SOME ALGEBRAIC UNIVERSAL SEMIGROUP
COMPACTIFICATIONS

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ABSTRACT. Universal compactifications of semitopological semigroups with respect to the properties satisfying the varieties of semigroups and groups are studied through two function algebras.

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1. Introduction. In the main approach to semigroup compactification, whose considerations goes back at least to the pioneering paper of de Leeuw and Glicksberg [2], the spectra of some $C^*$-algebras of functions on a semitopological semigroup are employed to construct certain universal compactifications. For instance, Junghenn in his elaborate study of distal functions characterized the universal group compactification, [7]. The main goal of the present work is to construct two function algebras $F_V$ and $F_Y$ whose corresponding compactifications are universal with respect to the properties satisfying a variety $V$ of semigroups and a variety $Y$ of groups (the structures of which in terms of subdirect products are given in [1, Section 3.3]).

2. Preliminaries. Here we highlight some required notions and notations from Berglund et al. [1], which is our ground rule. On $\mathcal{C}(S)$, the $C^*$-algebra of all continuous bounded complex-valued functions on a semitopological semigroup $S$, the left and right translations $L_s$ and $R_s$ are defined so that, $(L_s f)(t) = f(st) = (R_t f)(s)$. A unital $C^*$-subalgebra $F$ of $\mathcal{C}(S)$ which is a left translation invariant (i.e., $L_s f \in F$ for all $s \in S$ and $f \in F$) is called $m$-admissible if the function $s \rightarrow (T_\mu f)(s) = \mu(L_s f)$ lies in $F$ for all $f \in F$ and $\mu \in S^\circ$ (equal to the spectrum of $F$). Then the pair $(\varepsilon, S^\circ)$ is a (semigroup) compactification (called $F$-compactification) of $S$, in which $S^\circ$ is furnished with the Gelfand topology and the multiplication $\mu \nu = \mu \circ T_\nu$, and $\varepsilon : S \rightarrow S^\circ$ is the evaluation mapping.

The $m$-admissible algebras of the left multiplicatively continuous, (weakly) [strongly] almost periodic, and strongly minimal distal functions, are denoted by $\mathcal{L}\mathcal{M}\mathcal{C}$, $(W\mathcal{A}\mathcal{P})$ [$F\mathcal{A}\mathcal{P}$], $\mathcal{A}\mathcal{P}$, and $\mathcal{D}\mathcal{P}$, respectively; see [1, 7].

3. The main results. From now on, $F_A$ is a fixed free semigroup on the countable alphabet $A$, on which the identities of $V$ are expressed, and $p(\prod_{i=1}^m a_i) = q(\prod_{j=1}^n b_j)$ is a fixed arbitrary formal identity of $V$; see [6].

Definition 3.1. We define $F_V(S)$ as the set of those $f \in \mathcal{L}\mathcal{M}\mathcal{C}(S)$ such that the identities
\[
\lim_{\alpha} \left( \prod_{i=2}^{m} R_{\theta_\alpha(a_i)} \right) f (\theta_\alpha(a_1)) = \lim_{\alpha} \left( \prod_{j=2}^{n} R_{\theta_\alpha(b_j)} \right) f (\theta_\alpha(b_1)), \quad (3.1)
\]

\[
\lim_{\alpha} \left( \prod_{i=1}^{m} R_{\theta_\alpha(a_i)} \right) \left( \lim_{\alpha} R_{\alpha, f} \right) (\theta_\alpha(a_1)) = \lim_{\alpha} \left( \prod_{j=2}^{n} R_{\theta_\alpha(b_j)} \right) \left( \lim_{\alpha} R_{\alpha, f} \right) (\theta_\alpha(b_1)), \quad (3.2)
\]

\[
\left( \prod_{i=1}^{m} R_{\theta_\alpha(a_i)} \right) f = \left( \prod_{j=1}^{n} R_{\theta_\alpha(b_j)} \right) f, \quad (3.3)
\]

\[
\left( \prod_{i=1}^{m} R_{\theta_\alpha(a_i)} \right) \left( \lim_{\alpha} R_{\alpha, f} \right) = \left( \prod_{j=1}^{n} R_{\theta_\alpha(b_j)} \right) \left( \lim_{\alpha} R_{\alpha, f} \right), \quad (3.4)
\]

are valid for all identities $\prod_{i=1}^{m} a_i = \prod_{j=1}^{n} b_j$ of $V$, and all nets $\theta_\alpha$ (of mappings from $A$ into $S$), for which the involved pointwise limits exist. (Note that the notation $\prod$ is being used for the iterated multiplications in different semigroups, and one should realize that $\prod_{i=2}^{m} \lim_{\alpha} R_{\theta_\alpha(a_i)}$ plays the role of identity operator whenever $m = 1$; similarly for $n = 1$.)

The next lemma presents $\mathcal{F}_V$ in terms of $S^{x, y, e}$.

**Lemma 3.2.** A function $f \in \mathcal{F}_V$ belongs to $\mathcal{F}_V$ if and only if for every homomorphism $\Phi : F_A \to S^{x, y, e}$, each identity $p = q$ of $V$, and every $\mu \in S^{x, y, e}$, $\Phi(p)(f) = \Phi(q)(f)$, $\Phi(p)(\mu)(f) = \Phi(q)(\mu)(f)$, $T_{\Phi(p)}f = T_{\Phi(q)}f$, and $T_{\Phi(p)} \mu f = T_{\Phi(q)} \mu f$.

**Proof.** For the necessity, it is enough to show that $\Phi(p)(g) = \Phi(q)(g)$ and $T_{\Phi(p)}g = T_{\Phi(q)}g$, for all $g \in X_f \cup \{f\}$, where $X_f$ is the pointwise closure of $R_S f$ in $\mathcal{E}(S)$. There exists a net $\theta_\alpha : A \to S$ such that, for each $i$ and $j$, $\lim_{\alpha} \varepsilon(\theta_\alpha(a_i)) = \Phi(a_i)$ and $\lim_{\alpha} \varepsilon(\theta_\alpha(b_j)) = \Phi(b_j)$. Now using (3.1) and (3.2), we have

\[
\Phi(p)(g) = \Phi(a_1) \left( \prod_{i=2}^{m} T_{\Phi(a_i)} \right) g = \lim_{\alpha} \left( \prod_{i=2}^{m} \lim_{\alpha} R_{\theta_\alpha(a_i)} \right) g (\theta_\alpha(a_1)) = \lim_{\alpha} \left( \prod_{j=2}^{n} \lim_{\alpha} R_{\theta_\alpha(b_j)} \right) g (\theta_\alpha(b_1)) = \Phi(q)(g). \quad (3.5)
\]

A similar argument, using (3.3) and (3.4), shows that $T_{\Phi(p)}g = T_{\Phi(q)}g$. Conversely, for any net $\theta_\alpha$ of mappings from $A$ into $S$, consider the mapping $\theta : A \to S^{x, y, e}$ defined by $\theta(a) = \lim_{\alpha} \varepsilon(\theta_\alpha(a))$. Using the crucial property of free semigroups, $\theta$ can be extended to a homomorphism $\Phi : F_A \to S^{x, y, e}$. Therefore, for every $g \in X_f \cup \{f\}$,

\[
\left( \prod_{i=1}^{m} \lim_{\alpha} R_{\theta_\alpha(a_i)} \right) g = \left( \prod_{i=1}^{m} T_{\Phi(a_i)} \right) g = T_{\Phi(p)}g = T_{\Phi(q)}g = \left( \prod_{j=1}^{n} \lim_{\alpha} R_{\theta_\alpha(b_j)} \right) g; \quad (3.6)
\]
which is equivalent to (3.3) and (3.4), taken together. A similar argument can be used to obtain the other required identity, which is equivalent to (3.1) and (3.2) taken together.

Now we have the next result which describes the main property of $F_V$.

**Theorem 3.3.** $F_V$ is $m$-admissible and $F_V$-compactification is universal with respect to the property satisfying $V$.

**Proof.** Using Lemma 3.2, the $m$-admissibility of $F_V$ may be readily verified. Again Lemma 3.2 implies that, for each homomorphism $\Phi : F_A \to S^{\psi S}$ and $f \in F_V$, $\Phi(p)(f) = \Phi(q)(f)$; which means $S^{\psi S} \subseteq V$. It is enough to show that for every other compactification $(\psi, X)$ of $S$, with $X \subseteq V$, $\psi^*(\varepsilon X) \subseteq F_V(S)$. Let $\pi : S^{\psi S} \to X$ be the homomorphism for which $\pi \circ \varepsilon = \psi$. This implies that $\psi^*(\varepsilon X) \subseteq T\varepsilon(S)$. Furthermore, if $h \in \varepsilon X$, then for each homomorphism $\Phi : F_A \to S^{\psi S}$ and each $\mu \in S^{\psi S}$,

$$\left(\Phi(p)\mu\right)(\psi^*(h)) = h\left((\pi \circ \Phi)\mu\pi(\mu)\right) = (\Phi(q)\mu)(\psi^*(h)).$$

Similar arguments show that

$$\Phi(p)(\psi^*(h)) = \Phi(q)(\psi^*(h)),$$

$$T_{\Phi(p)}\psi^*(h) = T_{\Phi(q)}\psi^*(h),$$

$$T_{\Phi(p)\mu}\psi^*(h) = T_{\Phi(q)\mu}\psi^*(h).$$

Now Lemma 3.2 implies that $\psi^*(h) \in F_V(S)$, as claimed.

Let $F_{Vc}$ consist of those $f \in \varepsilon(S)$ such that $f(\Phi(p)) = f(\Phi(q))$, $f(\Phi(p)s) = f(\Phi(q)s)$, $f(s\Phi(p)) = f(s\Phi(q))$, and $f(s\Phi(p)t) = f(s\Phi(q)t)$, for all $s, t \in S$, for every homomorphism $\Phi : F_A \to S$, and all identities $p = q$ of $V$. Trivially, $F_{Vc} \subseteq F_{Vc}$ (with the equality holding in the compact setting). Due to joint continuity of the multiplication of $S^{\psi S}$, we get the next simplification of $F_{Vc} \cap A\mathcal{P}$.

**Proposition 3.4.** $F_{Vc} \cap A\mathcal{P} = F_{Vc} \cap A\mathcal{P}$.

**Proof.** According to Lemma 3.2 it is enough to show that if $f \in F_{Vc} \cap A\mathcal{P}$, then for each homomorphism $\Phi : F_A \to S^{\psi S}$, and all $\mu \in S^{\psi S}$, $\Phi(p)(f) = \Phi(q)(f)$, $(\Phi(p)\mu)(f) = (\Phi(q)\mu)(f)$, $T_{\Phi(p)}f = T_{\Phi(q)}f$, and $T_{\Phi(p)\mu}f = T_{\Phi(q)\mu}f$. Imitating the methods of the proof of Lemma 3.2, let $\Phi_\alpha : F_A \to S$ be that net (of homomorphisms) for which $\lim \alpha \varepsilon(\Phi_\alpha(a_i)) = \Phi(a_i)$, and $\lim \alpha \varepsilon(\Phi_\alpha(b_j)) = \Phi(b_j)$; and also $t_\alpha$ be a net in $S$ such that $\lim \alpha \varepsilon(t_\alpha) = \mu$. Now, for all $s \in S$,

$$\left(T_{\Phi(p)\mu}f\right)(s) = \left(\left(\prod_{i=1}^{m} \Phi(a_i)\right)\mu\right)(L_sf)$$

$$= \left(\left[\prod_{i=1}^{m} \lim \alpha \varepsilon(\Phi_\alpha(a_i))\right] \lim \alpha \varepsilon(t_\alpha)\right)(L_sf)$$

$$= \lim \alpha \varepsilon(\Phi_\alpha(p)t_\alpha)(L_sf) = \lim \alpha f(s\Phi_\alpha(p)t_\alpha)$$

$$= \lim \alpha f(s\Phi_\alpha(q)t_\alpha) = (T_{\Phi(q)\mu}f)(s).$$

Similar arguments may apply to the other required equalities.
In other words, \( \{ f \in \mathcal{L}M \mathcal{E} : f(st) = f(ts), \text{ and } f(stu) = f(sut) \forall s,t,u \in S \} \). (3.11)

In other words, \( \mathcal{F}_{AB} = \mathcal{F}_{AB, c} \cap \mathcal{W}A \mathcal{P} \).

Again, Definition 3.1 implies that \( \mathcal{F}_{BD} \) is the set of all \( f \in \mathcal{L}M \mathcal{E} \), for which

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{s,\alpha} f \right) (s) = \lim_{\alpha} f(s),
\]

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) (s) = \lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) (s),
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{s,\alpha} f \right) = \lim_{\alpha} R_{t,\alpha} f,
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) = \lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right);
\]

for all nets \( s_{\alpha} \) and \( t_{\alpha} \) in \( S \), for which the limits exist. In contrast to the situation for \( \mathcal{F}_{AB} \), in general \( \mathcal{F}_{BD} \neq \mathcal{W}A \mathcal{P} \) (e.g., a direct verification shows that the characteristic function of even numbers on \( \mathbb{N} \), with its maximum multiplication, lies in \( \mathcal{F}_{BD} \) but not in \( \mathcal{W}A \mathcal{P} \)). Furthermore, as it is seen by easy examples (e.g., consider \( [0,1] \times \mathbb{N} \) under its rectangular multiplication) \( \mathcal{F}_{BD} \neq \mathcal{F}_{BD, c} \), in general. It would be desirable to investigate the equality of \( \mathcal{F}_{BD} = \mathcal{F}_{BD, c} \cap \mathcal{L}M \mathcal{E} \), the left and right introversion of \( \mathcal{F}_{BD} \), the relation between \( \mathcal{F}_{BD} \) and the space similarly defined in terms of the left translations, and to characterize the topological center of \( S^\mathcal{F}_{BD} \). We believe that there are close connections among these problems.

By the joint continuity theorem of Lawson [8], \( \mathcal{F}_{SL} \subseteq \mathcal{A} \mathcal{P} \) and so \( \mathcal{F}_{SL} = \mathcal{F}_{SL, c} \cap \mathcal{A} \mathcal{P} \).

It would be desirable to examine \( \mathcal{F}_{V} \) for the variety of simple semigroups.

(b) It might be readily verified that the conditions (3.3) and (3.4) in the definition of \( \mathcal{F}_{V} \), taken together, are equivalent to the fact that the enveloping semigroup \( \sum (S, X_{f} \cup \{ f \}) \), of the natural flow \( (S, X_{f} \cup \{ f \}) \), lies in \( V \); and so the latter is satisfied whenever \( f \in \mathcal{F}_{V} \). A natural question that arises is whether the converse is also true. As a helpful answer, one can verify that; a function \( f \in \mathcal{G} \mathcal{P} \) lies in \( \mathcal{F}_{V} \) if and only if \( \sum (S, X_{f}) \) lies in \( V \); that is, \( \mathcal{F}_{V} \cap \mathcal{G} \mathcal{P} = \{ f \in \mathcal{G} \mathcal{P} : \sum (S, X_{f}) \in V \} \).

Now, for a variety \( V \) of groups, defined by the set of laws \( \Omega \) (see [9]), we define \( \mathcal{F}_{V} (S) \) as the set of all \( f \in \mathcal{G} \mathcal{P} (S) \) such that \( (\prod_{i=1}^{n} (\lim_{\alpha} R_{s,\alpha} f^{i})) (\lim_{\alpha} R_{s,\alpha} f) = \lim_{\alpha} R_{s,\alpha} f^{i} \), for all laws \( \prod_{i=1}^{n} X_{i}^{i} \) in \( \Omega \), and all nets \( s_{\alpha} \) and \( s_{\alpha} \) in \( S \) for which the required limits exist.

**Remarks.** (a) The present results are a generalization of what we have described in [4], for the familiar varieties of abelian semigroups, \( AB \), bands, \( BD \), and semilattices, \( SL \).

By Definition 3.1, \( \mathcal{F}_{AB} \) consists of those \( f \in \mathcal{L}M \mathcal{E} \) such that the identities

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{s,\alpha} f \right) (t) = \lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) (s),
\]

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{u,\alpha} f \right) \right) (t) = \lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} \left( \lim_{\alpha} R_{u,\alpha} f \right) \right) (s),
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) = \lim_{\alpha} R_{t,\alpha} \left( \lim_{\alpha} R_{s,\alpha} f \right),
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) = \lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right)
\]

hold for all nets \( s_{\alpha}, t_{\alpha} \), and \( u_{\alpha} \) in \( S \), for which the limits exist. Since \( S^\mathcal{F}_{AB} \) is semitopological, one realizes that the latter seemingly complicated limit process is preparatory, and can be condensed so that it presents \( \mathcal{F}_{AB} \) in the simple form

\[
\{ f \in \mathcal{W}A \mathcal{P} : f(st) = f(ts), \text{ and } f(stu) = f(sut) \forall s,t,u \in S \}.
\]

In other words, \( \mathcal{F}_{AB} = \mathcal{F}_{AB, c} \cap \mathcal{W}A \mathcal{P} \).

Again, Definition 3.1 implies that \( \mathcal{F}_{BD} \) is the set of all \( f \in \mathcal{L}M \mathcal{E} \), for which

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{s,\alpha} f \right) (s) = \lim_{\alpha} f(s),
\]

\[
\lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) (s) = \lim_{\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) (s),
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) = \lim_{\alpha} R_{s,\alpha} f,
\]

\[
\lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right) = \lim_{\alpha} R_{s,\alpha} \left( \lim_{\alpha} R_{t,\alpha} f \right);
\]

for all nets \( s_{\alpha} \) and \( t_{\alpha} \) in \( S \), for which the limits exist. In contrast to the situation for \( \mathcal{F}_{AB} \), in general \( \mathcal{F}_{BD} \neq \mathcal{W}A \mathcal{P} \) (e.g., a direct verification shows that the characteristic function of even numbers on \( \mathbb{N} \), with its maximum multiplication, lies in \( \mathcal{F}_{BD} \) but not in \( \mathcal{W}A \mathcal{P} \)). Furthermore, as it is seen by easy examples (e.g., consider \( [0,1] \times \mathbb{N} \) under its rectangular multiplication) \( \mathcal{F}_{BD} \neq \mathcal{F}_{BD, c} \), in general. It would be desirable to investigate the equality of \( \mathcal{F}_{BD} = \mathcal{F}_{BD, c} \cap \mathcal{L}M \mathcal{E} \), the left and right introversion of \( \mathcal{F}_{BD} \), the relation between \( \mathcal{F}_{BD} \) and the space similarly defined in terms of the left translations, and to characterize the topological center of \( S^\mathcal{F}_{BD} \). We believe that there are close connections among these problems.

By the joint continuity theorem of Lawson [8], \( \mathcal{F}_{SL} \subseteq \mathcal{A} \mathcal{P} \) and so \( \mathcal{F}_{SL} = \mathcal{F}_{SL, c} \cap \mathcal{A} \mathcal{P} \).

It would be desirable to examine \( \mathcal{F}_{V} \) for the variety of simple semigroups.
It is easy to verify that, a function $f \in \mathcal{F}_Y$ if and only if $T_{\omega \mu}f = T_{\mu}f$ for all $\mu \in S^{op}$ and all values in $S^{op}$ of the laws $\omega \in \Omega$; which is also equivalent to the fact that $\sum(S, X_f) \in \mathcal{V}$. Using these, one may obtain the next result, which is the group version of Theorem 3.3.

**Theorem 3.5.** $\mathcal{F}_Y$ is $m$-admissible and $\mathcal{F}_Y$-compactification is universal with respect to the property satisfying $\mathcal{V}$.

It should be mentioned that $\mathcal{F}_Y$ is not in $\mathcal{WAP}$ in general (e.g., for the variety of all groups we get $\mathcal{AP}$ which is not always in $\mathcal{WAP}$). Using the (joint continuity) theorem of Ellis [5], we have

$$\mathcal{F}_Y \cap \mathcal{WAP} = \mathcal{F}_Y \cap \mathcal{AP} = \mathcal{F}_Y \cap \mathcal{IA}. \quad (3.13)$$

More precisely, each side of (3.13) (and hence $\mathcal{F}_Y$, when $S$ is compact) consists of those $f \in \mathcal{IA}$ such that $\prod(R_{s_1}^{n_1} x_{i_1}) (R_{s_2} f) = R_{s_2} f$, for all laws $\prod x_{i_1}$ in $\Omega$, and all $s_1, s_2, \ldots, s_m$, and $s$ in $S$. (For instance, for the variety of abelian groups, $\mathcal{F}_Y$ consists of those $f \in \mathcal{IA}$ such that $f(stu) = f(stu)$ for all $s, t$ and $u$ in $S$, [3]; which, of course, is equal to $\mathcal{IA} \cap \mathcal{AP}$, see the previous remarks (b)). Hence, for every element of $\mathcal{V}$, each side of (3.13) is equal to $\mathcal{IA}$, and so for a compact element $S$ of $\mathcal{V}$, $\mathcal{F}_Y(S) = \mathcal{C}(S)$.

As we have shown in [3], for the variety of nilpotent groups, in the definition of $\mathcal{F}_Y$, $\lim_\alpha R_{s,\alpha} f$ can be replaced by $f$. However this seems not to be possible in general.

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**References**


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