IJMMS 31:8 (2002) 497–507
PII. S0161171202013169
http://ijmms.hindawi.com

A NOTE ON COMPUTING THE GENERALIZED INVERSE $A_{T,S}^{(2)}$ OF A MATRIX A

XIEZHANG LI and YIMIN WEI

Received 10 May 2001 and in revised form 1 February 2002

The generalized inverse $A_{T,S}^{(2)}$ of a matrix A is a $\{2\}$-inverse of A with the prescribed range T and null space S. A representation for the generalized inverse $A_{T,S}^{(2)}$ has been recently developed with the condition $\sigma(GA|_T) \subset (0,\infty)$, where G is a matrix with $R(G) = T$ and $N(G) = S$. In this note, we remove the above condition. Three types of iterative methods for $A_{T,S}^{(2)}$ are presented if $\sigma(GA|_T)$ is a subset of the open right half-plane and they are extensions of existing computational procedures of $A_{T,S}^{(2)}$, including special cases such as the weighted Moore-Penrose inverse $A_{M,N}^{\dagger}$ and the Drazin inverse A^D. Numerical examples are given to illustrate our results.

2000 Mathematics Subject Classification: 15A09, 65F20.

1. Introduction. Given a complex matrix $A \in \mathbb{C}^{m\times n}$, any matrix $X \in \mathbb{C}^{n\times m}$ satisfying $XAX = X$ is called a $\{2\}$-inverse of A. Let T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. A matrix $X \in \mathbb{C}^{n\times m}$ is called a $\{2\}$-inverse of A with the prescribed range T and null space S, denoted by $A_{T,S}^{(2)}$, if the following conditions are satisfied:

$$XAX = X, \qquad R(X) = T, \qquad N(X) = S, \tag{1.1}$$

where $R(X)$ is the range of X and $N(X)$ is the null space of X. It is a well-known fact [1] that if $\dim T = \dim S^{\perp} \le \operatorname{rank}(A)$, then there exists a unique $A_{T,S}^{(2)}$ if and only if $AT \oplus S = \mathbb{C}^m$. It is obvious from the definition above that $AA_{T,S}^{(2)} = P_{AT,S}$ and $A_{T,S}^{(2)}A = P_{T,(A^*S^{\perp})^{\perp}}$, where P_{S_1,S_2} is the projector on the subspace S_1 along the subspace S_2.

There are seven types of important $\{2\}$-inverses of A: the Moore-Penrose inverse $A^{\dagger}$, the weighted Moore-Penrose inverse $A_{M,N}^{\dagger}$, the W-weighed Drazin inverse $A_{d,w}$, the Drazin inverse A^D, the group inverse $A^{\#}$, the Bott-Duffin inverse $A_{(L)}^{(-1)}$, and the generalized Bott-Duffin inverse $A_{(L)}^{(\dagger)}$. All of them are the special cases of the generalized inverse $A_{T,S}^{(2)}$ of A for specific T and S.

Lemma 1.1. (a) *Let $A \in \mathbb{C}^{m\times n}$ [1]. Then, for the Moore-Penrose inverse $A^{\dagger}$ and the weighted Moore-Penrose inverse $A_{MN}^{\dagger}$,*

(i) $A^{\dagger} = A_{R(A^*),N(A^*)}^{(2)}$;

(ii) $A_{M,N}^{\dagger} = A_{R(N^{-1}A^*M),N(N^{-1}A^*M)}^{(2)}$, *where N and M are Hermitian positive definite matrices of order n and m, respectively;*

(iii) $A_{d,w} = (WAW)_{R(A(WA)^q),N(A(WA)^q)}^{(2)}$, *where $W \in \mathbb{C}^{n\times m}$ and $q = \operatorname{Ind}(WA)$, the index of WA.*

(b) [1, 2, 3] *Let* $A \in \mathbb{C}^{n\times n}$. *Then, for the Drazin inverse* A^D, *the group inverse* $A^\#$, *the Bott-Duffin inverse* $A^{(-1)}_{(L)}$, *and the generalized Bott-Duffin inverse* $A^{(\dagger)}_{(L)}$,

(iv) $A^D = A^{(2)}_{R(A^k),N(A^k)}$, *where* $k = \mathrm{Ind}(A)$;

(v) *in particular, when* $\mathrm{Ind}(A) = 1$, $A^\# = A^{(2)}_{R(A),N(A)}$;

(vi) $A^{(-1)}_{(L)} = P_L(AP_L + P_{L^\perp})^{-1} = A^{(2)}_{L,L^\perp}$, *where* L *is a subspace of* $\mathbb{C}^n$ *such that* $AL \oplus L^\perp = \mathbb{C}^n$ *and* P_L *is the orthogonal projector on* L;

(vii) $A^{(\dagger)}_{(L)} = A^{(-1)}_{(S)} = A^{(2)}_{S,S^\perp}$, *where* $S = R(P_LA)$.

The $\{2\}$-inverse has many applications, for example, the application in the iterative methods for solving nonlinear equations [1, 9] and the applications to statistics [6, 7]. In particular, $\{2\}$-inverse plays an important role in stable approximations of ill-posed problems and in linear and nonlinear problems involving rank-deficient generalized inverse [8, 12]. In literature, researchers have proposed many numerical methods for computing $A^{(2)}_{T,S}$, see [2, 3, 11, 13, 15, 16, 18].

As usual, we denote the spectrum and the spectral radius of A by $\sigma(A)$ and $\rho(A)$, respectively. The notation $\|\cdot\|$ stands for the spectral norm. The following theorem applied in this note is from the theory of semi-iterative method.

Theorem 1.2 (see [5]). *Let* $B \in \mathbb{C}^{n\times n}$ *be a nonsingular matrix and let* $\sigma(B) \subset \Omega$, *where* Ω *is a simply connected compact set excluding origin. If a sequence of polynomials* $\{s_m(z)\}_{m=0}^\infty$ *uniformly converges to* $1/z$ *on* Ω, *then* $\{s_m(B)\}$ *converges to* B^{-1}.

In this note, a representation for the generalized inverse $A^{(2)}_{T,S}$ with a condition $\sigma(GA|_T) \subset \{z : \mathrm{Re}(z) > 0\}$, where G is a matrix with $R(G) = T$ and $N(G) = S$ is presented in Section 2. Euler-Knopp iterative method and semi-iterative methods for $A^{(2)}_{T,S}$ with linear convergence are derived in Section 3. Quadratically convergent methods for $A^{(2)}_{T,S}$ are developed in Section 4. Finally, numerical examples are given to illustrate our results.

2. Representation.

In this section, we give a representation for the generalized inverse $A^{(2)}_{T,S}$, which may be viewed as an application of the classical theory summability to the representation of generalized inverse.

Lemma 2.1 (see [13]). *Suppose* $A \in \mathbb{C}^{m\times n}$. *Let* T *and* S *be subspaces of* $\mathbb{C}^n$ *and* $\mathbb{C}^m$, *respectively, such that* $AT \oplus S = \mathbb{C}^m$. *Suppose that* $G \in \mathbb{C}^{n\times m}$ *satisfies* $R(G) = T$ *and* $N(G) = S$. *Denote by* $\tilde{A} = (GA)|_T$ *the restriction of* GA *on* T. *Then* $\mathrm{Ind}(GA) = 1$ *and*

$$A^{(2)}_{T,S} = \tilde{A}^{-1}G. \tag{2.1}$$

It follows from Lemma 1.1 that the existence of G is assured for each of the common seven types of generalized inverses: A^*, $N^{-1}A^*M$, $A(WA)^q$, A^k, A, P_L, and P_S. Now we are in a position to establish a presentation theorem.

Theorem 2.2. *Let* A, T, S, G, *and* $\tilde{A}$ *be as in Lemma 2.1. If* $\sigma(\tilde{A})$ *is contained in a simply connected compact set* Ω *excluding origin and a polynomial sequence* $\{s_m(z)\}$ *uniformly converges to* $1/z$ *on* Ω, *then*

$$A^{(2)}_{T,S} = \lim_{m\to\infty} s_m(\tilde{A})G. \tag{2.2}$$

Furthermore,

$$\frac{\|s_m(\tilde{A})G - A^{(2)}_{T,S}\|_P}{\|A^{(2)}_{T,S}\|_P} \leq \max_{z\in\sigma(\tilde{A})} |zs_m(z) - 1| + O(\epsilon), \tag{2.3}$$

where P is invertible such that $P^{-1}GAP$ is the ϵ-Jordan canonical form of GA and $\|B\|_P = \|P^{-1}B\|$ for each $B \in \mathbb{C}^{n\times m}$.

Proof. Assume that $\sigma(\tilde{A}) \subset \Omega$. With applying Theorem 1.2, we get

$$\lim_{m\to\infty} s_m(\tilde{A}) = \tilde{A}^{-1} \tag{2.4}$$

uniformly on Ω. It follows from Lemma 2.1 that

$$\lim_{m\to\infty} s_m(\tilde{A})G = \tilde{A}^{-1}G = A^{(2)}_{T,S}. \tag{2.5}$$

The error can be written as

$$s_m(\tilde{A})G - A^{(2)}_{T,S} = (s_m(\tilde{A})\tilde{A} - I)A^{(2)}_{T,S}. \tag{2.6}$$

Since P is nonsingular such that $P^{-1}GAP$ is the ϵ-Jordan canonical form of GA, it is well known that

$$\|P^{-1}GAP\| \leq \rho(GA) + \epsilon. \tag{2.7}$$

Thus

$$\begin{aligned} \|s_m(\tilde{A})G - A^{(2)}_{T,S}\|_P &= \|P^{-1}(s_m(\tilde{A})\tilde{A} - I)PP^{-1}A^{(2)}_{T,S}\| \\ &\leq \|P^{-1}(s_m(\tilde{A})\tilde{A} - I)P\| \, \|A^{(2)}_{T,S}\|_P \\ &\leq \left[\max_{z\in\sigma(\tilde{A})} |s_m(z)z - 1| + O(\epsilon)\right] \|A^{(2)}_{T,S}\|_P. \end{aligned} \tag{2.8}$$

The last inequality is based on the spectrum mapping since $s_m(z)$ is a polynomial in z. This completes the proof. □

In order to make use of this general error estimate in Theorem 2.2 on specific approximation procedures, it will be convenient to have lower and upper bounds for $\sigma(\tilde{A})$. This is given in the next lemma.

Lemma 2.3. *Let A, T, S, G, and $\tilde{A}$ be as in Lemma 2.1. Then for each $\lambda \in \sigma(\tilde{A})$,*

$$\frac{1}{\|(GA)^{\#}\|} \leq |\lambda| \leq \|GA\|. \tag{2.9}$$

Proof. We only show the first inequality since the second is trivial. It follows from Lemma 2.1 that $\operatorname{Ind}(GA) = 1$. Then the Jordan canonical form of GA is

$$GA = P\begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \qquad (GA)^{\#} = P\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}P^{-1}, \tag{2.10}$$

where C is invertible. For each $\lambda \in \sigma(\tilde{A})$, $1/\lambda \in \sigma(\tilde{A}^{-1})$ since $\tilde{A}$ is invertible. Consequently, we have

$$\frac{1}{|\lambda|} \le \rho(\tilde{A}^{-1}) = \rho(C^{-1}) \le \|(GA)^{\#}\|, \tag{2.11}$$

which leads to (2.9). This completes the proof. □

Remark 2.4. Theorem 2.2 extends the representation of $A^{(2)}_{T,S}$ in [15] in which $\sigma(GA|_T) \subset (0,\infty)$ is required. The theorem also recovers the representations of A^D in [16] and $A^{\dagger}_{M,N}$ in [17] as special cases.

3. Iterative methods for $A^{(2)}_{T,S}$. In this section, we present applications of Theorem 2.2 and Lemma 2.3 in developing specific computational procedures for the generalized inverse $A^{(2)}_{T,S}$ and estimating corresponding error bounds.

A well-known summability method is called the Euler-Knopp method. A series $\sum_{m=0}^{\infty} a_m$ is said to be Euler-Knopp summable with parameter $\alpha > 0$ to the value a if the sequence defined by

$$s_m = \alpha \sum_{i=0}^{m} \sum_{j=0}^{i} \binom{i}{j} (1-\alpha)^{i-j} \alpha^j a_j \tag{3.1}$$

converges to a. If we choose $a_m = (1-z)^m$, $m \ge 0$, then as the Euler-Knopp transform of the series $\sum_{m=0}^{\infty}(1-z)^m$, we obtain a sequence $\{s_m(z)\}$, where

$$s_m(z) = \alpha \sum_{j=0}^{m} (1-\alpha z)^j. \tag{3.2}$$

Clearly, $\lim_{m\to\infty} s_m(z) = 1/z$ uniformly on any compact subset of an open set $E_\alpha := \{z : |1-\alpha z| < 1\}$. We assume that $\sigma(\tilde{A}) \subset \{z : \operatorname{Re}(z) > 0\}$. Denote

$$\phi := \max_{\lambda \in \sigma(\tilde{A})} \left\{ |\operatorname{Arg}\lambda| : -\frac{\pi}{2} < \operatorname{Arg}\lambda < \frac{\pi}{2} \right\}. \tag{3.3}$$

It follows from Lemma 2.3 that

$$\sigma(\tilde{A}) \subset \{z = re^{i\theta} : r_1 \le r \le r_2,\ -\phi \le \theta \le \phi\} =: F, \tag{3.4}$$

where $r_1 = 1/\|(GA)^{\#}\|$ and $r_2 = \|GA\|$. It can be shown with the law of Sines that

$$F \subset \{w : |w-g| \le g\}, \quad \text{for } g = \frac{\|GA\|}{2\cos\phi}. \tag{3.5}$$

If a parameter α satisfies

$$0 < \alpha < \frac{2\cos\phi}{\|GA\|}, \tag{3.6}$$

then $\sigma(\tilde{A}) \subset E_\alpha$. There is always a simply connected compact set Ω such that $\sigma(\tilde{A}) \subset \Omega \subset E_\alpha$. Hence $s_m(z)$ of (3.2) uniformly converges to $1/z$ on Ω. It follows from Theorem 2.2 that

$$A_{T,S}^{(2)} = \alpha \sum_{n=0}^{\infty} (I - \alpha GA)^n G. \tag{3.7}$$

Notice that if A_m is the mth partial sum, that is, $A_m = \alpha \sum_{j=0}^{m} (I - \alpha GA)^j G$, then an iteration form for $\{A_m\}$ is given by

$$A_0 = \alpha G, \qquad A_{m+1} = (I - \alpha GA)A_m + \alpha G, \quad m \geq 0. \tag{3.8}$$

For an error bound, we note that the sequence of polynomials $\{s_m(z)\}$ satisfies

$$z s_{m+1}(z) - 1 = (1 - \alpha z)\big(z s_m(z) - 1\big). \tag{3.9}$$

Thus

$$\big| z s_m(z) - 1 \big| = |1 - \alpha z|^{m+1} \leq \beta^{m+1} \longrightarrow 0, \quad (m \longrightarrow \infty), \tag{3.10}$$

where

$$\beta = \max_{z \in \sigma(\tilde{A})} |1 - \alpha z| \leq \max_{z \in F} |1 - \alpha z| < 1. \tag{3.11}$$

Actually, by the maximum modular theorem, $\max_{z \in F} |1 - \alpha z| = \max_{z \in \partial F} |1 - \alpha z|$. We denote four parts of ∂F as follows:

$$\begin{aligned} \Gamma_1 &= \{r_1 e^{i\theta} : -\phi \leq \theta \leq \phi\}, & \Gamma_2 &= \{r e^{i\phi} : r_1 \leq r \leq r_2\}, \\ \Gamma_3 &= \{r_2 e^{i\theta} : -\phi \leq \theta \leq \phi\}, & \Gamma_4 &= \{r e^{-i\phi} : r_1 \leq r \leq r_2\}. \end{aligned} \tag{3.12}$$

If $z \in \Gamma_1$, then $|1 - \alpha z|^2 = 1 - 2\alpha r_1 \cos\theta + \alpha^2 r_1^2$ and it is obvious that

$$\max_{z \in \Gamma_1} |1 - \alpha z| = \big| 1 - \alpha r_1 e^{i\phi} \big|. \tag{3.13}$$

With an analogous argument, we have

$$\max_{z \in \Gamma_3} |1 - \alpha z| = \big| 1 - \alpha r_2 e^{i\phi} \big|. \tag{3.14}$$

If $z \in \Gamma_2 \cup \Gamma_4$, then $|1 - \alpha z|^2 = 1 - 2\alpha r \cos\phi + \alpha^2 r^2$ is a quadratic function of r on $[r_1, r_2]$, which achieves its maximum at either $r = r_1$ or $r = r_2$. So

$$\max_{z \in \Gamma_2 \cup \Gamma_4} |1 - \alpha z| = \max\big\{ \big| 1 - \alpha r_1 e^{i\phi} \big|, \big| 1 - \alpha r_2 e^{i\phi} \big| \big\}. \tag{3.15}$$

It follows from Theorem 2.2 that an error bound is given by

$$\frac{\|A_m - A_{T,S}^{(2)}\|_P}{\|A_{T,S}^{(2)}\|_P} \leq \beta^{m+1} + O(\varepsilon), \tag{3.16}$$

where

$$\beta \leq \max\{|1-\alpha e^{i\phi}/\|(GA)^{\#}\||, |1-\alpha e^{i\phi}\|GA\||\}. \tag{3.17}$$

Therefore, we have shown the following general convergence theorem.

Theorem 3.1. *Let A, T, S, and G be as in Lemma 2.1. Suppose the spectrum of $GA|_T$ is contained in the open right half-plane. Then the sequence $\{A_m\}$ of (3.8) linearly converges to $A^{(2)}_{T,S}$, if $0 < \alpha < 2\cos\phi/\|GA\|$, where ϕ is given by (3.3). Moreover, the relative error is bounded by (3.16).*

We remark that Theorem 3.1 is an extension of corresponding results in [15, 16].

The procedure of semi-iterative methods [5, 10] for solving a linear system can easily be extended to solve

$$X = HX + C, \quad \text{for } C \in \mathbb{C}^{n\times n}. \tag{3.18}$$

If $\rho(H) < 1$, then a sequence of matrices $\{X_m\}$, yielded by

$$X_0 = C; \qquad X_{m+1} = HX_m + C \quad (m \geq 0), \tag{3.19}$$

converges to $(I-H)^{-1}C$. In general, let $1 \notin \sigma(H)$. As usual, based on a sequence of polynomials $\{p_m(z)\}$ given by

$$p_m(z) = \sum_{i=0}^{m} \pi_{m,i} z^i, \quad \text{where } \sum_{i=0}^{m} \pi_{m,i} = 1, \tag{3.20}$$

the corresponding semi-iterative method induced by $\{p_m(z)\}$ for the computation of $(I-H)^{-1}C$ is defined as

$$Y_m = \sum_{i=0}^{m} \pi_{m,i} X_i, \quad m \geq 0. \tag{3.21}$$

Moreover, the matrices Y_m and the corresponding residual matrices R_m are given by

$$Y_m = p_m(H)Y_0 + q_{m-1}(H)C, \qquad R_m = p_m(H)(C-(I-H)Y_0), \tag{3.22}$$

where

$$q_{m-1}(z) = (1-p_m(z))/(1-z) \quad \text{with } q_{-1}(z) = 0. \tag{3.23}$$

If $\{q_m(H)\}$ converges to $(I-H)^{-1}$, or equivalently, if $\{p_m(H)\}$ converges to 0, then the sequence $\{Y_m\}$ of (3.21) converges to $(I-H)^{-1}C$. Especially, for $H = I - GA|_T$ and $C = G$, $\{Y_m\}$ converges to $A^{(2)}_{T,S}$. With an application of Theorem 1.2, we have the following corollary.

Corollary 3.2. *Let A, T, S, and G be as in Lemma 2.1 and let $H = I - GA|_T$. If $\sigma(H)$ is contained in Ω_1, a simply connected compact set excluding 1, and $\{q_m(z)\}$ of (3.23) uniformly converges to $1/(1-z)$ on Ω_1, then the sequence $\{Y_m\}$ of (3.21) converges to $A^{(2)}_{T,S}$ for $Y_0 = G$.*

Especially, Ω_1 is either a complex segment $[\alpha,\beta]$ excluding 1 or a closed ellipse in the left half-plane $\{z:\ \mathrm{Re}(z)<1\}$ with foci α and β. Let a sequence of polynomials $\{p_m(z)\}$ given by

$$p_m(z)=\frac{T_m((z-\delta)/\xi)}{T_m((1-\delta)/\xi)},\quad \left(\delta=\frac{\alpha+\beta}{2},\ \xi=\frac{\beta-\alpha}{2}\right), \tag{3.24}$$

where T_m is the mth Chebyshev polynomial. The semi-iterative method induced by $\{p_m(z)\}$ is the Chebyshev iterative method optimal for ellipse Ω_1. The corresponding two-step stationary method with the same asymptotically optimal convergence rate is given by

$$\begin{gathered}Y_0=G;\qquad Y_1=\mu(HY_0+G);\\ Y_{m+1}=\mu_0(HY_m+G)+\mu_1Y_m+\mu_2Y_{m-1},\quad (m\ge 1),\end{gathered} \tag{3.25}$$

where

$$\mu_0=\frac{4}{(\sqrt{1-\beta}+\sqrt{1-\alpha})^2},\qquad \mu_1=\frac{\alpha+\beta}{2}\mu_0,\qquad \mu_2=1-\mu_0-\mu_1. \tag{3.26}$$

The sequence $\{Y_m\}$ converges asymptotically optimally to $A_{T,S}^{(2)}$.

4. Quadratically convergent methods. Newton-Raphson method for finding the root $1/z$ of the function $s(w)=w^{-1}-z$ is given by

$$w_{m+1}=w_m(2-zw_m),\quad \text{for a suitable } w_0. \tag{4.1}$$

For $\alpha>0$, a sequence of functions $\{s_m(z)\}$ is defined by

$$s_0(z)=\alpha,\qquad s_{m+1}(z)=s_m(z)[2-zs_m(z)]. \tag{4.2}$$

Let $z\in\sigma(GA|_T)$ and $0<\alpha<2\cos\phi/\|GA\|$. It follows from the recursive form $zs_{m+1}(z)-1=-[zs_m(z)-1]^2$ that

$$|zs_m(z)-1|=|\alpha z-1|^{2^m}\le\beta^{2^m}\longrightarrow 0,\quad \text{as } m\longrightarrow\infty, \tag{4.3}$$

where an upper bound of β is given by (3.17).

The great attraction of the Newton-Raphson method is the generally quadratic nature of the convergence. Using the above facts in conjunction with Lemma 2.3, we see that a sequence $\{s_m(\tilde{A})\}$ defined by

$$s_0(\tilde{A})=\alpha I,\qquad s_{m+1}(\tilde{A})=s_m(\tilde{A})[2I-\tilde{A}s_m(\tilde{A})] \tag{4.4}$$

has the property that $\lim_{m\to\infty}s_m(\tilde{A})G=A_{T,S}^{(2)}$. If we set $A_m=s_m(\tilde{A})G$, then

$$A_0=\alpha G,\qquad A_{m+1}=A_m(2I-AA_m). \tag{4.5}$$

Thus we have the following corollary.

Corollary 4.1. *Let A, T, S, and G be as in Lemma 2.1. Suppose that the spectrum of $\sigma(GA|_T) \subset \{z : \mathrm{Re}(z) > 0\}$. Then the sequence $\{A_m\}$ of (4.5) quadratically converges to $A^{(2)}_{T,S}$, for $0 < \alpha < 2\cos\phi/\|GA\|$. Furthermore, an error bound is given by*

$$||A_m - A^{(2)}_{T,S}||_P \le (\beta^{2^m} + O(\epsilon))||A^{(2)}_{T,S}||_P, \tag{4.6}$$

where an upper bound of β is given in (3.17).

We remark that Corollary 4.1 is an extension of [4, 13, 15]. It covers iterative methods for $A^{\dagger}_{M,N}$ in [17].

The Newton-Raphson procedure can be speeded up by the successive matrix squaring technique in [14] if two parallel processors are available. In fact, the sequence in (4.5) is mathematically equivalent to

$$\begin{aligned} A_0 &= \alpha G, \qquad P_0 = I - \alpha GA, \\ A_{m+1} &= (I + P_m)A_m, \qquad P_{m+1} = P_m^2. \end{aligned} \tag{4.7}$$

There are two matrix multiplications each step both in (4.5) and (4.7). However, A_{m+1} and P_{m+1} in (4.7) can be calculated simultaneously.

Two algorithms given by (3.8) and (4.5) are also valid in the case when the spectrum of $\tilde{A}$ is contained in the left half-plane with slight modification.

Moreover, all results in the previous two sections are valid without the restriction on $\sigma(GA)$ if G is substituted by another matrix. This is stated as the following corollary.

Corollary 4.2. *Let A, T, S, and G be as in Lemma 2.1. Then Theorem 3.1 and Corollaries 3.2 and 4.1 are valid without any restriction on the spectrum of $GA|_T$ if G is substituted by*

$$G_0 = G(GAG)^*G. \tag{4.8}$$

Proof. It suffices to show that

$$R(G_0) = R(G), \qquad N(G_0) = N(G), \qquad \sigma(G_0A|_T) \subset (0, \infty). \tag{4.9}$$

As a matter of fact (4.9) is a direct result of [4, Lemma 3.4]. □

We remark a disadvantage of the choice G_0 of (4.8). In the case of computing A^D with $\mathrm{Ind}(A) = k \ge 3$, $G_0 = A^k(A^{2k+1})^*A^k$, the condition number of $G_0A|_T$ will be extremely large since $\mathrm{cond}(G_0A|_T) = \mathrm{cond}(A|_T)^{4k+2}$. An accurate numerical solution cannot be obtained if there is any round-off error in A.

5. Examples.

Three examples are given in this section to illustrate the computations of three types of $A^{(2)}_{T,S}$. All calculations were performed on a PC with MATLAB.

Example 5.1. Let A and W be 20 by 10 and 10 by 10 random matrices with entries on $[-1, 1]$, respectively. We choose M and N as random symmetric and positive definite matrices of order 20 and 10, respectively. The stop criterion in (4.5) is

$\|A_m - A_{m-1}\|_\infty \le \epsilon = 10^{-10}$. Three special cases $A^\dagger$, $A^\dagger_{M,N}$, and $A_{d,w}$ are computed in this example. The choices of G, the number of iterations required and the norm of errors are listed in Table 5.1.

TABLE 5.1. Newton-Raphson method for $A^\dagger$, $A^\dagger_{M,N}$, and $A_{d,w}$.

$A^{(2)}_{T,S}$	G	m	$\|A_m - A_{m-1}\|_\infty$	$\|A_m - A^{(2)}_{T,S}\|_\infty$
$A^\dagger$	A^*	11	3.25E-14	2.56E-15
$A^\dagger_{M,N}$	$N^{-1}A^*M$	25	1.03E-15	3.09E-15
$A_{d,w}$	$AWA((WA)^*)^4WA$	36	1.96E-13	1.76E-07

It is remarked that the better accuracy of $A_{d,w}$ never be achieved and 1.7E-07 is the best error of $\|A_m - A_{d,w}\|_\infty$ even if $\|A_m - A_{m-1}\|_\infty \le \epsilon = 10^{-10}$ is used as a stop criterion. This is because the condition number of $GWAW|_T$ is as large as 10^{10}. If 2-step semi-iterative method of (3.25) is applied to compute $A^{(2)}_{T,S}$, then $\{Y_m\}$ converges to $A^\dagger$ after 54 iterations. However, the method fails to converge after 1500 iterations in other two cases because the segments $[\alpha,\beta]$ containing $\sigma(GA|_T)$ are $[-187970, 0.796]$ and $[-355800, 0.9997]$, respectively, so that the rate of asymptotic convergence is too slow.

EXAMPLE 5.2. Let A be 8 by 8 matrix with a complex spectrum given by

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & \frac{3}{4} & -\frac{3}{4} & 0 & 0 & 0 & 0 \\ -1 & -1 & -\frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{4} & -\frac{3}{4} & -1 & -1 \\ 0 & 0 & -1 & 0 & -\frac{3}{4} & \frac{3}{4} & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{3}{2} \end{bmatrix}. \tag{5.1}$$

In order to compute A^D, we choose $G = A^2$ since $\operatorname{Ind}(A) = 2$. The spectrum of $GA|_T$, $\sigma(GA|_T) = \{1.875 \pm 0.674i, 1.875 \pm 0.674i, 3.375, 3.375\}$, is located on the right half-plane. The foci $\alpha = -2.3$ and $\beta = -0.5$ of an ellipse containing $\sigma(I - GA|_T)$ is selected. It requires 28 iterations of 2-step method of (3.25) to compute A^D with the ∞-norm of the error less than 10^{-10}. As expected, Newton-Raphson algorithm of (4.5) converges much faster. It achieves the same accuracy with only 8 iterations.

TABLE 5.2. $A^{(2)}_{T,S}$ of a Toeplitz matrix.

$A^{(2)}_{T,S}$	G	No. of it. by Newton's	No. of it. by SIM
$A^\dagger$	A^*	10	63
$A^\dagger_{M,N}$	$N^{-1}A^*M$	11	75
$A_{d,w}$	$AWA((WA)^*)^4WA$	31	>1500

EXAMPLE 5.3. Let r and c be a row vector and column vector, respectively, such that

$$\begin{aligned} &r_1 = c_1 = 2.5, \\ &r_j = \frac{(-1)^j j}{16} + \frac{i(j-1)}{j}, \quad \text{for } j = 2,3,\ldots,16, \\ &c_k = \frac{(-1)^k k}{10}, \quad \text{for } k = 2,3,\ldots,10. \end{aligned} \tag{5.2}$$

A 10×16 complex Toeplitz matrix A is constructed by r and c. The stop criterion is the same as in Example 5.1. M and N are chosen positive definite diagonal matrix related to A, and W is a random matrix. The numbers of iterations by Newton's method and SIM method for $A^{(2)}_{T,S}$ are shown in Table 5.2.

The data shows that Newton's method is much faster than that of SIM.

ACKNOWLEDGMENTS. The authors thank the referees for their important suggestions. The second author was supported by the National Natural Science Foundation of China under grant No. 19901006 and Doctoral point Foundation of China. Part of the work was finished when Yimin Wei was visiting Harvard University and was supported by China Scholarship Council.

REFERENCES

[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1974.

[2] Y.-L. Chen, *Finite algorithms for the* (2)*-generalized inverse* $A^{(2)}_{T,S}$, Linear and Multilinear Algebra **40** (1995), no. 1, 61–68.

[3] ———, *Iterative methods for computing the generalized inverses* $A^{(2)}_{T,S}$ *of a matrix* A, Appl. Math. Comput. **75** (1996), no. 2-3, 207–222.

[4] Y.-L. Chen and X. Chen, *Representation and approximation of the outer inverse* $A^{(2)}_{T,S}$ *of a matrix* A, Linear Algebra Appl. **308** (2000), no. 1–3, 85–107.

[5] M. Eiermann, W. Niethammer, and R. S. Varga, *A study of semi-iterative methods for nonsymmetric systems of linear equations*, Numer. Math. **47** (1985), no. 4, 505–533.

[6] A. J. Getson and F. C. Hsuan, $\{2\}$*-Inverses and Their Statistical Application*, Lecture Notes in Statistics, vol. 47, Springer-Verlag, New York, 1988.

[7] F. Hsuan, P. Langenberg, and A. J. Getson, *The* $\{2\}$*-inverse with applications in statistics*, Linear Algebra Appl. **70** (1985), 241–248.

[8] M. Z. Nashed, *Generalized Inverses and Applications*, Academic Press, New York, 1976.

[9] M. Z. Nashed and X. Chen, *Convergence of Newton-like methods for singular operator equations using outer inverses*, Numer. Math. **66** (1993), no. 2, 235–257.

[10] W. Niethammer and R. S. Varga, *The analysis of* k*-step iterative methods for linear systems from summability theory*, Numer. Math. **41** (1983), no. 2, 177–206.

[11] P. S. Stanimirović, *Limit representations of generalized inverses and related methods*, Appl. Math. Comput. **103** (1999), no. 1, 51–68.

[12] P-Å. Wedin, *Perturbation results and condition numbers for outer inverses and especially for projections*, Tech. Report UMINF 124.85, S-901 87, Inst. of Info. Proc. University of Umeå, Umeå, Sweden, 1985.

[13] Y. Wei, *A characterization and representation of the generalized inverse $A_{T,S}^{(2)}$ and its applications*, Linear Algebra Appl. **280** (1998), no. 2-3, 87–96.

[14] ———, *Successive matrix squaring algorithm for computing the Drazin inverse*, Appl. Math. Comput. **108** (2000), no. 2-3, 67–75.

[15] Y. Wei and H. Wu, *The representation and approximation for the generalized inverse $A_{T,S}^{(2)}$*, to appear in Appl. Math. Comput.

[16] ———, *The representation and approximation for Drazin inverse*, J. Comput. Appl. Math. **126** (2000), no. 1-2, 417–432.

[17] ———, *The representation and approximation for the weighted Moore-Penrose inverse*, Appl. Math. Comput. **121** (2001), no. 1, 17–28.

[18] ———, *$(T-S)$ splittings methods for computing the generalized inverse $A_{T,S}^{(2)}$ and rectangular systems*, Int. J. Comput. Math. **77** (2001), 401–424.

Xiezhang Li: Department of Mathematics and Computer Science, Georgia Southern University, Statesboro, GA 30460, USA
E-mail address: xli@gsu.cs.gasou.edu

Yimin Wei: Department of Mathematics, Fudan University, Shanghai 200433, China
E-mail address: ymwei@fudan.edu.cn