcal{V}$.*

It should be mentioned that $\mathcal{F}_{\mathcal{V}}$ is not in $\mathcal{WAP}$ in general (e.g., for the variety of all groups we get $\mathcal{GP}$ which is not always in $\mathcal{WAP}$). Using the (joint continuity) theorem of Ellis [5], we have

$$\mathcal{F}_{\mathcal{V}} \cap \mathcal{WAP} = \mathcal{F}_{\mathcal{V}} \cap \mathcal{AP} = \mathcal{F}_{\mathcal{V}} \cap \mathcal{SAP}. \tag{3.13}$$

More precisely, each side of (3.13) (and hence $\mathcal{F}_{\mathcal{V}}$, when S is compact) consists of those $f \in \mathcal{SAP}$ such that, $(\prod_{i=1}^{m}(R_{s_i})^{l_i})(R_s f) = R_s f$, for all laws $\prod_{i=1}^{m} x_i{}^{l_i}$ in Ω, and all $s_1, s_2, \ldots, s_m$, and s in S. (For instance, for the variety of abelian groups, $\mathcal{F}_{\mathcal{V}}$ consists of those $f \in \mathcal{SAP}$ such that $f(stu) = f(sut)$ for all s, t and u in S, [3]; which, of course, is equal to $\mathcal{F}_{AB} \cap \mathcal{GP}$, see the previous remarks (b)). Hence, for every element of $\mathcal{V}$, each side of (3.13) is equal to $\mathcal{SAP}$, and so for a compact element S of $\mathcal{V}$, $\mathcal{F}_{\mathcal{V}}(S) = \mathcal{C}(S)$.

As we have shown in [3], for the variety of nilpotent groups, in the definition of $\mathcal{F}_{\mathcal{V}}$, $\lim_{\alpha} R_{s_\alpha} f$ can be replaced by f. However this seems not to be possible in general.

Acknowledgement. This work was in part supported by a grant from IPM.

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