RANDOM SUBGRAPHS OF CERTAIN GRAPH POWERS

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Received 4 March 2002

We determine the limiting probability that a random subgraph of the Cartesian power $K_a^n$ or of $K_{a,a}^n$ is connected.

2000 Mathematics Subject Classification: 05C80.

1. Introduction. A finite, simple, undirected graph $G$ has vertex set $V(G)$ and edge set $E(G)$. The order of $G$ is $|V(G)|$ and the size $e(G)$ of $G$ is $|E(G)|$. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ induced by $S$ and $G[S,V(G) - S]$ denote the spanning subgraph of $G$ with edges $xy$ where $x \in S$ and $y \in V(G) - S$. For $U \subseteq V(G)$, let $NG(U) = \{y \in V(G) : \exists x y \in E(G) \text{ with } x \in U\}$ and $\tilde{NG}(U) = NG(U) \cup U$. Of course, $NG(v) = NG(\{v\})$ and the degree $d_G(v)$ of $v$ in $G$ is $|NG(v)|$ for $v \in V(G)$. For $S \subseteq V(G)$, let $b_G(S) = |\{xy \in E(G) : x \in S, y \in V(G) - S\}|$ and $b_G(s) = \min\{b_G(S) : S \subseteq V(G), |S| = s\}$ ($0 \leq s \leq |V(G)|$).

The Cartesian product $G \Box H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ where vertices $(g_1,h_1)$ and $(g_2,h_2)$ are adjacent if and only if $g_1 = g_2$ and $h_1,h_2 \in E(H)$, or, $h_1 = h_2$ and $g_1,g_2 \in E(G)$. For a graph $G$, define $G^1 = G$ and $G^n = G^{n-1} \Box G$ for $n \geq 2$. We use the following recent isoperimetric result of Tillich [6]. Here $K_a$ denotes the complete graph of order $a$ and $K_{a,a}$ denotes the complete bipartite graph with parts of order $a$.

**Lemma 1.1** (see [6]). For $G = K_a^n$ with $a \geq 2$ and $n \geq 1$,

$$b_G(s) \geq (a - 1)s(n - \log_a s) \quad \text{for } 1 \leq s \leq a^n$$

(1.1)

and, for $G = K_{a,a}^n$ with $a \geq 1$ and $n \geq 1$,

$$b_G(s) \geq as(n - \log_{2a} s) \quad \text{for } 1 \leq s \leq (2a)^n.$$  

(1.2)

Let $G$ be a graph of order $n$ and size $N$. The probability space $\mathcal{G}(G,p)$ consists of all spanning subgraphs $H$ of $G$ where edges of $G$ are chosen for $H$ independently with probability $0 \leq p = p(n) \leq 1$, so that, $\Pr(H) = p^{e(H)}q^{N-e(H)}$ where $q = q(n) = 1 - p(n)$. (We denote the random graphs in $\mathcal{G}(G,p)$ generally by $G_p$.)

In this paper, we determine the limiting probability that $G_p$ is connected for $G = K_a^n$ and $K_{a,a}^n$. Specifically, we show that

$$\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K_a^n,p) \text{ is connected}) = e^{-\lambda}$$

(1.3)
for fixed \( a \geq 2 \) where \( p = p(n) = 1 - q(n) \) with \( q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)} \) and \( \lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty) \). In addition, we show that

\[
\lim_{n \to \infty} \Pr \left( G_p \in \mathcal{G}(K^n_{a,a}, p) \text{ is connected} \right) = e^{-\lambda} \tag{1.4}
\]

for fixed \( a \geq 1 \) where \( p = p(n) = 1 - q(n) \) with \( q(n) = [(\lambda(n))^{1/n}/2a]^{1/a} \) and \( \lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty) \). Our first result includes those of Burtin [3], Erdős and Spencer [5], and Bollobás [1] as a special case (\( a = 2 \)). Our approach is similar to [1].

The \( r \)th factorial moment of a random variable (r.v.) \( X \) is denoted by \( \mathbb{E}(X)_r \). Our approach is similar to [1].

For fixed \( a \geq 2 \), \( q = q(n) = [(\lambda(n))^{1/n}/a]^{1/(a-1)} \) where \( \lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty) \), and \( p = p(n) = 1 - q(n) \), we have

\[
\lim_{n \to \infty} \Pr \left( G_p \in \mathcal{G}(K^n_{a,a}, p) \text{ has no isolated vertices} \right) = e^{-\lambda}. \tag{2.2}
\]

**Proof.** Recall that \( G = K^n_{a,a} \). Let \( X_n(G_p) \) denote the number of isolated vertices in \( G_p \). Fix \( r \in \mathbb{P} \) and let \( \mathcal{A}_r \) denote the set of \( r \)-tuples of \( V \) with distinct coordinates; \( \mathbb{B}_r = \{(v_1, \ldots, v_r) \in \mathcal{A}_r : e(G([v_1, \ldots, v_r]) \neq 0) \} \) and \( \mathbb{E}_r = \mathbb{B}_r - \mathbb{B}_r = \{(v_1, \ldots, v_r) \in \mathcal{A}_r : e(G([v_1, \ldots, v_r]) = 0) \} \). Then \( |\mathbb{B}_r| \leq (a^n)^{r-1}r \), \( |\mathbb{E}_r| = (a^n)^{r-1}r \), and \( |\mathbb{E}_r| = a^{n(r-1)}r \). Observe that the number of edges in \( G \) incident with \( \{v_1, \ldots, v_r\} \) is at least \( (a - 1)r(n - r) \) for all \( (v_1, \ldots, v_r) \in \mathcal{A}_r \).
First,
\begin{align}
0 & \leq \sum_{(v_1, \ldots, v_r) \in \mathcal{E}_r} \Pr \left( d_{G^p}(v_1) = \cdots = d_{G^p}(v_r) = 0 \right) \leq |B_r| q^{(a-1)r(n-r)} \\
& \leq a^{n(r-1)}\text{ran} \frac{(\lambda(n))^{r-2/n}r}{a^{r(n-r)}} = \frac{(\lambda(n))^{r-2/n}r}{a^{r-2}}.
\end{align}

Next,
\begin{align}
\sum_{(v_1, \ldots, v_r) \in \mathcal{E}_r} \Pr \left( d_{G^p}(v_1) = \cdots = d_{G^p}(v_r) = 0 \right) & \geq \left[ a^{nr}e^{-r^2/a^n} - a^{r(n-1)}\text{ran} \frac{\lambda^r(n)}{a^{nr}} \right] \\
& = \lambda^r(n)e^{-r^2/a^n} - \frac{\lambda^r(n)\text{ran}}{a^n}.
\end{align}

while,
\begin{align}
\sum_{(v_1, \ldots, v_r) \in \mathcal{E}_r} \Pr \left( d_{G^p}(v_1) = \cdots = d_{G^p}(v_r) = 0 \right) & \leq a^{nr}q^{(a-1)nr} = \lambda^r(n).
\end{align}

Hence,
\begin{align}
\lambda^r(n)e^{-r^2/a^n} - \frac{\lambda^r(n)\text{ran}}{a^n} & \leq E_r(X_n) \leq \lambda^r(n) + \frac{(\lambda(n))^{r-2/n}r}{a^n}.
\end{align}

so that,
\begin{align}
\lim_{n \to \infty} E_r(X_n) = \lambda^r
\end{align}

and $X_n \overset{d}{\to} P_{\lambda}$ (see [4]).

**Lemma 2.3.** For fixed $a \geq 2$, $q = q(n) = [\ln n]^{1/n}/a^{1/(a-1)}$, and $p = p(n) = 1 - q(n)$, we have
\begin{align}
\Pr \left( G^p \in \mathcal{G}(K^{n,a}_a, p) \right) \text{ has a component of order } s \text{ with } 2 \leq s \leq a^n/2 = o(1) \text{ as } n \to \infty.
\end{align}

**Proof.** Recall that $G = K^{n,a}_a$. Let $\mathcal{A}_s = \{ S \subseteq V(G) : |S| = s \} (1 \leq s \leq a^n).$ We consider four cases.

**Case 1** ($2 \leq s \leq s_1 = [a^{n/2}/n]$). We have
\begin{align}
| \{ S \in \mathcal{A}_s : G[S] \text{ is connected} \} | & \leq a^n \cdot (a-1)n \cdot 2(a-1)n \cdots (s-1)(a-1)n \\
& \leq a^{n+s}n^{s-1}s^s,
\end{align}

so that (Lemma 1.1)
\begin{align}
\sum_{S \in \mathcal{A}_s} \Pr \left( G^p[S] \text{ is a component} \right) & \leq a^{n+s}n^{s-1}s^s q^{b_G(s)} \\
& = \frac{1}{n} \left[ \frac{\ln n}{a} \right]^{s(n-n^{1/s})}.
\end{align}
By examining the derivative $f(s) \ln(c e^2 s^2 / a^n)$ with respect to $s$ of $f(s) = e^s s^2 / a^n(s-1)$ with $c = an \ln n$, we see that $f(s)$ is decreasing for $s \in [2, a^{n/2} / e c^{1/2}]$. Here $f(s) \leq f(2) = 16a^2 n^2 \ln^2 n / a^n$. Hence,

$$\sum_{s=2}^{s_1} \sum_{S \in \mathcal{A}_s} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s=2}^{s_1} \frac{16a^2 n \ln^2 n}{a^n} = O(1) \quad \text{as } n \to \infty. \quad (2.11)$$

**Case 2** ($s_1 + 1 \leq s \leq s_3 = \lfloor a^n / 2 \rfloor$). Let $\mathcal{B}_s = \{ S \in \mathcal{A}_s : b_G(S) \geq (a-1)s(n-\log_a(s/n)) \}$ and $\mathcal{C}_s = \mathcal{A}_s - \mathcal{B}_s = \{ S \in \mathcal{A}_s : b_G(S) < (a-1)s(n-\log_a(s/n)) \}$.

First,

$$\sum_{S \in \mathcal{B}_s} \Pr(G_p[S] \text{ is a component}) \leq \left(\frac{a^n}{s}\right)^{(a-1)s(n-\log_a(s/n))} \leq \left(\frac{e a^n}{s}\right)^{(s(n-\log_a(s/n))}\sum_{n} \left[\frac{(\ln n)^{1/n}}{a}\right]^{s(n-\log_a(s/n))} = \left[\frac{e}{(\ln n)^{(1/(1/n) \log_a(s/n))}} \right]^n s \leq \left(\frac{e \ln n}{n}\right)^n. \quad (2.12)$$

Hence,

$$\sum_{s=s_1+1}^{s_3} \sum_{S \in \mathcal{B}_s} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s=s_1+1}^{s_3} \left(\frac{e \ln n}{n}\right)^n = o(1) \quad \text{as } n \to \infty. \quad (2.13)$$

Next, for $S \in \mathcal{C}_s$, let $H = G[S]$. Then

$$(a-1)s n = \sum_{v \in S} d_G(v) = 2e(H) + b_G(S) < 2e(H) + (a-1)s\left(n - \frac{s}{\log_a(s/n)}\right), \quad (2.14)$$

so that

$$2e(H) \geq (a-1)s \log_a \frac{s}{n} \quad (2.15)$$

and the average degree $d$ in $H$ satisfies

$$d > (a-1) \log_a \frac{s}{n} \quad (2.16)$$

**Case 3** ($s_1 + 1 \leq s \leq s_2 = \lfloor a^n / \ln^2 n \rfloor$). Let $u = \lfloor s/n \rfloor$, so that $(a-1)n + 1 < u < s - (a-1)n - 1$, and by Lemma 2.1, for sufficiently large $n$, there exists $U \subseteq S$, $|U| = u$, and

$$|\tilde{N}_H(U)| \geq \frac{s}{n} \log_a \frac{s}{n} \left[1 - \exp \left(-\frac{u[(a-1)n+1]}{s}\right)\right] \geq \frac{\delta s}{n} \log_a \frac{s}{n} \quad \text{with } \delta = 1 - e^{-1} = 0.631. \quad (2.17)$$

Let $t = \lfloor (\delta s/n) \log_a(s/n) \rfloor$, so that $u < t < s$, and let $w = s - t = s(1 - x) - \tau$ with $x = (\delta/n) \log_a(s/n)$ and $0 \leq \tau < 1$. Observe that $\delta/4 < x < \delta$ here. For sufficiently
large \( n \), take the smallest such \( u \)-set \( U = \{d_1, \ldots, d_u\} \) in \( S \subseteq V(G) \) which is totally ordered; take the (uniquely determined) first \( t - u \) vertices of \( (N_G(d_1) \cap (S - U)) \cup \cdots \cup (N_G(d_u) \cap (S - U)) \subseteq V(G) \); and add the remaining \( w \) vertices \( W \) of \( S \). Then
\[
S \mapsto (\{d_1, \ldots, d_u\}; N_G(d_1) \cap (S - U), \ldots, N_G(d_u) \cap (S - U); W)
\] (2.18)
is an injection. Hence,
\[
|\{s\}^s \leq \left(\frac{a^n}{u}\right)^{2(a-1)nu} \left(\frac{a^n}{w}\right)^w \leq \left(\frac{e a^n}{u}\right)^{u} 2^{(a-1)nu} \left(\frac{e a^n}{w}\right)^w \leq \left(\frac{e a^n}{s}\right)^{s/n} 2^{(a-1)s} \left(\frac{e a^n}{s(1-x)}\right)^{s(1-x)}.
\]
(2.19)

Then (where \( x - 1/n > 0 \), Lemma 1.1)
\[
\sum_{S \in \xi_s} \Pr(G_p[S] \text{ is a component}) \leq |\{s\}^s q^{b_G(s)} \leq \left(\frac{e a^n}{s}\right)^{s/n} 2^{(a-1)s} \left(\frac{e a^n}{s(1-x)}\right)^{s(1-x)} \left[\frac{(\ln n)^{1/n}}{a} s^{n-\log a s}\right]^s
\]
(2.20)

Here
\[
2x + \frac{1}{n} \log a s - 1 - \frac{2}{n} \geq \delta - \frac{1}{2} - \frac{4}{n} \log a n - \frac{2}{n} \geq \frac{1}{10},
\]
(2.21)
so that
\[
\sum_{S \in \xi_s} \Pr(G_p[S] \text{ is a component}) \leq \left[\left(\frac{e a^n}{s}\right)^{1/n} 2^{a-1} \left(\frac{e}{1-x}\right)^{1-x} (\ln n)^{-0.1}\right]^s.
\]
(2.22)

Hence,
\[
\sum_{s=s_1+1}^{s_2} \sum_{S \in \xi_s} \Pr(G_p[S] \text{ is a component}) \leq \left[\left(\frac{e a^n}{s}\right)^{1/n} 2^{a-1} \left(\frac{e}{1-x}\right)^{1-x} (\ln n)^{-0.1}\right]^s
\]
(2.23)

\[
= o(1) \quad \text{as } n \to \infty.
\]
\textbf{Case 4} \((s_2 + 1 \leq s \leq s_3)\). For \(S \in \ell_k\) and \(H = G[S]\), let \(T = \{v \in S : d_H(v) \geq (a - 1)n - \log_a^2 n\}\), \(t = |T|\) and \(H_1 = H[T] = G[T]\). Then

\[
2e(H_1) = 2e(H) - 2e(H[S - T, T]) - 2e(H[S - T]) \\
> (a - 1)s \log_a \frac{s}{n} - 2(a - 1)n(s - t) \\
= (a - 1)s \left[ \log_a \frac{s}{n} - \frac{2n}{s}(s - t) \right].
\]

Here

\[
\log_a \frac{s}{n} \geq n - 2\log_a n,
\]

so that

\[
s(a - 1)n - (s - t)\log_a^2 n \geq \sum_{v \in T} d_H(v) + \sum_{v \in S - T} d_H(v) > (a - 1)s \log_a \frac{s}{n} \\
\geq (a - 1)s(n - 2\log_a n),
\]

hence,

\[
t^* \geq s \left( 1 - \frac{2(a - 1)}{\log_a n} \right).
\]

We take the first \(t\) vertices of \(T\) for \(H_1\) where \(t = s(1 - \epsilon)\) with \(s\epsilon = \lfloor 2(a - 1)s/\log_a n \rfloor\)

\[
2e(H_1) \geq (a - 1)s [(1 - 2\epsilon)n - 2\log_a n]
\]

and the average degree \(d_1\) in \(H_1\) satisfies

\[
d_1 > (a - 1) \left[ n - \frac{\epsilon}{1 - \epsilon} n - \frac{2}{1 - \epsilon} \log_a n \right] \geq (a - 1)(1 - 3\epsilon)n.
\]

Let \(u = \lfloor a^n / \ln^6 n \rfloor\), so that \((a - 1)n + 1 < u < t - (a - 1)n - 1\), and by Lemma 2.1, for all sufficiently large \(n\), there exists \(U \subseteq T\), \(|U| = u\), and

\[
|\tilde{N}_H(U)| \geq |\tilde{N}_{H_1}(U)| \geq s(1 - \epsilon)(1 - 3\epsilon) \left\{ 1 - \exp \left( - \frac{u[(a - 1)n + 1]}{t} \right) \right\} \\
\geq s(1 - \epsilon)^2(1 - 3\epsilon) \geq s(1 - 4\epsilon).
\]

Let \(t = s - [4\epsilon s]\), so that \(u^* \geq s^* < s\), and \(w = [4\epsilon s]\). For sufficiently large \(n\), take the smallest such \(u\)-set \(U = \{d_1, \ldots, d_u\}\) in \(S = \subseteq V(G)\); take the (uniquely determined) first \(t - u\) vertices of \((N_G(d_1) \cap (S - U)) \cup \cdots \cup (N_G(d_u) \cap (S - U)) \subseteq V(G)\); and add the remaining \(w\) vertices \(W\) of \(S\). Then

\[
S \mapsto \{d_1, \ldots, d_u; N_G(d_1) - S, \ldots, N_G(d_u) - S; W\}
\]
is an injection with $|N_G(d_i) - S| \leq \lfloor \log_\alpha n \rfloor$ ($1 \leq i \leq u$). Hence, with $y = \lfloor \log_\alpha n \rfloor$,

$$
|\mathcal{E}_s| \leq \binom{a^n}{u} \sum_{(k_1, \ldots, k_u) \in \{0, \ldots, y\}^u} \prod_{i=1}^u \binom{(a-1)n}{k_i} \binom{a^n}{w}.
$$

(2.32)

since

$$
\binom{(a-1)n}{k} \leq \binom{(a-1)n}{y+1}, \quad \forall k \in \{0, \ldots, y\}.
$$

(2.33)

Then

$$
|\mathcal{E}_s| \leq \binom{e\alpha^n}{u} (y+1)^u \binom{e\alpha^n}{y+1}^{u(y+1)} \binom{e\alpha^n}{w}^w.
$$

(2.34)

Hence, (Lemma 1.1)

$$
\sum_{S \in \mathcal{E}_s} \Pr(G_p[S] \text{ is a component}) \leq |\mathcal{E}_s| q^{|G(s)}| = (e^2 an \ln^6 n)^u \binom{e\alpha^n}{
\frac{\ln^2 n}{4\epsilon}} (\frac{\ln n}{\alpha n})^{4\epsilon s} (\frac{\ln n}{\alpha n})^{s(n - \log_\alpha s)} = \left[ (e^2 an \ln^6 n)^{u/s} (e\alpha^n a)^{y/s} \left( \frac{\alpha n}{\ln n} \right)^{4\epsilon} (\frac{s}{a^n})^{1-4\epsilon} \right]^{s(n-(1/n)\log_\alpha s - 2uy/s)}.
$$

(2.35)

Here

$$
1 \leq ean\ln^2 a \leq e^2 an \ln^6 n, \quad 0 < \frac{u}{s} \leq \frac{uy}{s} \leq \frac{5}{\ln^2 n},
$$

(2.36)

$$
1 - \frac{1}{n} \log_\alpha s - \frac{2uy}{s} \leq \frac{2}{n} \log_\alpha \ln n - \frac{4}{\ln^2 n} \leq 0,
$$

so that

$$
\sum_{S \in \mathcal{E}_s} \Pr(G_p[S] \text{ is a component}) \leq \left[ (e^3 a^2 n^2 \ln^2 a \ln^6 n)^{5/\ln^2 n} \left( \frac{\alpha n}{\ln n} \right)^{4\epsilon} \right]^{s(n-(1/n)\log_\alpha s - 2uy/s)}.
$$

(2.37)

since $(e^3 a^2 n^2 \ln^2 a \ln^6 n)^{5/\ln^2 n} \rightarrow 1$, $(e/4\epsilon)^{4\epsilon} - 1$ and $\epsilon \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$
\sum_{s=s_2+1}^{s_3} \Pr(G_p[S] \text{ is a component}) \leq \sum_{s=s_2+1}^{s_3} \left( \frac{2}{3} \right)^s = o(1) \quad \text{as } n \rightarrow \infty.
$$

(2.38)
**Remark 2.4.** For all \( a \geq 2 \) and \( n \geq 2 \), \( b_G(s) \geq 2 \) when \( 2 \leq s \leq a^n/2 \). Hence, \( 0 < \tilde{q}(n) \leq q(n) \) implies \( \tilde{q}(n)^{b_G(s)} \leq q(n)^{b_G(s)} \) when \( 2 \leq s \leq a^n/2 \). Then (2.10), (2.12), (2.20), and (2.35) hold for \( G_{\tilde{p}(n)} \) where \( \tilde{p}(n) = 1 - \tilde{q}(n) \) (the exponent in (2.12) is larger than \( b_G(s) \)). Hence, Lemma 2.3 holds for \( G_{\tilde{p}(n)} \) as well. The inequalities in the proof of Lemma 2.3 hold for all sufficiently large \( n \) which can be determined from nineteen appropriate inequalities there.

**Theorem 2.5.** For fixed \( a \geq 2 \), \( q = q(n) = [\lambda(n)^{1/n} / 2a]^{1/(a-1)} \) where \( \lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty) \), and \( p = p(n) = 1 - q(n) \), we have

\[
\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K^n_{a,a}) \text{ is connected}) = e^{-\lambda}. \tag{2.39}
\]

**Proof.** We have

\[
0 \leq \Pr(G_p \text{ is disconnected}) - \Pr(G_p \text{ has isolated vertices}) \leq \Pr(G_p \text{ has a component of order } s \text{ with } 2 \leq s \leq a^{n/2}) = o(1) \quad \text{as } n \to \infty, \tag{2.40}
\]

by Remark 2.4. Hence, Lemma 2.2 gives

\[
\lim_{n \to \infty} \Pr(G_p \text{ is disconnected}) = \lim_{n \to \infty} \Pr(G_p \text{ has isolated vertices}) = 1 - e^{-\lambda}. \tag{2.41}
\]

We state the result for \( G = K^n_{a,a} \) since its proof is similar to the proof of Theorem 2.5.

**Theorem 2.6.** For fixed \( a \geq 1 \), \( q = q(n) = [\lambda(n)^{1/n} / 2a]^{1/a} \) where \( \lim_{n \to \infty} \lambda(n) = \lambda \in (0, \infty) \), and \( p = p(n) = 1 - q(n) \), we have

\[
\lim_{n \to \infty} \Pr(G_p \in \mathcal{G}(K^n_{a,a}) \text{ is connected}) = e^{-\lambda}. \tag{2.42}
\]

**References**


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