EVALUATION OF EULER-ZAGIER SUMS

KHRISTO BOYADZHIEV

(Received 2 May 2000)

ABSTRACT. We present a simple method for evaluation of multiple Euler sums in terms of single and double zeta values.

2000 Mathematics Subject Classification. 11M99, 40B05.

1. Introduction. We give a short evaluation of the triple sums

$$w(p, q, r) = \sum_{n, m=1}^{\infty} \frac{1}{n^p m^q (n+m)^r} \quad (1.1)$$

in terms of single zeta values $\zeta(p)$

$$\zeta(p) = \sum_{n=1}^{\infty} n^{-p} \quad (1.2)$$

and double zeta values (Euler sums) $S(p, q)$

$$S(p, q) = \sum_{n=1}^{\infty} H_n^{(p)} n^{-q}, \quad H_n^{(p)} = 1^{-p} + 2^{-p} + \cdots + n^{-p}, \quad (1.3)$$

where $p \geq 1, q > 1$.

Multiple Euler sums have been discussed and evaluated in a number of papers of which we want to point out [1, 2, 3, 4, 5, 6, 7, 10]. Also [8, Sections 18 and 19]. We refer to these publications for general comments and details.

2. Euler sums

LEMMA 2.1. *For any integer $p > 1$ and any $x > 0$,*

$$\sigma(p; x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^p (n+x)} = \sum_{k=1}^{p-1} \frac{(-1)^{k-1} \zeta(p-k+1)}{x^k} + \frac{(-1)^{p-1}}{x^p} (\psi(x+1) + \gamma), \quad (2.1)$$

where $\psi = \Gamma' / \Gamma$ is the psi function and γ is Euler's constant.

PROOF. We have

$$\begin{aligned} \sigma(p; x) &= \sum_{n=1}^{\infty} \frac{x+n-n}{n^p x (n+x)} = \sum_{n=1}^{\infty} \frac{1}{n^p x} - \sum_{n=1}^{\infty} \frac{1}{n^{p-1} x (n+x)} \\ &= \frac{1}{x} (\zeta(p) - \sigma(p-1; x)). \end{aligned} \quad (2.2)$$

Equation (2.1) follows by repeating the procedure $p-1$ times in view of the fact that (see [9, page 665])

$$\sigma(1; x) = \sum_{n=1}^{\infty} \frac{1}{n(n+x)} = \frac{1}{x} (\psi(x+1) + \gamma). \quad (2.3)$$

Now we differentiate (2.1) $r-1$ times, where $r > 1$. With $D = d/dx$ we have

$$\begin{aligned} D^{r-1} \frac{1}{n+x} &= \frac{(-1)^{r-1} (r-1)!}{(n+x)^r}, \\ D^{r-1} \frac{1}{x^k} &= (-1)^{r-1} \frac{(r+k-2)!}{(k-1)!} \frac{1}{x^{r+k-1}} = (-1)^{r-1} (r-1)! \binom{r+k-2}{r-1} \frac{1}{x^{r+k-1}}, \\ D^{r-1} (x^{-p} (\psi(x+1) + \gamma)) &= \sum_{k=0}^{r-1} \binom{r-1}{k} (D^{r-1-k} x^{-p}) (D^k (\psi(x+1) + \gamma)). \end{aligned} \quad (2.4)$$

Therefore,

$$\begin{aligned} D^{r-1} \sigma(p; x) &= (-1)^{r-1} (r-1)! \sum_{n=1}^{\infty} \frac{1}{n^p (n+x)^r} \\ &= (-1)^{r-1} (r-1)! \sum_{k=1}^{p-1} (-1)^{k-1} \binom{r+k-2}{r-1} \frac{\zeta(p-k+1)}{x^{k+r-1}} \\ &\quad + \frac{(-1)^{p-1} (-1)^{r-1} (r-1)!}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k (r+p-k-2)!}{k! (r-k-1)!} \frac{(\psi(x+1) + \gamma)^{(k)}}{x^{r+p-k-1}}. \end{aligned} \quad (2.5)$$

□

We summarize this result in the following lemma.

LEMMA 2.2. *For any integers $p > 1$, $r \geq 1$ and any $x > 0$,*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^p (n+x)^r} &= \sum_{k=1}^{p-1} (-1)^{k-1} \binom{r+k-2}{r-1} \frac{\zeta(p-k+1)}{x^{k+r-1}} \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k (r+p-k-2)!}{k! (r-k-1)!} \frac{(\psi(x+1) + \gamma)^{(k)}}{x^{r+p-k-1}}. \end{aligned} \quad (2.6)$$

Next, we replace here x by mx and multiply both sides by m^{-q} , $q \geq 1$. This gives

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n+mx)^r} &= \sum_{k=1}^{p-2} \frac{(-1)^k}{x^{k+r}} \binom{r+k-1}{r-1} \frac{\zeta(p-k)}{m^{k+q+r}} \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k (r+p-k-2)!}{k! (r-k-1)!} \frac{1}{x^{r+p-k-1}} \frac{(\psi(mx+1) + \gamma)^{(k)}}{m^{r+p+q-k-1}}. \end{aligned} \quad (2.7)$$

Summing for $m = 1, 2, \dots$ we obtain our main representation.

THEOREM 2.3. For all integers $p > 1$, $r \geq 1$ and all $q \geq 0$, $q + r > 1$, $x > 0$,

$$\begin{aligned} \sigma(p, q, r; x) &\equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n+mx)^r} \\ &= \sum_{k=0}^{p-2} \frac{(-1)^k}{x^{k+r}} \binom{r+k-1}{r-1} \zeta(p-k) \zeta(k+q+r) \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(-1)^k (r+p-k-2)!}{k! (r-k-1)!} \frac{1}{x^{r+p-k-1}} \sum_{m=1}^{\infty} \frac{(\psi(mx+1)+\gamma)^{(k)}}{m^{r+p+q-k-1}}. \end{aligned} \quad (2.8)$$

The case $p = 1$ can be derived directly from (2.3), namely,

$$\begin{aligned} \sigma(1, q, r; x) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^q (n+mx)^r} \\ &= \sum_{k=0}^{r-1} \frac{(-1)^k}{k!} \frac{1}{x^{r-k}} \sum_{m=1}^{\infty} \frac{(\psi(mx+1)+\gamma)^{(k)}}{m^{r+q-k}}. \end{aligned} \quad (2.9)$$

We remind the reader that the expression $(\psi(mx+1)+\gamma)^{(k)}$ stands for the k th derivative of the function $\psi(x+1)+\gamma$ evaluated at mx .

By setting $x = 1$ we get the desired representation of $w(p, q, r)$. Making use of

$$\begin{aligned} \psi(m+1)+\gamma &= H_m^{(1)} = 1 + 2^{-1} + \cdots + m^{-1}, \\ \psi^{(k)}(m+1) &= (-1)^k k! [H_m^{(k+1)} - \zeta(k+1)], \end{aligned} \quad (2.10)$$

(see [9, page 775]), and with the agreement to read $\zeta(1) = 0$, one obtains the following corollary.

COROLLARY 2.4. For all integers $p > 1$, $r \geq 1$ and all $q \geq 0$ with $q + r > 1$,

$$\begin{aligned} w(p, q, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n+m)^r} \\ &= \sum_{k=0}^{p-2} (-1)^k \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+q+k) \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-1} \frac{(r+p-k-2)!}{(r-k-1)!} [S(k+1, r+p+q-k-1) - \zeta(k+1) \zeta(r+p+q-k-1)], \end{aligned} \quad (2.11)$$

in particular,

$$\begin{aligned} w(1, q, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^q (n+m)^r} \\ &= \sum_{k=0}^{r-1} [S(k+1, r+q-k) - \zeta(k+1) \zeta(r+q-k)]. \end{aligned} \quad (2.12)$$

When $q > 0$ (or $q \geq 1, p = 1$) we also have

$$\begin{aligned} w(p, q, 1) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p m^q (n+m)} \\ &= \sum_{k=1}^{p-1} (-1)^{k-1} \zeta(p-k+1) \zeta(q+k) + (-1)^{p-1} S(1, p+q). \end{aligned} \quad (2.13)$$

3. Remarks. Our notation $S(p, q)$ corresponds to $S_{p,q}$ in [5]. The authors of [2] use the sums $\zeta(p, q)$, which equal $S(q, p) - \zeta(p+q)$.

The representation (2.11) has strong and weak points. One good feature is that q need not be an integer. A weak point is that the right-hand side in (2.11) is not explicitly symmetrical in p and q , while obviously $w(p, q, r) = w(q, p, r)$. Moreover, the right-hand side has too many terms. For instance, when $q = 0$ (2.11) becomes

$$\begin{aligned} w(p, 0, r) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^p (n+m)^r} \\ &= (-1)^{p-1} S(r, p) + \sum_{k=0}^{p-2} (-1)^k \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+k) \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-2} \frac{(r+p-k-2)!}{(r-k-1)!} [S(k+1, r+p-k-1) - \zeta(k+1) \zeta(r+p-k-1)] \end{aligned} \quad (3.1)$$

(here the term $(-1)^{p-1} S(r, p)$ is written separately on purpose).

At the same time

$$w(p, 0, r) = \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{m=1}^{\infty} \frac{1}{(n+m)^r} = \sum_{n=1}^{\infty} \frac{1}{n^p} (\zeta(r) - H_n^{(r)}) = \zeta(p) \zeta(r) - S(r, p) \quad (3.2)$$

which is much shorter. However, we can benefit from this situation if we compare the two representations of $w(p, 0, r)$ and derive relations for the single and double Euler sums. For instance, when p is odd, we can solve for $S(r, p)$ to get

$$\begin{aligned} 2S(r, p) &= \sum_{k=1}^{p-2} (-1)^{k+1} \binom{r+k-1}{r-1} \zeta(p-k) \zeta(r+k) \\ &\quad + \frac{(-1)^{p-1}}{(p-1)!} \sum_{k=0}^{r-2} \frac{(r+p-k-2)!}{(r-k-1)!} [S(k+1, r+p-k-1) - \zeta(k+1) \zeta(r+p-k-1)], \end{aligned} \quad (3.3)$$

that is, $S(r, p)$ can be expressed in terms of single zeta values and $S(k, l)$, with $k < r$, $k+l = r+p$.

4. Other sums. It is interesting to consider also the sum

$$u(p, q, r) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^p m^q (n^r + m^r)} \quad (4.1)$$

and compare it to $w(p, q, r)$. Here one can write

$$u(p, q, r) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{n^r + m^r - n^r}{n^{p+r} m^q (n^r + m^r)} = \zeta(p+r) \zeta(q) - u(p+r, q-r, r). \quad (4.2)$$

Let $q > p$. We observe that if $(q-p)/r$ is odd, repeating this step $(q-p)/r$ times, we get

$$u(p, q, r) = \sum_{j=1}^{(q-p)/r} (-1)^{j-1} \zeta(p+jr) \zeta(q-(j-1)r) - u(q, p, r) \quad (4.3)$$

from where, because of the symmetry $u(p, q, r) = u(q, p, r)$, we obtain the following proposition.

PROPOSITION 4.1. *For all $q > p \geq 1$, $r \geq 1$ with $(q-p)/r$ odd,*

$$u(p, q, r) = \frac{1}{2} \sum_{j=1}^{(q-p)/r} (-1)^{j-1} \zeta(p+jr) \zeta(q-(j-1)r). \quad (4.4)$$

Note that p, q, r need not be integers. The only restrictions are those listed above. When $r = 1$ we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^p m^q (n+m)} = \frac{1}{2} \sum_{j=1}^{q-p} (-1)^{j-1} \zeta(p+j) \zeta(q-j+1) \quad (4.5)$$

which can be compared to (2.13). This gives the well-known expression of $S(1, p+q)$ in terms of zeta values. To make this more explicit we set $p = 1$ and $q \geq 2$. Then from (2.13),

$$w(1, q-1, 1) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{nm^{q-1}(n+m)} = S(1, q). \quad (4.6)$$

This is the same as $u(1, q-1, 1)$. When q is odd we find from (4.5) (with $p = 1$ and $q-1$ in the place of q)

$$S(1, q) = \frac{1}{2} \sum_{j=1}^{q-2} (-1)^{j-1} \zeta(j+1) \zeta(q-j) \quad (4.7)$$

which is a variant of Euler's formula for the sum $S(1, q)$ (see [5, Theorem 2.2]).

REFERENCES

- [1] J. M. Borwein, D. M. Bradley, and D. J. Broadhurst, *Evaluations of k -fold Euler/Zagier sums: a compendium of results for arbitrary k* , J. Combin **4** (1997), no. 2, 31-49. [Zbl 884.40004](#).
- [2] J. M. Borwein and R. Girgensohn, *Evaluation of triple Euler sums*, Electron. J. Combin. **3** (1996), no. 1, 1-27. [MR 97d:11137](#). [Zbl 884.40005](#).
- [3] R. E. Crandall, *Fast evaluation of multiple zeta sums*, Math. Comp. **67** (1998), no. 223, 1163-1172. [MR 98j:11066](#). [Zbl 901.11036](#).
- [4] R. E. Crandall and J. P. Buhler, *On the evaluation of Euler sums*, Experiment. Math. **3** (1994), no. 4, 275-285. [MR 96e:11113](#). [Zbl 833.11045](#).
- [5] P. Flajolet and B. Salvy, *Euler sums and contour integral representations*, Experiment. Math. **7** (1998), no. 1, 15-35. [MR 99c:11110](#). [Zbl 920.11061](#).

- [6] A. Granville, *A decomposition of Riemann's zeta-function*, Analytic Number Theory (Kyoto, 1996) (Y. Motohashi, ed.), London Math. Soc. Lecture Note Ser., vol. 247, Cambridge Univ. Press, Cambridge, 1997, pp. 95–101. [MR 2000c:11134](#). [Zbl 907.11024](#).
- [7] C. Markett, *Triple sums and the Riemann zeta function*, J. Number Theory **48** (1994), no. 2, 113–132. [MR 95f:11067](#). [Zbl 0810.11047](#).
- [8] N. Nielsen, *Die Gammafunktion*, Chelsea, New York, 1965 (German). [MR 32#2622](#).
- [9] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Elementary functions, vol. 1, Gordon and Breach Science Publishers, New York, 1986. [MR 88f:00013](#). [Zbl 733.00004](#).
- [10] D. Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, vol. II (Paris, 1992) (A. Joseph et al., ed.), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 497–512. [MR 96k:11110](#). [Zbl 822.11001](#).

KHRISTO BOYADZHIEV: DEPARTMENT OF MATHEMATICS, OHIO NORTHERN UNIVERSITY, ADA, OH 45810, USA

E-mail address: k-boyadzhiev@onu.edu