CERTAIN INTEGRAL OPERATOR AND STRONGLY STARLIKE FUNCTIONS

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Let $S^*(\rho, \gamma)$ denote the class of strongly starlike functions of order $\rho$ and type $\gamma$ and let $C(\rho, \gamma)$ be the class of strongly convex functions of order $\rho$ and type $\gamma$. By making use of an integral operator defined by Jung et al. (1993), we introduce two novel families of strongly starlike functions $S_{\alpha \beta}(\rho, \gamma)$ and $C_{\alpha \beta}(\rho, \gamma)$. Some properties of these classes are discussed.

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1. Introduction. Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

which are analytic in the unit disc $E = \{z : |z| < 1\}$. A function $f(z)$ belonging to $A$ is said to be starlike of order $\gamma$ if it satisfies

$$\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > \gamma \quad (z \in E)$$

(1.2)

for some $\gamma$ ($0 \leq \gamma < 1$). We denote by $S^*(\gamma)$ the subclass of $A$ consisting of functions which are starlike of order $\gamma$ in $E$. Also, a function $f(z)$ in $A$ is said to be convex of order $\gamma$ if it satisfies $zf''(z) \in S^*(\gamma)$, or

$$\Re \left\{ 1 + \frac{zf'''(z)}{f''(z)} \right\} > \gamma \quad (z \in E)$$

(1.3)

for some $\gamma$ ($0 \leq \gamma < 1$). We denote by $C(\gamma)$ the subclass of $A$ consisting of all functions which are convex of order $\gamma$ in $E$.

If $f(z) \in A$ satisfies

$$\left| \arg \left( \frac{zf'(z)}{f(z)} - \gamma \right) \right| < \frac{\pi}{2} \rho \quad (z \in E)$$

(1.4)

for some $\gamma$ ($0 \leq \gamma < 1$) and $\rho$ ($0 < \rho \leq 1$), then $f(z)$ is said to be strongly starlike of order $\rho$ and type $\gamma$ in $E$, and denoted by $f(z) \in S^*(\rho, \gamma)$. If $f(z) \in A$ satisfies

$$\left| \arg \left( 1 + \frac{zf'''(z)}{f''(z)} - \gamma \right) \right| < \frac{\pi}{2} \rho \quad (z \in E)$$

(1.5)
for some \( \gamma (0 \leq \gamma < 1) \) and \( \rho (0 < \rho \leq 1) \), then we say that \( f(z) \) is strongly convex of order \( \rho \) and type \( \gamma \) in \( E \), and we denote by \( C(\rho, \gamma) \) the class of such functions. It is clear that \( f(z) \in A \) belongs to \( C(\rho, \gamma) \) if and only if \( zf'(z) \in S^*(\rho, \gamma) \). Also, we note that \( S^*(1, y) = S^*(y) \) and \( C(1, y) = C(y) \).

For \( c > -1 \) and \( f(z) \in A \), we recall the generalized Bernardi-Libera-Livingston integral operator \( L_c(f) \) as

\[
L_c(f) = \frac{c + 1}{z^c} \int_0^z t^{c-1} f(t) dt.
\]  

The operator \( L_c(f) \) when \( c \in \mathbb{N} = \{1, 2, 3, \ldots \} \) was studied by Bernardi [1]. For \( c = 1 \), \( L_1(f) \) was investigated by Libera [4].

Recently, Jung et al. [2] introduced the following one-parameter family of integral operators:

\[
Q_{\alpha}^{\beta} f(z) = \left( \frac{\alpha + \beta}{\beta} \right) \frac{\alpha}{z^\beta} \int_0^z \left( 1 - \frac{t}{z} \right)^{\alpha-1} t^{\beta-1} f(t) dt \quad (\alpha > 0, \beta > -1, f \in A).
\]  

They showed that

\[
Q_{\alpha}^{\beta} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)\Gamma(\alpha+\beta+1)}{\Gamma(\beta+\alpha+n)\Gamma(\beta+1)} a_n z^n,
\]  

where \( \Gamma(x) \) is the familiar Gamma function. Some properties of this operator have been studied (see [2, 3]). From (1.7) and (1.8), one can see that

\[
z(Q_{\alpha}^{\beta} f(z))' = (\alpha + \beta + 1)Q_{\alpha}^{\beta} f(z) - (\alpha + \beta)Q_{\alpha}^{\beta+1} f(z).
\]  

It should be remarked in passing that the operator \( Q_{\alpha}^{\beta} \) is related rather closely to the Beta or Euler transformation.

Using the operator \( Q_{\alpha}^{\beta} \), we now introduce the following classes:

\[
S_{\alpha}^{\beta}(\rho, \gamma) = \left\{ f(z) \in A : Q_{\alpha}^{\beta} f(z) \in S^*(\rho, \gamma), \frac{z(Q_{\alpha}^{\beta} f(z))'}{Q_{\alpha}^{\beta} f(z)} \neq \gamma \forall z \in E \right\},
\]

\[
C_{\alpha}^{\beta}(\rho, \gamma) = \left\{ f(z) \in A : Q_{\alpha}^{\beta} f(z) \in C(\rho, \gamma), \frac{(z(Q_{\alpha}^{\beta} f(z))')'}{(Q_{\alpha}^{\beta} f(z))'} \neq \gamma \forall z \in E \right\}.
\]  

It is obvious that \( f(z) \in C_{\alpha}^{\beta}(\rho, \gamma) \) if and only if \( z f'(z) \in S_{\alpha}^{\beta}(\rho, \gamma) \).

In this note, we investigate some properties of the classes \( S_{\alpha}^{\beta}(\rho, \gamma) \) and \( C_{\alpha}^{\beta}(\rho, \gamma) \). The basic tool for our investigation is the following lemma which is due to Nunokawa [5].

**Lemma 1.1.** Let a function \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) be analytic in \( E \) and \( p(z) \neq 0 \) \( (z \in E) \). If there exists a point \( z_0 \in E \) such that

\[
|\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|), \quad |\arg p(z_0)| = \frac{\pi}{2} \rho \quad (0 < \rho \leq 1),
\]  

\[
\text{Lemma} 1.1. \quad \text{Let a function } p(z) = 1 + c_1 z + c_2 z^2 + \cdots \text{ be analytic in } E \text{ and } p(z) \neq 0 \quad (z \in E). \text{ If there exists a point } z_0 \in E \text{ such that}
\]
then
\[ \frac{z_0 p'(z_0)}{p(z_0)} = ik\rho, \quad (1.12) \]
where
\[ k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \left( \text{when } \arg p(z_0) = \frac{\pi}{2} \rho \right), \quad (1.13) \]
\[ k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \left( \text{when } \arg p(z_0) = -\frac{\pi}{2} \rho \right), \]
and \( p(z_0)^{1/\rho} = \pm ia (a > 0). \)

2. Main results. Our first inclusion theorem is stated as follows.

**Theorem 2.1.** The class \( S_\beta^{\alpha}(\rho, \gamma) \subset S_\beta^{\alpha+1}(\rho, \gamma) \) for \( \alpha > 0, \beta > -1, 0 \leq \gamma < 1 \) and \( \alpha + \beta \geq -\gamma \).

**Proof.** Let \( f(z) \in S_\beta^{\alpha}(\rho, \gamma) \). Then we set
\[ \frac{z(Q_\beta^{\alpha+1}f(z))'}{Q_\beta^{\alpha+1}f(z)} = (1-\gamma)p(z) + y, \quad (2.1) \]
where \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is analytic in \( E \) and \( p(z) \neq 0 \) for all \( z \in E \). Using (1.9) and (2.1), we have
\[ (\alpha + \beta + 1) \frac{Q_\beta^\alpha f(z)}{Q_\beta^{\alpha+1}f(z)} = (\alpha + \beta + \gamma) + (1-\gamma)p(z). \quad (2.2) \]
Differentiating both sides of (2.2) logarithmically, it follows from (2.1) that
\[ \frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)} - y = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(\alpha + \beta + \gamma) + (1-\gamma)p(z)}. \quad (2.3) \]

Suppose that there exists a point \( z_0 \in E \) such that
\[ |\arg p(z)| < \frac{\pi}{2} \rho \quad (|z| < |z_0|), \quad |\arg p(z_0)| = \frac{\pi}{2} \rho. \quad (2.4) \]
Then, by applying Lemma 1.1, we can write that \( z_0 p'(z_0)/p(z_0) = ik\rho \) and that \( (p(z_0))^{1/\rho} = \pm ia (a > 0). \)
Therefore, if \( \arg p(z_0) = -(\pi/2)\rho \), then
\[ \frac{z_0(Q_\beta^\alpha f(z_0))'}{Q_\beta^\alpha f(z_0)} - y = (1-\gamma)p(z_0) \left[ 1 + \frac{z_0 p'(z_0)/p(z_0)}{(\alpha + \beta + \gamma) + (1-\gamma)p(z)} \right] \]
\[ = (1-\gamma)a^\rho e^{-i\pi\rho/2} \left[ 1 + \frac{ik\rho}{(\alpha + \beta + \gamma) + (1-\gamma)a^\rho e^{-i\pi\rho/2}} \right]. \quad (2.5) \]
From (2.5) we have
\[
\arg \left\{ \frac{z_0 (Q_{\alpha}^\rho f(z_0))'}{Q_{\alpha}^\rho f(z_0)} - \gamma \right\} = -\frac{\pi}{2} \rho + \arg \left\{ 1 + \frac{ik\rho}{(\alpha + \beta + \gamma) + (1 - \gamma)a^\rho e^{-i\pi/2}} \right\} \\
= -\frac{\pi}{2} \rho + \tan^{-1} \left\{ \frac{(\alpha + \beta + \gamma) + (1 - \gamma)a^\rho \cos \frac{\pi\rho}{2}}{k}\left[ (\alpha + \beta + \gamma)^2 + 2(\alpha + \beta + \gamma)(1 - \gamma)a^\rho \cos \frac{\pi\rho}{2} \right. \right. \\
+ (1 - \gamma)^2a^{2\rho} - k(1 - \gamma)a^\rho \sin \frac{\pi\rho}{2} \left. \right] \right\} \\
\leq -\frac{\pi}{2} \rho,
\]
where \(k \leq -(1/2)(a + 1/a) \leq -1, \alpha + \beta \geq -\gamma\), which contradicts the condition \(f(z) \in S_{\rho}^\alpha(\rho, \gamma)\).

Similarly, if \(\arg p(z_0) = (\pi/2)\rho\), then we have
\[
\arg \left\{ \frac{z_0 (Q_{\alpha}^\rho f(z_0))'}{Q_{\alpha}^\rho f(z_0)} - \gamma \right\} \geq \frac{\pi}{2} \rho,
\]
which also contradicts the hypothesis that \(f(z) \in S_{\rho}^\alpha(\rho, \gamma)\).

Thus the function \(p(z)\) has to satisfy \(|\arg p(z)| < (\pi/2)\rho\ \ (z \in E)\), which leads us to the following:
\[
\left| \arg \left\{ \frac{z (Q_{\alpha+1}^\rho f(z))'}{Q_{\alpha+1}^\rho f(z)} - \gamma \right\} \right| < \frac{\pi}{2} \rho \ \ (z \in E).
\]

This evidently completes the proof of Theorem 2.1.

We next state the following theorem.

**Theorem 2.2.** The class \(C_{\alpha}^\beta(\rho, \gamma) \subset C_{\alpha+1}^\beta(\rho, \gamma)\) for \(\alpha > 0, \beta > -1, 0 \leq \gamma < 1, \) and \(\alpha + \beta \geq -\gamma\).

**Proof.** By definition (1.10), we have
\[
f(z) \in C_{\rho}^\alpha(\rho, \gamma) \iff Q_{\rho}^\alpha f(z) \in C(\rho, \gamma) \iff z (Q_{\rho}^\alpha f(z))' \in S^\ast(\rho, \gamma) \\
\iff z (Q_{\rho}^\alpha f(z))' \in S^\ast(\rho, \gamma) \iff z f'(z) \in S_{\rho}^\alpha(\rho, \gamma) \\
\Rightarrow z f'(z) \in S_{\rho}^{\alpha+1}(\rho, \gamma) \iff Q_{\rho}^{\alpha+1} f(z) \in S^\ast(\rho, \gamma) \\
\iff z (Q_{\rho}^{\alpha+1} f(z))' \in S^\ast(\rho, \gamma) \iff Q_{\rho}^{\alpha+1} f(z) \in C(\rho, \gamma) \\
\iff f(z) \in C_{\rho}^{\alpha+1}(\rho, \gamma).
\]

The following theorem involves the generalized Bernardi-Libera-Livingston integral operator \(L_c(f)\) given by (1.6).
Theorem 2.3. Let $c > -\gamma$ and $0 \leq \gamma < 1$. If $f(z) \in A$ and $z(Q^\alpha_bl_c f(z))'/Q^\alpha_bl_c f(z) \neq \gamma$ for all $z \in E$, then $f(z) \in S^\alpha_{\beta}(\rho,\gamma)$ implies that $L_c(f) \in S^\alpha_{\beta}(\rho,\gamma)$.

Proof. Let $f(z) \in S^\alpha_{\beta}(\rho,\gamma)$. Put

$$\frac{z(Q^\alpha_bl_c f(z))'}{Q^\alpha_bl_c f(z)} = y + (1 - y)p(z), \quad (2.10)$$

where $p(z)$ is analytic in $E$, $p(0) = 1$ and $p(z) \neq 0$ ($z \in E$). From (1.6) we have

$$z(Q^\alpha_bl_c f(z))' = (c + 1)Q^\alpha_bl_c f(z) - cQ^\alpha_bl_c f(z). \quad (2.11)$$

Using (2.10) and (2.11), we get

$$(c + 1)\frac{Q^\alpha_bl_c f(z)}{Q^\alpha_bl_c f(z)} = (c + y) + (1 - y)p(z). \quad (2.12)$$

Differentiating both sides of (2.12) logarithmically, we obtain

$$\frac{z(Q^\alpha_bl_c f(z))'}{Q^\alpha_bl_c f(z)} = (1 - y)p(z) + \frac{(1 - y)p'(z)}{(c + y) + (1 - y)p(z)}.$$ \quad (2.13)

Suppose that there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z| < |z_0|), \quad |\arg p(z_0)| = \frac{\pi}{2}\rho. \quad (2.14)$$

Then, applying Lemma 1.1, we can write that $z_0p'(z_0)/p(z_0) = ik\rho$ and $(p(z_0))^{1/p} = \pm ia$ ($a > 0$).

If $\arg p(z_0) = (\pi/2)\rho$, then

$$\frac{z_0(Q^\alpha_bl_c f(z_0))'}{Q^\alpha_bl_c f(z_0)} - y = (1 - y)p(z_0) \left[ 1 + \frac{z_0p'(z_0)/p(z_0)}{(c + y) + (1 - y)p(z_0)} \right] = (1 - y)a^pe^{i\pi\rho/2} \left[ 1 + \frac{ik\rho}{(c + y) + (1 - y)a^pe^{i\pi\rho/2}} \right]. \quad (2.15)$$

This shows that

$$\arg \left\{ \frac{z_0(Q^\alpha_bl_c f(z_0))'}{Q^\alpha_bl_c f(z_0)} - y \right\}$$

$$= \frac{\pi}{2}\rho + \arg \left\{ 1 + \frac{ik\rho}{(c + y) + (1 - y)a^pe^{i\pi\rho/2}} \right\}$$

$$= \frac{\pi}{2}\rho + \tan^{-1} \left\{ \left( k\rho \left[ (c + y) + (1 - y)a^p \cos \frac{\pi\rho}{2} \right] \right) \right.$$ 

$$\times \left( (c + y)^2 + 2(c + y)(1 - y)a^p \cos \frac{\pi\rho}{2} \right)$$

$$+ (1 - y)^2a^{2\rho} + k\rho(1 - y)a^p \sin \frac{\pi\rho}{2} \} \right\}^{-1} \right\} \quad (2.16)$$

$$\approx \frac{\pi}{2}\rho,$$  

where $k \geq (1/2)(a + 1/a) \geq 1$, which contradicts the condition $f(z) \in S^\alpha_{\beta}(\rho,\gamma)$. 


Similarly, we can prove the case \( \arg p(z_0) = -\left(\frac{\pi}{2}\right)\rho \). Thus we conclude that the function \( p(z) \) has to satisfy \( |\arg p(z)| < \left(\frac{\pi}{2}\right)\rho \) for all \( z \in E \). This shows that

\[
\left| \arg \left\{ \frac{z(Q_{\alpha}^\delta L_c f(z))'}{Q_{\beta}^\delta L_c f(z)} - \gamma \right\} \right| < \frac{\pi}{2} \rho \quad (z \in E).
\]

(2.17)

The proof is complete.

**Theorem 2.4.** Let \( c > -\gamma \) and \( 0 \leq \gamma < 1 \). If \( f(z) \in A \) and \( \frac{(z(Q_{\alpha}^\delta L_c f(z)))'}{Q_{\beta}^\delta L_c f(z)} \neq \gamma \) for all \( z \in E \), then \( f(z) \in C_\alpha^\delta \rho, \gamma \) implies that \( L_c(f) \in C_\alpha^\delta \rho, \gamma \).

**Proof.** Using the same method as in Theorem 2.2 we have

\[
f(z) \in C_\alpha^\delta \rho, \gamma \iff zf'(z) \in S_\alpha^\delta \rho, \gamma \iff L_c(zf'(z)) \in S_\alpha^\delta \rho, \gamma
\]

\[
\iff z(L_c f(z))' \in S_\alpha^\delta \rho, \gamma \iff L_c f(z) \in C_\alpha^\delta \rho, \gamma.
\]

(2.18)

**References**


