THE GALOIS ALGEBRAS AND THE AZUMAYA GALOIS EXTENSIONS

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Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$, $C$ the center of $B$, $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$, $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in K$, and $B_K = (\oplus_{g \in K} J_g)$. Then $B_K$ is a central weakly Galois algebra with Galois group induced by $K$. Moreover, an Azumaya Galois extension $B$ with Galois group $K$ is characterized by using $B_K$.

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1. Introduction. Let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$ and $C$ the center of $B$. The class of Galois algebras has been investigated by DeMeyer [2], Kanzaki [6], Harada [4, 5], and the authors [7]. In [2], it was shown that if $R$ contains no idempotents but 0 and 1, then $B$ is a central Galois algebra with Galois group $K$ and $C$ is a commutative Galois algebra with Galois group $G/K$ where $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$ [2, Theorem 1]. This fact was extended to the Galois algebra $B$ over $R$ containing more than two idempotents [6, Proposition 3], and generalized to any Galois algebra $B$ [7, Theorem 3.8] by using the Boolean algebra $B_a$ generated by $\{ 0, e_g \mid g \in G \}$ for a central idempotent $e_g$ where $BJ_g = Be_g$ and $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ for each $g \in G$ [6]. The purpose of this paper is to show that there exists a subalgebra $B_K$ of $B$ such that $B_K$ is a central weakly Galois algebra with Galois group $K$ induced by $K$ where a weakly Galois algebra was defined in [8] and that $B_K B_K$ is an Azumaya weakly Galois extension with Galois group $K|B_K B_K$ where an Azumaya Galois extension was studied in [1]. Thus some characterizations of an Azumaya Galois extension $B$ of $B_K$ with Galois group $K$ are obtained, and the results as given in [2, 6] are generalized.

2. Definitions and notations. Throughout, let $B$ be a Galois algebra over a commutative ring $R$ with Galois group $G$, $C$ the center of $B$, and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$. We keep the definitions of a Galois extension, a Galois algebra, a central Galois algebra, a separable extension, and an Azumaya algebra as defined in [7]. An Azumaya Galois extension $A$ with Galois group $G$ is a Galois extension $A$ of $A^G$ which is a $C^G$-Azumaya algebra where $C$ the center of $A$ [1]. A weakly Galois extension $A$ with Galois group $G$ is a finitely generated projective left module $A$ over $A^G$ such that $A_G \cong \text{Hom}_{A^G}(A, A)$ where $A_I = \{ a_I \}$, a left multiplication map by $a \in A$ [8]. We call that $A$ is a weakly Galois algebra with Galois group $G$ if $A$ is a weakly Galois extension with Galois group $G$ such that $A^G$ is contained in the center of $A$ and that
A is a central weakly Galois algebra with Galois group $G$ if $A$ is a weakly Galois extension with Galois group $G$ such that $A^G$ is the center of $A$. An Azumaya weakly Galois extension $A$ with Galois group $G$ is a weakly Galois extension $A$ of $A^G$ which is a $C^G$-Azumaya algebra where $C$ the center of $A$.

3. A weakly Galois algebra. In this section, let $B$ be a Galois algebra over $R$ with Galois group $G$, $C$ the center of $B$, $B^G = \{ b \in B \mid g(b) = b \text{ for all } g \in G \}$, and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$. Then, $B = \oplus_{g \in G} J_g = (\oplus_{g \in K} J_g) \oplus (\oplus_{g \notin K} J_g)$ where $J_g = \{ b \in B \mid bx = g(x)b \text{ for all } x \in B \}$ [6, Theorem 1]. We denote $\oplus_{g \in K} J_g$ by $B_K$ and the center of $B_K$ by $Z$. Clearly, $K$ is a normal subgroup of $G$. We show that $B_K$ is an Azumaya algebra over $Z$ and a central weakly Galois algebra with Galois group $K|_{B_K}$.

**Theorem 3.1.** The algebra $B_K$ is an Azumaya algebra over $Z$.

**Proof.** By the definition of $B_K$, $B_K = \oplus_{g \in K} J_g$, so $C(=J_1) \subset B_K$. Since $B$ is a Galois algebra with Galois group $G$ and $K = \{ g \in G \mid g(c) = c \text{ for all } c \in C \}$, the order of $K$ is a unit in $C$ by [6, Proposition 5]. Moreover, $K$ is an $C$-automorphism group of $B$, so $B_K$ is a $C$-separable algebra by [5, Proposition 5]. Thus $B_K$ is an Azumaya algebra over $Z$. \hfill \Box

In order to show that $B_K$ is a central weakly Galois algebra with Galois group $K|_{B_K}$, we need two lemmas.

**Lemma 3.2.** Let $L = \{ g \in K \mid g(a) = a \text{ for all } a \in B_K \}$. Then, $L$ is a normal subgroup of $K$ such that $K(=K/L)$ is an automorphism group of $B_K$ induced by $K$ (i.e., $K|_{B_K} \cong K$).

**Proof.** Clearly, $L$ is a normal subgroup of $K$, so for any $h \in K$,

$$h(B_K) = \oplus_{g \in K} h(J_g) = \oplus_{g \in K} J_{g h^{-1}} = \oplus_{g \in hK} J_g = \oplus_{g \in K} J_g = B_K. \quad (3.1)$$

Thus $K|_{B_K} \cong K$. \hfill \Box

**Lemma 3.3.** The fixed ring of $B_K$ under $K$, $(B_K)^K = Z$.

**Proof.** Let $x$ be any element in $(B_K)^K$ and $b$ any element in $B_K$. Then $b = \sum_{g \in K} b_g$ where $b_g \in J_g$ for each $g \in K$. Hence $bx = \sum_{g \in K} b_g x = \sum_{g \in K} g(x) b_g = \sum_{g \in K} x b_g = x \sum_{g \in K} b_g = xb$. Therefore $x \in Z$. Thus $(B_K)^K \subset Z$. Conversely, for any $z \in Z$ and $g \in K$, we have that $xz = xz = g(z)x$ for any $x \in J_g$, so $(g(z) - z)x = 0$ for any $x \in J_g$. Hence $(g(z) - z)J_g = \{0\}$. Noting that $BJ_g = J_g B = B$, we have that $(g(z) - z)B = \{0\}$, so $g(z) = z$ for any $z \in Z$ and $g \in K$. Thus $Z \subset (B_K)^K$. Therefore $(B_K)^K = Z$. \hfill \Box

**Theorem 3.4.** The algebra $B_K$ is a central weakly Galois algebra with Galois group $K|_{B_K} \cong K$.

**Proof.** By Lemma 3.3, it suffices to show that (1) $B_K$ is a finitely generated projective module over $Z$, and (2) $(B_K|_{\overline{K}}) \cong \text{Hom}_Z(B_K, B_K)$. Part (1) is a consequence of Theorem 3.1. For part (2), since $B_K$ is an Azumaya algebra over $Z$ by Theorem 3.1 again, $B_K \otimes_Z B_K^\varphi \cong \text{Hom}_Z(B_K, B_K)$ [3, Theorem 3.4, page 52] by extending the map $(a \otimes b)(x) = axb$ linearly for $a \otimes b \in B_K \otimes_Z B_K^\varphi$ and each $x \in B_K$ where $B_K^\varphi$ is the
opposite algebra of $B_K$. By denoting the left multiplication map with $a \in B_K$ by $a_l$ and the right multiplication map with $b \in B_K$ by $b_r$, $(a \circ b)(x) = (a_l b_r)(x) = a x b$. Since $B_K = \oplus \sum_{g \in K} J_g$, $B_K \otimes_B B_K^\circ = \sum_{g \in K} (B_K)_l (J_g)_r$. Observing that $(J_g)_r = (J_g)\mathfrak{g}^{-1}$ where $\mathfrak{g} = g|_{B_K} \in K|_{B_K} \cong K$, we have that $B_K \otimes_B B_K^\circ = \sum_{g \in K} (B_K)_l (J_g)_r = \sum_{g \in K} (B_K)_l (J_g)\mathfrak{g}^{-1} = \sum_{g \in K} (B_K)_l J_g$. Moreover, since $B J_g = B$ for each $g \in K$ and $B = \oplus \sum_{h \in G} J_h = B_K \oplus (\oplus \sum_{h \notin K} J_h)$, $B_K \otimes_B (\oplus \sum_{h \notin K} J_h) = B = B J_g = B K J_g \oplus (\oplus \sum_{h \notin K} J_h J_g)$ such that $B K J_g \subset B_K$ and $\oplus \sum_{h \notin K} J_h J_g \subset (\oplus \sum_{h \notin K} J_h)$. Hence $B K J_g = B_K$ for each $g \in K$. Therefore $B_K \otimes_B B_K^\circ = \sum_{g \in K} (B_K)_l J_g \mathfrak{g}^{-1} = \sum_{g \in K} (B_K)_l \mathfrak{g}^{-1} = (B_K)_l K$. Thus $(B_K)_l K \cong \text{Hom}_Z(B_K, B_K)$. This completes the proof of part (2). Thus $B_K$ is an Azumaya algebra with Galois group $K|_{B_K} \cong \overline{K}$.

Recall that an algebra $A$ is called an Azumaya weakly Galois extension of $A^K$ with Galois group $K$ if $A$ is a weakly Galois extension of $A^K$ which is a $C^K$-Azumaya algebra where $C$ is the center of $A$. Next, we show that $B_K B^K$ is an Azumaya weakly Galois extension with Galois group $K|_{B_K B^K} \cong \overline{K}$. We begin with the following two lemmas about $B_K$.

**Lemma 3.5.** The fixed ring of $B$ under $K$, $B^K = V_{B}(B_K)$.

**Proof.** For any $b \in B^K$ and $x \in J_g$, for any $g \in K$, we have that $xb = g(b) x = bx$, so $b \in V_{B}(B_K)$ for any $g \in K$. Thus $b \in V_{B}(B_K)$. Conversely, for any $b \in V_{B}(B_K)$ and $g \in K$, we have that $bx = x b = g(b) x$ for any $x \in J_g$, so $g(b) - b x = 0$ for any $x \in J_g$. Hence $(g(b) - b) J_g = \{0\}$. But $J_g J_g B = B$ for any $g \in K$, so $g(b) - b B = \{0\}$. Thus $g(b) - b$ for any $g \in K$; and so $b \in B^K$. Therefore $B^K = V_{B}(B_K)$.

**Lemma 3.6.** The algebra $B^K$ is an Azumaya algebra over $Z$ where $Z$ is the center of $B_K$.

**Proof.** Since $B$ is a Galois algebra over $R$ with Galois group $G$, $B$ is an Azumaya algebra over its center $C$. By the proof of Theorem 3.1, $B_K$ is a $C$-separable subalgebra of $B$, so $V_{B}(B_K)$ is a $C$-separable subalgebra of $B$ and $V_{B}(V_{B}(B_K)) = B_K$ by the commutator theorem for Azumaya algebras [3, Theorem 4.3, page 57]. This implies that $B_K$ and $V_{B}(B_K)$ have the same center $Z$. Thus $V_{B}(B_K)$ is an Azumaya algebra over $Z$. But, by Lemma 3.5, $B^K = V_{B}(B_K)$, so $B^K$ is an Azumaya algebra over $Z$.

**Theorem 3.7.** Let $A = B_K B^K$. Then $A$ is an Azumaya weakly Galois extension with Galois group $K|_{A} \cong \overline{K}$.

**Proof.** Since $B$ is a central weakly Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ by Theorem 3.4, $B_K$ is a finitely generated projective module over $Z$ and $(B_K)_l \overline{K} \cong \text{Hom}_Z(B_K, B_K)$. By Lemma 3.6, $B^K$ is an Azumaya algebra over $Z$, so $A \cong B_K \otimes Z B^K$ is a finitely generated projective module over $B^K (= A^K)$. Moreover, since $B^K = V_{B}(B_K)$ by Lemma 3.5 and $(B_K)_l \overline{K} \cong \text{Hom}_Z(B_K, B_K)$,

\[A|_K = (B_K B^K)_l \overline{K} = (B_K)_l \overline{K}(B^K)_r \cong B_K \otimes Z B^K \cong \text{Hom}_Z(B_K, B_K) \otimes Z B^K \cong \text{Hom}_{B^K}(B_K \otimes Z B^K, B_K \otimes Z B_K) \cong \text{Hom}_{B^K}(B_K B^K, B_K B_K) \cong \text{Hom}_{A^K}(A, A).\]
Thus $A$ is a weakly Galois extension of $A^K$ with Galois group $K|_A \cong \overline{K}$. Next, we claim that $A$ has center $Z$ and $A^K$ is an Azumaya algebra over $Z^K$. In fact, $B_K$ and $B^K$ are Azumaya algebras over $Z$ by Theorem 3.1 and Lemma 3.6, respectively, so $A(=B_K B^K)$ has center $Z$ and $A^K = (B_K B^K)^K = B^K$. Noting that $B^K$ is an Azumaya algebra over $Z$, we conclude that $A^K$ is an Azumaya algebra over $Z^K$. Thus $A$ is an Azumaya weakly Galois extension with Galois group $K|_A \cong \overline{K}$. □

4. An Azumaya Galois extension. In this section, we give several characterizations of an Azumaya Galois extension $B$ by using $B_K$. This generalizes the results in [2, 6]. The $Z$-module $\{b \in B_K \mid bx = g(x) b\}$ for all $x \in B_K$ is denoted by $J_{\overline{g}}^{(B_K)}$ for $\overline{g} \in \overline{K}$ where $\overline{K}(=K/L)$ is defined in Lemma 3.2.

**Lemma 4.1.** The algebra $B_K$ is a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ if and only if $J_{\overline{g}}^{(B_K)} = \oplus \sum_{l \in L} J_{\overline{g}l}$ for each $\overline{g} \in \overline{K}$.

**Proof.** Let $B_K$ be a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$. Then $B_K = \oplus \sum_{\overline{g} \in \overline{K}} J_{\overline{g}}^{(B_K)}$, [6, Theorem 1]. Next it is easy to check that $\oplus \sum_{l \in L} J_{\overline{g}l} \subseteq J_{\overline{g}}^{(B_K)}$. But $B_K = \oplus \sum_{\overline{g} \in \overline{K}} J_{\overline{g}} = \oplus \sum_{l \in L} J_{\overline{g}l}$ where $\oplus \sum_{l \in L} J_{\overline{g}l} \subset J_{\overline{g}}^{(B_K)}$. Thus $J_{\overline{g}}^{(B_K)} = \oplus \sum_{l \in L} J_{\overline{g}l}$ for each $\overline{g} \in \overline{K}$. Conversely, since $J_{\overline{g}}^{(B_K)} = \oplus \sum_{l \in L} J_{\overline{g}l}$ for each $\overline{g} \in \overline{K}$, $B_K = \oplus \sum_{\overline{g} \in \overline{K}} J_{\overline{g}} = \oplus \sum_{l \in L} J_{\overline{g}l}$. Moreover, by Lemma 3.3, $(B_K)^K = Z$, so $\overline{K}$ is a $Z$-automorphism group of $B_K$. Hence $J_{\overline{g}}^{(B_K)} J_{\overline{g}^{-1}}^{(B_K)} = Z$ for each $\overline{g} \in \overline{K}$. Thus $B_K$ is a central Galois algebra with Galois group $K|_{B_K} \cong \overline{K}$ because $B_K$ is an Azumaya $Z$-algebra by Theorem 3.1 (see [4, Theorem 1]). □

Next, we characterize an Azumaya Galois extension $B$ with Galois group $K$.

**Theorem 4.2.** The following statements are equivalent:

1. $B$ is an Azumaya Galois extension with Galois group $K$;
2. $Z = C$;
3. $B = B_K B^K$;
4. $B_K$ is a central Galois algebra over $C$ with Galois group $K|_{B_K} \cong \overline{K}$.

**Proof.** (1)⇒(2). Since $B$ is an Azumaya Galois extension with Galois group $K$, $B^K$ is a $C^K$-Azumaya algebra. But, by Lemma 3.6, $B^K$ is an Azumaya algebra over $Z$, so $Z = C^K \subset C$. Hence $C \subset Z = C^K \subset C$. Thus $Z = C$.

(2)⇒(3). Suppose that $Z = C$. Then, by Theorem 3.1, $B_K$ is an Azumaya algebra over $C$. Hence by the commutator theorem for Azumaya algebras, $B = B_K V_{B_K}(B_K)$ [3, Theorem 4.3, page 57]. But, by Lemma 3.6, $B^K = V_{B_K}(B_K)$, so $B = B_K B^K$.

(3)⇒(4). By hypothesis, $B = B_K B^K$, so $L = \{1\}$ where $L$ is given in Lemma 3.2. By the proofs of Theorem 3.1 and Lemma 3.6, $B_K$ and $B^K$ are $C$-separable subalgebras of the Azumaya $C$-algebra $B$ such that $B = B_K B^K$, so $B_K$ and $B^K$ are Azumaya algebras over $C$ [3, Theorem 4.4, page 58]. Thus $C$ is the center of $B_K$. Next, we claim that $J_{\overline{g}} = J_{\overline{g}}^{(B_K)}$ for each $g \in K$. In fact, it is clear that $J_{\overline{g}} \subset J_{\overline{g}}^{(B_K)}$. Conversely, for each $a \in J_{\overline{g}}^{(B_K)}$ and $x \in B$ such that $x = yz$ for some $y \in B_K$ and $z \in B^K$, noting that $B^K = V_{B_K}(B_K)$, we have that $ax = ayz = g(y)az = g(y)za = g(yz)a = g(x)a$. Thus $J_{\overline{g}}^{(B_K)} \subset J_{\overline{g}}$. This proves that $J_{\overline{g}} = J_{\overline{g}}^{(B_K)} (=J_{\overline{g}}^{(B_K)}$ since $L = \{1\}$) for each $g \in K$. Hence, $B_K$ is a central Galois algebra over $C$ with Galois group $K|_{B_K} \cong \overline{K}$ by Lemma 4.1.
(4)⇒(1). Since $B$ is a Galois algebra with Galois group $G$, $B$ is a Galois extension with Galois group $K$. By hypothesis, $B_K$ is a central Galois algebra over $C$ with Galois group $K|B_K \equiv K$, so the center of $B_K$ is $C$, that is, $Z = C$. Hence $B^K$ is an Azumaya algebra over $C (= C^K)$ by Lemma 3.6. Thus $B$ is an Azumaya Galois extension with Galois group $K$. 

\textbf{Theorem 4.2} generalizes the following result of Kanzaki [6, Proposition 3].

\textbf{Corollary 4.3.} If $J_g = \{0\}$ for each $g \notin K$, then $B$ is a central Galois algebra with Galois group $K$ and $C$ is a Galois algebra with Galois group $G/K$.

\textbf{Proof.} This is the case in \textbf{Theorem 4.2} that $B = B_K B^K = B_K$ where $B^K = C$.

We conclude the present paper with two examples, one to illustrate the result in \textbf{Theorem 4.2}, and another to show that $Z \neq C$.

\textbf{Example 4.4.} Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = (A \otimes \mathbb{R}) \oplus A \oplus A \oplus A \oplus A$, and $G$ the group generated by the elements in $\{g_1, k_1, k_j, k_k, h_1, h_j, h_k\}$ where $g_1$ is the identity of $G$ and for all $(a \otimes b, a_1, a_2, a_3, a_4) \in B$,

$$
k_i(a \otimes b, a_1, a_2, a_3, a_4) = (iai^{-1} \otimes b, ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}, ia_4i^{-1}),$$

$$
k_j(a \otimes b, a_1, a_2, a_3, a_4) = (ja^{-1} \otimes b, ja_1j^{-1}, ja_2j^{-1}, ja_3j^{-1}, ja_4j^{-1}),$$

$$
k_k(a \otimes b, a_1, a_2, a_3, a_4) = (kak^{-1} \otimes b, ka_1k^{-1}, ka_2k^{-1}, ka_3k^{-1}, ka_4k^{-1}),$$

$$
h_1(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes bi^{-1}, a_2, a_1, a_4, a_3),$$

$$
h_j(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes jb^{-1}, a_3, a_4, a_1, a_2),$$

$$
h_k(a \otimes b, a_1, a_2, a_3, a_4) = (a \otimes kb^{-1}, a_4, a_3, a_2, a_1).
$$

Then,

(1) we can check that $B$ is a Galois algebra over $B^G$ with Galois group $G$ where $B^G = \{(r_1 \otimes r_2, r, r, r, r) \mid r_1, r_2, r \in \mathbb{R}\} \subset C$, and $C = (\mathbb{R} \oplus \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of $B$;

(2) $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{g_1, k_1, k_j, k_k\}$;

(3) $J_1 = C$, $J_{k_1} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R}$, $J_{k_j} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R}$, $J_{k_k} = (\mathbb{R} \otimes 1) \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R} \oplus \mathbb{R} \otimes \mathbb{R}$, so $B_K = (A \otimes \mathbb{R}) \oplus A \oplus A \oplus A \oplus A$. Hence $B_K$ has center $C$, that is $Z = C$, and $B_K$ is a central Galois algebra over $C$ with Galois group $K|B_K \equiv K$;

(4) $B^K = (\mathbb{R} \oplus \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and $B = B_K B^K$, that is, $B$ is an Azumaya Galois extension with Galois group $K$.

\textbf{Example 4.5.} Let $A = \mathbb{R}[i,j,k]$, the real quaternion algebra over the field of real numbers $\mathbb{R}$, $B = A \oplus A \oplus A$, $G = \{1, g_1, g_j, g_k\}$, and for all $(a_1, a_2, a_3) \in B$,

$$
g_i(a_1, a_2, a_3) = (ia_1i^{-1}, ia_2i^{-1}, ia_3i^{-1}),$$

$$
g_j(a_1, a_2, a_3) = (ja_1j^{-1}, ja_2j^{-1}, ja_3j^{-1}),$$

$$
g_k(a_1, a_2, a_3) = (ka_1k^{-1}, ka_3k^{-1}, ka_2k^{-1}).
$$
Then,

(1) $B$ is a Galois algebra over $B^G$ where $B^G = \{(r_1, r, r) \mid r_1, r \in \mathbb{R}\} \subset C$, and $C = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, the center of $B$. The $G$-Galois system is $\{a_i; b_i \mid i = 1, 2, \ldots, 8\}$ where

$$a_1 = (1, 0, 0), \quad a_2 = (i, 0, 0), \quad a_3 = (j, 0, 0), \quad a_4 = (k, 0, 0),$$

$$a_5 = (0, 1, 0), \quad a_6 = (0, j, 0), \quad a_7 = (0, 0, 1), \quad a_8 = (0, 0, k);$$

$$b_1 = \frac{1}{4} a_1, \quad b_2 = -\frac{1}{4} a_2, \quad b_3 = -\frac{1}{4} a_3, \quad b_4 = -\frac{1}{4} a_4, \quad b_5 = \frac{1}{2} a_5, \quad b_6 = -\frac{1}{2} a_6, \quad b_7 = \frac{1}{2} a_7, \quad b_8 = -\frac{1}{2} a_8,$$

(4.3)

(2) $K = \{g \in G \mid g(c) = c \text{ for all } c \in C\} = \{1, g_1\}$ where $J_{g_1} = \mathbb{R} i \oplus \mathbb{R} i \oplus \mathbb{R} i$, so $B_K = \mathbb{R} [i] \oplus \mathbb{R} [i] \oplus \mathbb{R} [i]$ which is a commutative ring not equal to $C$, that is, $Z \neq C$.

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