ON A THEOREM OF SCHUR

PETER HILTON

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To the memory of a dear friend and colleague, Paul Olum

ABSTRACT. We study the ramifications of Schur's theorem that, if G is a group such that G/ZG is finite, then G' is finite, if we restrict attention to nilpotent group. Here ZG is the center of G , and G' is the commutator subgroup. We use localization methods and obtain relativized versions of the main theorems.

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1. Introduction. The theorem to which we refer is that which asserts that if G is a group and ZG is its center, then

$$G/ZG \text{ finite} \implies G' \text{ finite}, \quad (1.1)$$

where G' is the commutator subgroup of G . This theorem has a nice homological proof, using the 5-term exact sequence

$$H_2G \xrightarrow{\alpha_4} H_2(G/ZG) \xrightarrow{\alpha_3} ZG \xrightarrow{\alpha_2} G_{ab} \xrightarrow{\alpha_1} (G/ZG)_{ab} \quad (1.2)$$

derived from the short exact sequence $ZG \rightarrow G \rightarrow G/ZG$. For if G/ZG is finite then $H_2(G/ZG)$ is finite. Thus $G' \cap ZG = \ker \alpha_2 = \text{im } \alpha_3$ is finite. But $G'/G' \cap ZG \subseteq G/ZG$ is also finite, so, finally, G' is finite.

We remark that Schur's theorem has a converse which is valid if G is finitely generated (fg). We include a proof for completeness.

THEOREM 1.1. *Let G be an fg group such that G' is finite. Then G/ZG is finite.*

PROOF. Let $G = \langle x_1, x_2, \dots, x_k \rangle$. Now, for any $x \in G$, there can only be *finitely* many distinct conjugates of x . For there is a one-one correspondence

$$y^{-1}xy \longleftrightarrow x^{-1}y^{-1}xy \quad (1.3)$$

between the set of conjugates of x and a subset of G' ; and G' is finite. Thus $[G : C_G x]$ is finite for all $x \in G$, where $C_G S$ is the centralizer in G of the subset S of G . But if each $[G : C_G x_i]$, $1 \leq i \leq k$, is finite, so is $[G : \cap_i C_G x_i]$. On the other hand, $\cap_i C_G x_i = ZG$, so $[G : ZG]$ is finite. Thus, as claimed, G/ZG is a finite group. $\square$

Schur's theorem, and its converse, take on a particular significance in the localization theory of nilpotent groups [1]. For it is one of the main problems in that theory to

calculate the *Mislin genus* $\mathcal{G}(G)$ of an $f\mathcal{G}$ nilpotent group G and to identify its members. Here $\mathcal{G}(G)$ is the set of isomorphism classes of $f\mathcal{G}$ nilpotent groups H such that G and H localize at every prime p to isomorphic groups, $G_p \cong H_p$ for all primes p . It is shown in [2, 3] that if G' is finite then $\mathcal{G}(G)$ may itself be given the structure of a (finite) abelian group, a fact which very much facilitates the study of $\mathcal{G}(G)$.

In the category of nilpotent groups (not necessarily $f\mathcal{G}$) it makes sense to consider P -torsion groups, where P is a family of primes, and to study such groups by the techniques of localization. In this way we are able to prove a P -torsion variant of Schur's theorem, namely,

THEOREM 1.2. *Let G be a nilpotent group such that G/ZG is a P -group. Then G' is a P -group.*

We may also prove a converse of Theorem 1.2; as with Schur's theorem itself, it is necessary to impose a supplementary finiteness condition.

THEOREM 1.3. *Let G be a nilpotent group such that G' is a P -group of exponent m . Then G/ZG is a P -group of exponent dividing m^{c-1} , where $\text{nil } G = c$.*

Actually we regard Theorems 1.2 and 1.3 as the *absolute* forms of our results and emphasize the *relative* forms which appear to be quite new. In our relativization we replace the group G by a pair (G, N) consisting of a nilpotent group G and a normal subgroup N of G . Then the absolute case is given by $N = G$; moreover, in our relativization, ZG is replaced by $C_G(N)$, which is easily seen to be a normal subgroup of G ; and G' is replaced by the commutator group $[G, N]$.

We remark that Theorem 1.2 also has a variant in which a finiteness condition is imposed just as in Theorem 1.3. Precisely, we have the following theorem.

THEOREM 1.4. *Let G be a nilpotent group such that G/ZG is a P -group of exponent m . Then G' is a P -group of exponent dividing m^{c-1} , where $\text{nil } G = c$.*

We will prove the relativizations of Theorems 1.2, 1.3, and 1.4 in Section 2. Proofs of the absolute forms, that is, of Theorems 1.3 and 1.4 are to be found in [4]. For Warfield proves (the case $n = 1$ is the critical case).

- (a) If Γ_{n+1} has exponent m , then $G/Z_n G$ has exponent dividing m^{c-n} (see [4, Corollary 2.6]); and
- (b) if $G/Z_n G$ has exponent m , then Γ_{n+1} has exponent dividing m^{c-n} (see [4, Corollary 3.16]).

Here we adopt Warfield's convention that $\Gamma_2 = G'$ and $Z_1 = ZG$.

We do not have available to us a homological proof of a relative version of Schur's theorem. However we do show in the appendix how we may use homological arguments to obtain Theorem 1.4 with a small loss of sharpness in our bound on the exponent of G' .

A key tool in our proof of the *relative* version of Theorem 1.3 is a theorem on the localization of nilpotent groups due to Karl Lorensen (Theorem 2.6). This theorem is of considerable interest in its own right. It is a pleasure to acknowledge the crucial help the author received from his friend (and erstwhile student) Karl Lorensen, not only in the provision of Theorem 2.6.

2. Localization methods. Let P be a family of primes and let Q be the complementary family. We first state and prove the relativization of [Theorem 1.2](#).

THEOREM 2.1. *Let G be a nilpotent group and N a normal subgroup such that $G/C_G(N)$ is a P -group. Then $[G, N]$ is a P -group.*

PROOF. Let $e : G \rightarrow G_Q$ localize at the family Q . Now e maps $C_G(N)$ into $C_{G_Q}(N_Q)$; moreover, $C_{G_Q}(N_Q)$ is Q -local. Thus, in fact, the Q -localization $C_G(N)_Q$ of $C_G(N)$ must be a subgroup of $C_{G_Q}(N_Q)$, that is,

$$C_G(N)_Q \subseteq C_{G_Q}(N_Q) \subseteq G_Q. \quad (2.1)$$

Now since $G/C_G(N)$ is a P -group, $(G/C_G(N))_Q = 1$, so that $G_Q = C_G(N)_Q$. Hence, by (2.1) $G_Q = C_{G_Q}(N_Q)$. Thus every element of G_Q commutes with every element of N_Q , so that $[G_Q, N_Q] = 1$. But $[G_Q, N_Q] = [G, N]_Q$, so $[G, N]$ is a P -group. $\square$

It is clear from this line of proof that, if we want a result in the opposite direction to that of [Theorem 2.1](#), we will have to establish conditions under which

$$C_{G_Q}(N_Q) = C_G(N)_Q. \quad (2.2)$$

Put another way, we ask when the restriction $e_0 : C_G(N) \rightarrow C_{G_Q}(N_Q)$ of the Q -localization $e : G \rightarrow G_Q$ itself Q -localizes. Now certainly e_0 is Q -injective and $C_{G_Q}(N_Q)$ is Q -local. Thus e_0 Q -localizes if and only if it is Q -surjective.

In seeking conditions under which e_0 is Q -surjective—and again in proving Lorenzen's theorem ([Theorem 2.6](#)), we need to apply a basic result in [1], namely, Theorem 6.1. We quote that result here as [Lemma 2.2](#).

LEMMA 2.2 (see [1, Theorem 6.1]). *Let G be a nilpotent group with $\text{nil } G = c$ and let $a, b \in G$ with $b^m = 1$. Then $(ab)^{m^c} = a^{m^c}$.*

However, we can, in fact, refine this result and it will be valuable to do so. Thus we may enunciate

LEMMA 2.3. *If, in addition, $b \in \Gamma^i G$, then $(ab)^{m^{c-i+1}} = a^{m^{c-i+1}}$.*

(Recall that we adopt Warfield's convention for enumerating the terms of the lower central series of G , so that $\Gamma^1 G = G$, $\Gamma^2 G = G'$.)

PROOF OF LEMMA 2.3. We apply [Lemma 2.2](#), but replace G by $\langle a, b \rangle$. However, if $b \in \Gamma^i G$ then $\text{nil } \langle a, b \rangle \leq c - i + 1$. $\square$

We now apply [Lemma 2.2](#) (we will need the more refined [Lemma 2.3](#) later) to prove the following theorem.

THEOREM 2.4. *Let G, H be nilpotent groups with subgroups $\tilde{G} \subseteq G$, $\tilde{H} \subseteq H$. Let φ be a Q -bijective homomorphism from G to H sending $\tilde{G}$ into $\tilde{H}$, and let $\tilde{\varphi} : \tilde{G} \rightarrow \tilde{H}$ be obtained by restricting φ . Then $\tilde{\varphi}$ is Q -surjective (and hence Q -bijective) if and only if, for all $x \in G$ such that $\varphi x \in \tilde{H}$, there exists a P -number m such that $x^m \in \tilde{G}$.*

PROOF. We for brevity, describe the property that, for all $x \in G$ such that $\varphi x \in \tilde{H}$, there exists a P -number m such that $x^m \in \tilde{G}$ as *property S*. Suppose

then that $\bar{\varphi}$ is Q -surjective, and let $x \in G$ satisfy $\varphi x \in \bar{H}$. Since $\bar{\varphi}$ is Q -surjective, there exists a P -number n and an element $\bar{x} \in \bar{G}$ such that $\bar{\varphi}\bar{x} = \varphi x^n$. But then $x^n = \bar{x}z$, $z \in G$ with $z^k = 1$ for some P -number k , since φ is Q -injective. Let $\text{nil } G = c$. Then, by [Lemma 2.2](#), $x^{nk^c} = \bar{x}^{k^c} \in \bar{G}$ and nk^c is a P -number, establishing property S .

Suppose, conversely, that property S holds and let $y \in \bar{H}$. Since φ is Q -surjective, there exists a P -number n and $x \in G$ such that $\varphi x = y^n$. Thus, by property S , there exists a P -number m such that $x^m \in \bar{G}$. Then $\bar{\varphi}(x^m) = y^{mn}$ and mn is a P -number, so $\bar{\varphi}$ is Q -surjective. $\square$

COROLLARY 2.5. *The restriction $e_0 : C_G(N) \rightarrow C_{G_Q}(N_Q)$ Q -localizes if and only if, for all $x \in G$ such that $ex \in C_{G_Q}(N_Q)$, there exists a P -number n such that $x^n \in C_G(N)$.*

This result enables us to exploit the following theorem due to Karl Lorensen. With G a nilpotent group, N a normal subgroup of G , and $x \in G$, we write $T_P \Gamma_{(x)}^2 N$ for the P -primary component of the torsion subgroup of $\Gamma_{(x)}^2 N$, which is a subgroup of N generated by commutators $[x^r, a]$, $a \in N$. We then prove the following theorem.

THEOREM 2.6 (Lorensen). *Let $e_0 : C_G(N) \rightarrow C_{G_Q}(N_Q)$ be obtained by restricting the Q -localization $e : G \rightarrow G_Q$. Then e_0 Q -localizes provided that, for all $x \in G$, $T_P \Gamma_{(x)}^2 N$ has finite exponent.*

PROOF. (This is a small but significant modification of Lorensen's proof, since it exploits [Lemma 2.3](#).) We will apply [Corollary 2.5](#). Thus we must show that, for all $x \in G$ such that $ex \in C_{G_Q}(N_Q)$, there exists a P -number n such that $x^n \in C_G(N)$. Now let $m = \exp T_P \Gamma_{(x)}^2 N$, and let $y \in N$. Then m is a P -number and $e[x, y] = [ex, ey] = 1$, since $ex \in C_{G_Q}(N_Q)$. Hence $[x, y] \in T_P \Gamma_{(x)}^2 N$, so $[x, y]^m = 1$.

Now $x[x, y] = y^{-1}xy$. Hence, by [Lemma 2.3](#), noting that $[x, y] \in \Gamma^2 G$, we conclude that $x^{m^{c-1}} = (y^{-1}xy)^{m^{c-1}} = y^{-1}x^{m^{c-1}}y$, where $\text{nil } G = c$. Since y is an arbitrary element of N , it follows that $x^{m^{c-1}} \in C_G(N)$ and [Theorem 2.6](#) is proved. $\square$

REMARK 2.7. Notice that it would have sufficed to assume that $T_P \Gamma_{(x)}^2 N$ has finite exponent for all $x \in G$ such that $ex \in C_{G_Q}(N_Q)$.

Lorensen's theorem is the key to our relativization of [Theorem 1.3](#), which we now state.

THEOREM 2.8. *Let G be a nilpotent group and N a normal subgroup of G . Then if $[G, N]$ is a P -group of exponent m , $G/C_G(N)$ is a P -group of exponent dividing m^{c-1} , where $\text{nil } G = c$.*

PROOF. Since $\Gamma_{(x)}^2 N \subseteq [G, N]$, and $[G, N]$ is a P -group of exponent m , it follows that we have the conditions for applying Lorensen's theorem, so that $e_0 : C_G(N) \rightarrow C_{G_Q}(N_Q)$ Q -localizes. Now since $[G, N]$ is a P -group, its Q -localization vanishes, that is, $[G_Q, N_Q] = 1$. This means that $G_Q = C_{G_Q}(N_Q)$, so that every $x \in G$ has the property that $ex \in C_{G_Q}(N_Q)$. Moreover, $\exp \Gamma_{(x)}^2 N$ divides m . Thus, following the proof of [Theorem 2.6](#), we see that $x^{m^{c-1}} \in C_G(N)$ for all $x \in G$, so that $\exp(G/C_G(N)) \mid m^{c-1}$. This, of course, implies that $G/C_G(N)$ is a P -group. $\square$

REMARK 2.9. This last implication follows immediately from $G_Q = C_{G_Q}(N_Q) = C_G(N)_Q$.

It remains to provide the relativization of [Theorem 1.4](#). In fact, we may simply relativize each step in Warfield's argument in [4, Corollary 3.16], thus obtaining the following theorem.

THEOREM 2.10. *Let G be a nilpotent group and N a normal subgroup such that $G/C_G(N)$ is a P -group of exponent m . Then $[G, N]$ is a P -group of exponent dividing m^{c-1} , where $\text{nil } G = c$.*

Appendix

Homological methods. We show in this appendix how homological arguments may be used to obtain [Theorem 1.2](#), although the numerical estimate is marginally inferior to that given by [Theorem 1.4](#). We emphasize that we have only succeeded in developing a homological method in the absolute case.

We begin with a crucial homological lemma.

LEMMA A.11. *Let G be a nilpotent group with $\text{nil } G = c$ and let $n \geq 1$. If G is a torsion group with $\exp G = m$, then $m^{n(c-1)+1}H_n G = 0$.*

PROOF. We argue by induction on c . If $c = 1$, then G is commutative. If K is an arbitrary f -group, then K is a direct product of (finitely many) finite cyclic groups whose orders divide m , hence $mH_n K = 0$. Now $H_n G = \varprojlim_{\bar{K}} H_n K$, so that $mH_n G = 0$.

Now we assume $c \geq 2$, and assume the lemma proved for nilpotent groups of class $< c$. We consider the central extension

$$\Gamma \twoheadrightarrow G \twoheadrightarrow G/\Gamma, \quad (\text{A.3})$$

where $\Gamma = \Gamma^c G$, and we exploit the Lyndon-Hochschild-Serre spectral sequence associated with (A.3). In this spectral sequence

$$E_{pq}^2 = H_p(G/\Gamma; H_q \Gamma). \quad (\text{A.4})$$

Since the universal coefficient formula in homology splits, and since $\text{nil } \Gamma = 1$, $\text{nil } G/\Gamma = c - 1$, and $\exp \Gamma \mid m$, $\exp(G/\Gamma) \mid m$, it follows from the inductive hypothesis that, if $p + q > 0$,

$$mE_{pq}^2 = 0, \quad q > 0, \quad m^{p(c-2)+1}E_{p0}^2 = 0. \quad (\text{A.5})$$

(The form of writing in (A.5) and in what follows is acceptable since homology groups and E_{pq}^r are additive abelian groups.)

We may then pass to the limit of the spectral sequence, obtaining

$$mE_{pq}^\infty = 0, \quad q > 0, \quad m^{p(c-2)+1}E_{p0}^\infty = 0. \quad (\text{A.6})$$

Now $H_n G$ admits a finite filtration

$$0 = F^{-1} \subseteq F^0 \subseteq \dots \subseteq F^{p-1} \subseteq F^p \subseteq \dots \subseteq F^{n-1} \subseteq F^n = H_n G, \quad (\text{A.7})$$

such that

$$F^p / F^{p-1} = E_\infty^{pq}, \quad p + q = n, \quad 0 \leq p \leq n. \quad (\text{A.8})$$

From (A.6) and (A.8) an easy finite induction shows that

$$m^{p+1}F^p = 0, \quad 0 \leq p \leq n-1. \quad (\text{A.9})$$

Finally, we exploit the short exact sequence

$$F^{n-1} \twoheadrightarrow H_n G \twoheadrightarrow E_{n0}^\infty \quad (\text{A.10})$$

to infer that $m^{n+n(c-2)+1}H_n G = 0$, or $m^{n(c-1)+1}H_n G = 0$, completing the inductive step. $\square$

Armed with this lemma, we may prove the following theorem.

THEOREM A.12. *Let G a nilpotent group with $\text{nil} G = c$. Then if G/ZG is a torsion group of exponent m , G' is a torsion group of exponent dividing m^{2c-2} .*

PROOF. We exploit the exact sequence (1.2) and the argument used to prove Schur's theorem. Since $\text{nil} G/ZG = c-1$, we know from Lemma A.11 that

$$m^{2(c-2)+1}H_2(G/ZG) = 0. \quad (\text{A.11})$$

Thus

$$m^{2(c-2)+1}(G' \cap ZG) = 0. \quad (\text{A.12})$$

Now $G'/G' \cap ZG \subseteq G/ZG$, so $\exp(G'/G' \cap ZG) \mid m$. Putting this together with (A.12), we deduce finally that G' is a torsion group and $\exp G' \mid m^{2c-2}$. $\square$

We remark (again) that our estimate of $\exp G'$ is not best possible.

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PETER HILTON: DEPARTMENT OF MATHEMATICAL SCIENCES, SUNY BINGHAMTON, BINGHAMTON, NY 13902-6000, USA

Current address: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, FL 32816-1364, USA

E-mail address: marge@math.binghamton.edu