ON $\beta$-DUAL OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

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The $\beta$-dual of a vector-valued sequence space is defined and studied. We show that if an $X$-valued sequence space $E$ is a BK-space having AK property, then the dual space of $E$ and its $\beta$-dual are isometrically isomorphic. We also give characterizations of $\beta$-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

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1. Introduction. Let $(X, \| \cdot \|)$ be a Banach space and $p = (p_k)$ a bounded sequence of positive real numbers. Let $\mathbb{N}$ be the set of all natural numbers, we write $x = (x_k)$ with $x_k$ in $X$ for all $k \in \mathbb{N}$. The $X$-valued sequence spaces of Maddox are defined as

\begin{align*}
c_0(X,p) &= \left\{ x = (x_k) : \lim_{k \to \infty} \|x_k\|^{p_k} = 0 \right\}; \\
c(X,p) &= \left\{ x = (x_k) : \lim_{k \to \infty} \|x_k - a\|^{p_k} = 0 \text{ for some } a \in X \right\}; \\
\ell_\infty(X,p) &= \left\{ x = (x_k) : \sup_k \|x_k\|^{p_k} < \infty \right\}; \\
\ell(X,p) &= \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^{p_k} < \infty \right\}. \tag{1.1}
\end{align*}

When $X = \mathbb{K}$, the scalar field of $X$, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3, 4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p)$, $c(p)$, $\ell(p)$, and $\ell_\infty(p)$ and has given characterizations of $\beta$-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$, $1 < p < \infty$, is defined by

$$\ell_p[X] = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |f(x_k)|^p < \infty \text{ for each } f \in X' \right\}. \tag{1.2}$$

In this paper, the $\beta$-dual of a vector-valued sequence space is defined and studied and we give characterizations of $\beta$-dual of vector-valued sequence spaces of Maddox.
\( \ell(X,p), \ell_\infty(X,p), c_0(X,p), \) and \( c(X,p) \). Some results, obtained in this paper, are generalizations of some in [1, 3].

2. Notation and definitions. Let \((X,\| \cdot \|)\) be a Banach space. Let \( W(X) \) and \( \Phi(X) \) denote the space of all sequences in \( X \) and the space of all finite sequences in \( X \), respectively. A sequence space in \( X \) is a linear subspace of \( W(X) \). Let \( E \) be an \( X \)-valued sequence space. For \( x \in E \) and \( k \in \mathbb{N} \) we write that \( x_k \) stand for the \( k \)th term of \( x \). For \( x \in E \) and \( k \in \mathbb{N} \), we let \( e^{(k)}(x) \) be the sequence \((0,0,0,\ldots,0,x,0,\ldots)\) with \( x \) in the \( k \)th position and let \( e(x) \) be the sequence \((x,x,x,\ldots)\). For a fixed scalar sequence \( u = (u_k) \), the sequence space \( E_u \) is defined as

\[
E_u = \{ x = (x_k) \in W(X) : (u_kx_k) \in E \}. \tag{2.1}
\]

An \( X \)-valued sequence space \( E \) is said to be normal if \((y_k) \in E \) whenever \( \| y_k \| \leq \| x_k \| \) for all \( k \in \mathbb{N} \) and \((x_k) \in E \). Suppose that the \( X \)-valued sequence space \( E \) is endowed with some linear topology \( \tau \). Then \( E \) is called a \( K \)-space if, for each \( k \in \mathbb{N} \), the \( k \)th coordinate mapping \( p_k : E \rightarrow X \), defined by \( p_k(x) = x_k \), is continuous on \( E \). In addition, if \((E,\tau)\) is a Fréchet (Banach) space, then \( E \) is called an FK-(BK)-space. Now, suppose that \( E \) contains \( \Phi(X) \), then \( E \) is said to have property \( AK \) if \( \sum_{k=1}^{n} e^{(k)}(x_k) \rightarrow x \) in \( E \) as \( n \to \infty \) for every \( x = (x_k) \in E \).

The spaces \( c_0(p) \) and \( c(p) \) are FK-spaces. In \( c_0(X,p) \), we consider the function \( g(x) = \sup_k ||x_k||^{p_k/M} \), where \( M = \max\{1,\sup_k p_k\} \), as a paranorm on \( c_0(X,p) \), and it is known that \( c_0(X,p) \) is an FK-space having property \( AK \) under the paranorm \( g \) defined as above. In \( \ell(X,p) \), we consider it as a paranormed sequence space with the paranorm given by \( \| (x_k) \| = (\sum_{k=1}^{\infty} ||x_k||^{p_k})^{1/M} \). It is known that \( \ell(X,p) \) is an FK-space under the paranorm defined as above.

For an \( X \)-valued sequence space \( E \), define its Köthe dual with respect to the dual pair \((X,X')\) (see [2]) as follows:

\[
E^\times|_{(X,X')} = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \ \forall \ x = (x_k) \in E \right\}. \tag{2.2}
\]

In this paper, we denote \( E^\times|_{(X,X')} \) by \( E^\alpha \) and it is called the \( \alpha \)-dual of \( E \).

For a sequence space \( E \), the \( \beta \)-dual of \( E \) is defined by

\[
E^\beta = \left\{ (f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges } \forall (x_k) \in E \right\}. \tag{2.3}
\]

It is easy to see that \( E^\alpha \subseteq E^\beta \).

For the sake of completeness we introduce some further sequence spaces that will be considered as \( \beta \)-dual of the vector-valued sequence spaces of Maddox:

\[
M_0(X,p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} ||x_k||^{M^{-1/p_k}} < \infty \text{ for some } M \in \mathbb{N} \right\};
\]

\[
M_\infty(X,p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} ||x_k||^{n^{-1/p_k}} < \infty \ \forall \ n \in \mathbb{N} \right\};
\]
\[ \ell_0(X, p) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \|x_k\|^p M^{-p_k} < \infty \text{ for some } M \in \mathbb{N}, \quad p_k > 1 \ \forall k \in \mathbb{N} \right\}, \]

\[ cs[X'] = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges } \forall x \in X \right\}. \]

When \( X = \mathbb{K} \), the scalar field of \( X \), the corresponding first two sequence spaces are written as \( M_0 (p) \) and \( M_\infty (p) \), respectively. These two spaces were first introduced by Grosse-Erdmann [1].

### 3. Main results

We begin by giving some general properties of \( \beta \)-dual of vector-valued sequence spaces.

**Proposition 3.1.** Let \( X \) be a Banach space and let \( E, E_1, \) and \( E_2 \) be \( X \)-valued sequence spaces. Then

(i) \( E^\alpha \subseteq E^\beta \).

(ii) If \( E_1 \subseteq E_2 \), then \( E_2^\beta \subseteq E_1^\beta \).

(iii) If \( E = E_1 + E_2 \), then \( E^\beta = E_1^\beta \cap E_2^\beta \).

(iv) If \( E \) is normal, then \( E^\alpha = E^\beta \).

**Proof.** Assertions (i), (ii), and (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that \( E^\beta \subseteq E^\alpha \). Let \( (f_k) \in E^\beta \) and \( x = (x_k) \in E \). Then \( \sum_{k=1}^{\infty} f_k(x_k) \) converges. Choose a scalar sequence \( (t_k) \) with \( |t_k| = 1 \) and \( f_k(t_k x_k) = |f_k(x_k)| \) for all \( k \in \mathbb{N} \). Since \( E \) is normal, \( (t_k x_k) \in E \). It follows that \( \sum_{k=1}^{\infty} |f_k(x_k)| \) converges, hence \( (f_k) \in E^\alpha \).

If \( E \) is a BK-space, we define a norm on \( E^\beta \) by the formula

\[ \| (f_k) \|_{E^\beta} = \sup_{\| (x_k) \| \leq 1} \left\| \sum_{k=1}^{\infty} f_k(x_k) \right\| \]  

(3.1)

It is easy to show that \( \| \cdot \|_{E^\beta} \) is a norm on \( E^\beta \).

Next, we give a relationship between \( \beta \)-dual of a sequence space and its continuous dual. Indeed, we need a lemma.

**Lemma 3.2.** Let \( E \) be an \( X \)-valued sequence space which is an FK-space containing \( \Phi(X) \). Then for each \( k \in \mathbb{N} \), the mapping \( T_k : X \to E \), defined by \( T_k x = e^k(x) \), is continuous.

**Proof.** Let \( V = \{ e^k(x) : x \in X \} \). Then \( V \) is a closed subspace of \( E \), so it is an FK-space because \( E \) is an FK-space. Since \( E \) is a \( K \)-space, the coordinate mapping \( p_k : V \to X \) is continuous and bijective. It follows from the open mapping theorem that \( p_k \) is open, which implies that \( p_k^{-1} : X \to V \) is continuous. But since \( T_k = p_k^{-1} \), we thus obtain that \( T_k \) is continuous.

**Theorem 3.3.** If \( E \) is a BK-space having property AK, then \( E^\beta \) and \( E' \) are isometrically isomorphic.
Proof. We first show that for \( x = (x_k) \in E \) and \( f \in E' \),
\[
f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)).
\] (3.2)
To show this, let \( x = (x_k) \in E \) and \( f \in E' \). Since \( E \) has property AK,
\[
x = \lim_{n \to \infty} \sum_{k=1}^{n} e^{(k)}(x_k).
\] (3.3)
By the continuity of \( f \), it follows that
\[
f(x) = \lim_{n \to \infty} \sum_{k=1}^{n} f(e^k(x_k)) = \sum_{k=1}^{\infty} f(e^k(x_k)),
\] (3.4)
so (3.2) is obtained. For each \( k \in \mathbb{N} \), let \( T_k : X \to E \) be defined as in Lemma 3.2. Since \( E \) is a BK-space, by Lemma 3.2, \( T_k \) is continuous. Hence \( f \circ T_k \in X' \) for all \( k \in \mathbb{N} \). It follows from (3.2) that
\[
f(x) = \sum_{k=1}^{\infty} (f \circ T_k)(x) \quad \forall x = (x_k) \in E.
\] (3.5)
It implies, by (3.5), that \((f \circ T_k)_{k=1}^{\infty} \in E^\beta\). Define \( \varphi : E' \to E^\beta \) by
\[
\varphi(f) = (f \circ T_k)_{k=1}^{\infty} \quad \forall f \in E'.
\] (3.6)
It is easy to see that \( \varphi \) is linear. Now, we show that \( \varphi \) is onto. Let \((f_k) \in E^\beta\). Define \( f : E \to K \), where \( K \) is the scalar field of \( X \), by
\[
f(x) = \sum_{k=1}^{\infty} f_k(x_k) \quad \forall x = (x_k) \in E.
\] (3.7)
For each \( k \in \mathbb{N} \), let \( p_k \) be the \( k \)th coordinate mapping on \( E \). Then we have
\[
f(x) = \sum_{k=1}^{\infty} (f_k \circ p_k)(x) = \lim_{n \to \infty} \sum_{k=1}^{n} (f \circ p_k)(x).
\] (3.8)
Since \( f_k \) and \( p_k \) are continuous linear, so is also continuous \( f \circ p_k \). It follows by Banach-Steinhaus theorem that \( f \in E' \) and we have by (3.7) that, for each \( k \in \mathbb{N} \) and each \( z \in X \), \((f \circ T_k)(z) = f(e^k(z)) = f_k(z)\). Thus \( f \circ T_k = f_k \) for all \( k \in \mathbb{N} \), which implies that \( \varphi(f) = (f_k) \), hence \( \varphi \) is onto.

Finally, we show that \( \varphi \) is linear isometry. For \( f \in E' \), we have
\[
\|f\| = \sup_{\|x_k\| \leq 1} \|f((x_k))\| = \sup_{\|x_k\| \leq 1} \left| \sum_{k=1}^{\infty} f(e^k(x_k)) \right| (\text{by (3.2)}) = \sup_{\|x_k\| \leq 1} \left| \sum_{k=1}^{\infty} (f \circ T_k)(x_k) \right| (3.9)
\]
\[
= \|((f \circ T_k)_{k=1}^{\infty})\|_{E^\beta} = \|\varphi(f)\|_{E^\beta}.
\]
Hence $\varphi$ is isometry. Therefore, $\varphi : E' \to E^\beta$ is an isometrically isomorphism from $E'$ onto $E^\beta$. This completes the proof.

We next give characterizations of $\beta$-dual of the sequence space $\ell(X, p)$ when $p_k > 1$ for all $k \in \mathbb{N}$.

**Theorem 3.4.** Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$. Then $\ell(X, p)^\beta = \ell_0(X', q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$.

**Proof.** Suppose that $(f_k) \in \ell_0(X', q)$. Then $\sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. Then for each $x = (x_k) \in \ell(X, p)$, we have

$$\sum_{k=1}^{\infty} \|f_k(x_k)\| \leq \sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\|$$

$$\leq \sum_{k=1}^{\infty} \left(\|f_k\|^{q_k} M^{-q_k/p_k} + M \|x_k\|\right)^{p_k}$$

$$= \sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-(q_k-1)} + M \sum_{k=1}^{\infty} \|x_k\|\right)^{p_k}$$

$$= M \sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-q_k} + M \sum_{k=1}^{\infty} \|x_k\|^{p_k}$$

$$< \infty,$$  \hfill (3.10)

which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so $(f_k) \in \ell(X, p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X, p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in \ell(X, p)$. For each $x = (x_k) \in \ell(X, p)$, choose scalar sequence $(t_k)$ with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in \ell(X, p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in \ell(X, p).$$  \hfill (3.11)

We want to show that $(f_k) \in \ell_0(X', q)$, that is, $\sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. If it is not true, then

$$\sum_{k=1}^{\infty} \|f_k\|^{q_k} M^{-q_k} = \infty \quad \forall m \in \mathbb{N}.\quad \hfill (3.12)$$

It implies by (3.12) that for each $k \in \mathbb{N}$,

$$\sum_{i \geq k} |f_i|^{q_i} m^{-q_i} = \infty \quad \forall m \in \mathbb{N}.\quad \hfill (3.13)$$

By (3.12), let $m_1 = 1$, then there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{k=k_1}^{\infty} \|f_k\|^{q_k} m_1^{-q_k} > 1.\quad \hfill (3.14)$$
By (3.13), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that
\[
\sum_{k_1 < k < k_2} \left\| f_k \right\|_{q_k}^{q_k} m_2^{-q_k} > 1.
\] (3.15)

Proceeding in this way, we can choose sequences of positive integers $(k_i)$ and $(m_i)$ with $1 = k_0 < k_1 < k_2 < \cdots$ and $m_1 < m_2 < \cdots$, such that $m_i > 2^i$ and
\[
\sum_{k_{i-1} < k < k_i} \left\| f_k \right\|_{q_k}^{q_k} m_i^{-q_k} > 1.
\] (3.16)

For each $i \in \mathbb{N}$, choose $x_k$ in $X$ with $\| x_k \| = 1$ for all $k \in \mathbb{N}$, $k_{i-1} < k \leq k_i$ such that
\[
\sum_{k_{i-1} < k < k_i} \left\| f_k(x_k) \right\|_{q_k}^{q_k} m_i^{-q_k} > 1 \quad \forall i \in \mathbb{N}.
\] (3.17)

Let $a_i = \sum_{k_{i-1} < k < k_i} |f_k(x_k)|^{q_k} m_i^{-q_k}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-q_k} f_k(x_k) |q_k^{-1} x_k$ for all $k \in \mathbb{N}$ with $k_{i-1} < k \leq k_i$. By using the fact that $p_k q_k = p_k + q_k$ and $p_k (q_k - 1) = q_k$ for all $k \in \mathbb{N}$, we have that for each $i \in \mathbb{N}$,
\[
\sum_{k_{i-1} < k < k_i} \| y_k \|_{p_k}^{p_k} = \sum_{k_{i-1} < k < k_i} \left\| a_i^{-1} m_i^{-q_k} f_k(x_k) |q_k^{-1} x_k \right\|_{p_k}^{p_k}
= \sum_{k_{i-1} < k < k_i} a_i^{-p_k} m_i^{-p_k q_k} |f_k(x_k)|^{q_k}
= \sum_{k_{i-1} < k < k_i} a_i^{-p_k} m_i^{-p_k} m_i^{-q_k} |f_k(x_k)|^{q_k}
\leq a_i^{-1} m_i^{-1} \sum_{k_{i-1} < k < k_i} m_i^{-q_k} |f_k(x_k)|^{q_k}
\leq a_i^{-1} m_i^{-1} a_i
= m_i^{-1}
< \frac{1}{2^i},
\] (3.18)

so we have that $\sum_{k=1}^{\infty} \| y_k \|_{p_k}^{p_k} \leq \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X, p)$. For each $i \in \mathbb{N}$, we have
\[
\sum_{k_{i-1} < k < k_i} |f_k(y_k)| = \sum_{k_{i-1} < k < k_i} \left| f_k \left( a_i^{-1} m_i^{-q_k} f_k(x_k) |q_k^{-1} x_k \right) \right|
= \sum_{k_{i-1} < k < k_i} a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k}
= a_i^{-1} \sum_{k_{i-1} < k < k_i} m_i^{-q_k} |f_k(x_k)|^{q_k}
= 1,
\] (3.19)

so that $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.11). Hence $(f_k) \in \ell_0(X', q)$. The proof is now complete. \qed
The following theorem gives a characterization of $\beta$-dual of $\ell(X,p)$ when $p_k \leq 1$ for all $k \in \mathbb{N}$. To do this, the following lemma is needed.

**Lemma 3.5.** Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X,(n^{-1/p_k})$.

**Proof.** Let $x \in \ell_\infty(X,p)$, then there is some $n \in \mathbb{N}$ with $\|x_k\|^{p_k} \leq n$ for all $k \in \mathbb{N}$. Hence $\|x_k\|^{n^{-1/p_k}} \leq 1$ for all $k \in \mathbb{N}$, so that $x \in \ell_\infty(X,(n^{-1/p_k})$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} \ell_\infty(X,(n^{-1/p_k})$, then there are some $n \in \mathbb{N}$ and $M > 1$ such that $\|x_k\|^{n^{-1/p_k}} \leq M$ for every $k \in \mathbb{N}$. Then we have $\|x_k\|^{p_k} \leq nM^{p_k} \leq nM^\alpha$ for all $k \in \mathbb{N}$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$.

**Theorem 3.6.** Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \leq 1$ for all $k \in \mathbb{N}$. Then $\ell(X,p)^\beta = \ell_\infty(X',p)$.

**Proof.** If $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell(X,p). \tag{3.20}$$

If $(f_k) \notin \ell_\infty(X',p)$, it follows by Lemma 3.5 that $\sup_k \|f_k\| m_k^{-1/p_k} = \infty$ for all $m \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose sequences $(m_i)$ and $(k_i)$ of positive integers with $m_1 < m_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1. \tag{3.21}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some $i$, and 0 otherwise. Then $\sum_{i=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| = \sum_{i=1}^{\infty} |f_{k_i} \left( m_i^{-1/p_{k_i}} x_{k_i} \right)|$$

$$= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})|$$

$$= \infty \quad \text{(by (3.21))}, \tag{3.22}$$

and this is contradictory to (3.20), hence $(f_k) \in \ell_\infty(X',p)$.

Conversely, assume that $(f_k) \in \ell_\infty(X',p)$. By Lemma 3.5, there exists $M \in \mathbb{N}$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$, so there is a $K > 0$ such that

$$\|f_k\| \leq KM^{1/p_k} \quad \forall k \in \mathbb{N}. \tag{3.23}$$

Let $x = (x_k) \in \ell(X,p)$. Then there is a $k_0 \in \mathbb{N}$ such that $M^{1/p_k} \|x_k\| \leq 1$ for all $k \geq k_0$. By $p_k \leq 1$ for all $k \in \mathbb{N}$, we have that, for all $k \geq k_0$,

$$M^{1/p_k} \|x_k\| \leq (M^{1/p_k} \|x_k\|)^{p_k} = M \|x_k\|^{p_k}. \tag{3.24}$$
Then
\[
\sum_{k=1}^{\infty} |f_k(x_k)| \leq \sum_{k=1}^{k_0} \||f_k|||x_k|| + \sum_{k=k_0+1}^{\infty} \||f_k|||x_k||
\]
\[
\leq \sum_{k=1}^{k_0} \||f_k|||x_k|| + K \sum_{k=k_0+1}^{\infty} M^{1/p_k}||x_k|| \quad \text{(by (3.23))}
\]
\[
\leq \sum_{k=1}^{k_0} \||f_k|||x_k|| + KM \sum_{k=k_0+1}^{\infty} ||x_k||^{p_k} \quad \text{(by (3.24))}
\]
\[
< \infty.
\]
This implies that \(\sum_{k=1}^{\infty} f_k(x_k)\) converges, hence \((f_k) \in \ell(X,p)^\beta\).

**Theorem 3.7.** Let \(p = (p_k)\) be a bounded sequence of positive real numbers. Then \(\ell_\infty(X,p)^\beta = M_\infty(X',p)\).

**Proof.** If \((f_k) \in M_\infty(X',p)\), then \(\sum_{k=1}^{\infty} \||f_k|||x_k|| < \infty\) for all \(m \in \mathbb{N}\), we have that for each \(x = (x_k) \in \ell_\infty(X,p)\), there is \(m_0 \in \mathbb{N}\) such that \(||x_k|| \leq m_0^{1/p_k}\) for all \(k \in \mathbb{N}\), hence \(\sum_{k=1}^{\infty} \||f_k(x_k)|| \leq \sum_{k=1}^{\infty} \||f_k||||x_k|| \leq \sum_{k=1}^{\infty} ||f_k||m_0^{1/p_k} < \infty\), which implies that \(\sum_{k=1}^{\infty} f_k(x_k)\) converges, so that \((f_k) \in \ell_\infty(X,p)^\beta\).

Conversely, assume that \((f_k) \in \ell_\infty(X,p)^\beta\), then \(\sum_{k=1}^{\infty} f_k(x_k)\) converges for all \(x = (x_k) \in \ell_\infty(X,p)\), by using the same proof as in Theorem 3.4, we have
\[
\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell_\infty(X,p).
\]
(3.26)

If \((f_k) \notin M_\infty(X',p)\), then \(\sum_{k=1}^{\infty} ||f_k||M^{1/p_k} = \infty\) for some \(M \in \mathbb{N}\). Then we can choose a sequence \((k_i)\) of positive integers with \(0 = k_0 < k_1 < k_2 < \cdots\) such that
\[
\sum_{k_{i-1} < k \leq k_i} ||f_k||M^{1/p_k} > i \quad \forall i \in \mathbb{N}.
\]
(3.27)

And we choose \(x_k\) in \(X\) with \(||x_k|| = 1\) such that for all \(i \in \mathbb{N}\),
\[
\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|M^{1/p_k} > i.
\]
(3.28)

Put \(y = (y_k), y_k = M^{1/p_k}x_k\). Clearly, \(y \in \ell_\infty(X,p)\) and
\[
\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)|M^{1/p_k} > i \quad \forall i \in \mathbb{N}.
\]
(3.29)

Hence \(\sum_{k=1}^{\infty} |f_k(y_k)| = \infty\), which contradicts (3.26). Hence \((f_k) \in M_\infty(X',p)\). The proof is now complete.

**Theorem 3.8.** Let \(p = (p_k)\) be a bounded sequence of positive real numbers. Then \(c_0(X,p)^\beta = M_0(X',p)\).
**Proof.** Suppose \((f_k) \in M_0(X', p')\), then \(\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty\) for some \(M \in \mathbb{N}\). Let \(x = (x_k) \in c_0(X, p)\). Then there is a positive integer \(K_0\) such that \(\|x_k\|^{p_k} < 1/M\) for all \(k \geq K_0\), hence \(\|x_k\| < M^{-1/p_k}\) for all \(k \geq K_0\). Then we have

\[
\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} \|f_k\||x_k| \leq \sum_{k=K_0}^{\infty} \|f_k\| M^{-1/p_k} < \infty.
\]  

(3.30)

It follows that \(\sum_{k=1}^{\infty} f_k(x_k)\) converges, so that \((f_k) \in c_0(X, p)^\beta\).

On the other hand, assume that \((f_k) \in c_0(X, p)^\beta\), then \(\sum_{k=1}^{\infty} f_k(x_k)\) converges for all \(x = (x_k) \in c_0(X, p)\). For each \(x = (x_k) \in c_0(X, p)\), choose scalar sequence \((t_k)\) with \(|t_k| = 1\) such that \(f_k(t_kx_k) = |f_k(x_k)|\) for all \(k \in \mathbb{N}\). Since \((t_kx_k) \in c_0(X, p)\), by our assumption, we have \(\sum_{k=1}^{\infty} f_k(t_kx_k)\) converges, so that

\[
\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in c_0(X, p).
\]  

(3.31)

Now, suppose that \((f_k) \notin M_0(X', p)\). Then \(\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty\) for all \(m \in \mathbb{N}\). Choose \(m_1, k_1 \in \mathbb{N}\) such that

\[
\sum_{k=k_1}^{\infty} \|f_k\| m_1^{-1/p_k} > 1
\]  

(3.32)

and choose \(m_2 > m_1\) and \(k_2 > k_1\) such that

\[
\sum_{k_1 < k < k_2} \|f_k\| m_2^{-1/p_k} > 2.
\]  

(3.33)

Proceeding in this way, we can choose \(m_1 < m_2 < \cdots\), and \(0 = k_1 < k_2 < \cdots\) such that

\[
\sum_{k_{i-1} < k \leq k_i} \|f_k\| m_i^{-1/p_k} > 1.
\]  

(3.34)

Take \(x_k \in X\) with \(\|x_k\| = 1\) for all \(k, k_{i-1} < k \leq k_i\) such that

\[
\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i\quad \forall i \in \mathbb{N}.
\]  

(3.35)

Put \(y = (y_k)\), \(y_k = m_i^{-1/p_k} x_k\) for \(k_{i-1} < k \leq k_i\), then \(y \in c_0(X, p)\) and

\[
\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \forall i \in \mathbb{N}.
\]  

(3.36)

Hence we have \(\sum_{k=1}^{\infty} |f_k(y_k)| = \infty\), which contradicts (3.31), therefore \((f_k) \in M_0(X', p)\). This completes the proof.

**Theorem 3.9.** Let \(p = (p_k)\) be a bounded sequence of positive real numbers. Then \(c(X, p)^\beta = M_0(X', p) \cap c_0[X']\).

**Proof.** Since \(c(X, p) = c_0(X, p) + E\), where \(E = \{e(x) : x \in X\}\), it follows by Proposition 3.1(iii) and Theorem 3.8 that \(c(X, p)^\beta = M_0(X', p) \cap E^\beta\). It is obvious by definition that \(E^\beta = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x)\) converges for all \(x \in X\} = c_0[X']\). Hence we have the theorem.
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