GENERALIZED TRANSVERSELY PROJECTIVE STRUCTURE ON A TRANSVERSELY HOLOMORPHIC FOLIATION

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The results of Biswas (2000) are extended to the situation of transversely projective foliations. In particular, it is shown that a transversely holomorphic foliation defined using everywhere locally nondegenerate maps to a projective space $\mathbb{CP}^n$, and whose transition functions are given by automorphisms of the projective space, has a canonical transversely projective structure. Such a foliation is also associated with a transversely holomorphic section of $N^{\otimes -k}$ for each $k \in [3, n+1]$, where N is the normal bundle to the foliation. These transversely holomorphic sections are also flat with respect to the Bott partial connection.

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1. Introduction. A projective structure on a Riemann surface X is defined by giving a covering of X by holomorphic coordinate charts such that all the transition functions are restrictions of Möbius transformations. It is well known that the notion of a projective structure can be extended to the situation of foliations (cf. [10]). To define this generalization, let $\mathcal{F}$ be a foliation of codimension two on a real manifold M . Let $\{U_i\}_{i \in I}$ be an open covering of M , and let $\phi_i : U_i \rightarrow \mathbb{C}$ be submersions onto the image such that the fibers of ϕ_i are leaves for $\mathcal{F}$. A transversely projective structure on $\mathcal{F}$ is defined by imposing the condition that, for every $i, j \in I$, there is a commutative diagram

$$\begin{array}{ccc}
 U_i \cap U_j & \xlongequal{\quad} & U_i \cap U_j \\
 \downarrow \phi_i & & \downarrow \phi_j \\
 \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j)
 \end{array} \tag{1.1}$$

such that $f_{i,j}$ is a restriction of some Möbius transformation [10].

A holomorphic immersion $\gamma : X \rightarrow \mathbb{CP}^n$ of a Riemann surface X is called everywhere locally nondegenerate if for every $x \in X$, the order of contact of the image $\gamma(U)$ at $\gamma(x)$, where U is a neighborhood of x in X , with any hyperplane in $\mathbb{CP}^n$ passing through $\gamma(x)$ is at most $n-1$ (see [3, 9]). Two such immersions are called equivalent if they differ by an automorphism of $\mathbb{CP}^n$. A $\mathbb{CP}^n$ -structure on X is an equivalence class of an everywhere locally nondegenerate equivariant map of the universal cover of X into $\mathbb{CP}^n$. A $\mathbb{CP}^1$ -structure on X is the same as a projective structure on X .

If $f : X \rightarrow \mathbb{CP}^n$ is a holomorphic map such that the image of f is not contained in any hyperplane of $\mathbb{CP}^n$, then there is a finite subset $S \subset X$ such that the restriction of

f to the complement $X \setminus S$ defines a $\mathbb{CP}^n$ -structure on $X \setminus S$. Any Riemann surface has many $\mathbb{CP}^n$ -structures. In [3], it has been shown that the space of $\mathbb{CP}^n$ -structures on X , where $n \geq 2$, is canonically identified with the Cartesian product of the space of all projective structures on X with the direct sum $\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})$.

The notion of a $\mathbb{CP}^n$ -structure can be extended to the situation of foliations which will be called a *transversely $\mathbb{CP}^n$ -structure*; see Definition 2.3 for the definition of a transversely $\mathbb{CP}^n$ -structure.

Let $\mathcal{F}$ be a transversely holomorphic foliation of complex codimension one. So the normal bundle N is a transversely holomorphic line bundle. The normal bundle N is equipped with the Bott partial connection obtained from the Lie bracket operation of vector fields. The transversely holomorphic structure of N is compatible with the Bott partial connection.

We prove that, giving a transversely $\mathbb{CP}^n$ -structure on $\mathcal{F}$ is equivalent to giving a transversely projective structure on $\mathcal{F}$ together with a transversely holomorphic section ω_k of $N^{\otimes -k}$, for each $k \in [3, n+1]$, such that ω_k is flat with respect to the Bott partial connection (see Theorem 2.4). In particular, setting all ω_k to be zero we conclude that, for any transversely $\mathbb{CP}^n$ -structure on $\mathcal{F}$ there is a canonically associated transversely projective structure on $\mathcal{F}$. When the foliation is trivial, that is, $\mathcal{F} = 0$, then Theorem 2.4 is the main result of [3] (see [3, Theorem 5.5]).

It is not easy to directly construct a transversely $\mathbb{CP}^n$ -structure on a holomorphic foliation. In fact, when the foliation is trivial, namely we have a Riemann surface X , it is not easy to construct a map of the universal cover of X to $\mathbb{CP}^n$, which is everywhere locally nondegenerate. However, using Theorem 2.4 we can indirectly construct many examples of transversely $\mathbb{CP}^n$ -structures, just as using [3, Theorem 5.5], we can indirectly construct examples of everywhere locally nondegenerate maps of the universal cover of a Riemann surface to $\mathbb{CP}^n$.

2. Transversely projective foliations defined by maps to a projective space. Let M be a connected smooth real manifold of dimension $d+2$. Let $\mathcal{F}$ be a C^∞ -subbundle of rank d of the tangent bundle TM .

DEFINITION 2.1. A transversely holomorphic structure on $\mathcal{F}$ is defined by giving the following data (see [5]):

- (1) a covering of M by open subsets U_i , where i runs over an index set I . So we have $\bigcup_{i \in I} U_i = M$;
- (2) for each $i \in I$, a submersion ϕ_i of U_i to an open subset D_i of $\mathbb{C}$. The restriction $\mathcal{F}|_{U_i}$ is the kernel of the differential map $d\phi_i: TU_i \rightarrow \phi_i^* TD_i$;
- (3) for every pair $i, j \in I$, there is a commutative diagram of maps

$$\begin{array}{ccc}
 U_i \cap U_j & \xrightarrow{Id} & U_i \cap U_j \\
 \downarrow \phi_i & & \downarrow \phi_j \\
 \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j),
 \end{array} \tag{2.1}$$

where $f_{i,j}$ is a holomorphic map.

Two such data $\{U_i, \phi_i\}_{i \in I}$ and $\{U_i, \phi_i\}_{i \in J}$ are called *equivalent* if their union, namely

$$\{U_i, \phi_i\}_{i \in I \cup J}, \quad (2.2)$$

also satisfies the above conditions. A *transversely holomorphic* structure on $\mathcal{F}$ will mean an equivalence class of data of the above type satisfying the three conditions.

Next we recall the definition of a transversely projective foliation.

DEFINITION 2.2. A transversely projective structure on $\mathcal{F}$ is defined by giving a data $\{U_i, \phi_i\}_{i \in I}$ exactly as in Definition 2.1, but satisfying the extra condition (apart from the three conditions) that the holomorphic maps $f_{i,j}$ in condition (3) are of the form $z \mapsto (az + b)/(cz + d)$, where $a, b, c, d \in \mathbb{C}$ are constant scalars and $ad - bc = 1$, that is, each $f_{i,j}$ is the restriction of some Möbius transformation; the scalars a, b, c, d may depend on the index i . As before, two such data $\{U_i, \phi_i\}_{i \in I}$ and $\{U_i, \phi_i\}_{i \in J}$ are called *equivalent* if their union $\{U_i, \phi_i\}_{i \in I \cup J}$ is also a data for a transversely projective structure. A *transversely projective* structure on $\mathcal{F}$ will mean an equivalence class of such data.

Clearly, a transversely projective structure on $\mathcal{F}$ defines a transversely holomorphic structure on $\mathcal{F}$. If $\tilde{\mathcal{F}}$ is a transversely holomorphic structure on $\mathcal{F}$, then a transversely projective structure on $\tilde{\mathcal{F}}$ is a transversely projective structure on $\mathcal{F}$ such that, the transversely holomorphic structure defined by it coincides with $\tilde{\mathcal{F}}$.

We now recall the notion of a locally nondegenerate immersion of a Riemann surface into a projective space (see [3, 9]).

Let X be a Riemann surface, that is, a complex manifold of complex dimension one. Let $\mathbb{CP}^n$, $n \geq 1$, denote the n -dimensional projective space consisting of all lines in $\mathbb{C}^{n+1}$. A holomorphic immersion

$$\gamma : X \rightarrow \mathbb{CP}^n \quad (2.3)$$

is called *everywhere locally nondegenerate* if for every $x \in X$, the order of contact of the image $\gamma(U)$, where U is a neighborhood of x in X , at $\gamma(x)$ with any hyperplane in $\mathbb{CP}^n$ passing through $\gamma(x)$ is at most $n - 1$. We need to consider a neighborhood in the definition since γ may not be injective.

An alternative description of the above nondegeneracy condition following [9] is given below.

Let

$$0 \rightarrow S \rightarrow V \xrightarrow{q} Q \rightarrow 0 \quad (2.4)$$

be the universal exact sequence over $\mathbb{CP}^n$. The vector bundle V is the trivial vector bundle with $\mathbb{C}^{n+1}$ as fiber and S is the tautological line bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$. Consider the differential

$$d\gamma : T_X \rightarrow \gamma^* T_{\mathbb{CP}^n} = \gamma^* \text{Hom}(S, Q) \quad (2.5)$$

of the immersion γ ; here T_X is the holomorphic tangent bundle of X . Since γ is an immersion, the homomorphism $d\gamma$ is injective.

Now, the homomorphism $d\gamma$ gives a homomorphism

$$\overline{d\gamma}: T_X^* \otimes \gamma^* S \rightarrow \gamma^* Q, \quad (2.6)$$

where T_X^* is the holomorphic cotangent bundle of X . Let S_1 denote the inverse image $q^{-1}(\text{image}(\overline{d\gamma}))$, where the homomorphism q is defined in (2.4). The subbundle S_1 of $\gamma^* V$ defines a map

$$\gamma_1: X \rightarrow G(n+1, 2) \quad (2.7)$$

of X into the Grassmannian of two planes in $\mathbb{C}^{n+1}$.

Now assume that γ_1 is an immersion. Then repeating the above argument we get a map

$$\gamma_2: X \rightarrow G(n+1, 3) \quad (2.8)$$

of X into the Grassmannian of three planes in $\mathbb{C}^{n+1}$.

More generally, inductively we have a map

$$\gamma_i: X \rightarrow G(n+1, i+1), \quad (2.9)$$

where $i \in [1, n-1]$, by assuming that γ_{i-1} is an immersion. (See also [9, Section 1] for the details of the construction of the maps γ_i described above.)

The condition that the map γ , together with each map γ_i , where $i \in [1, n-1]$, is an immersion, is equivalent to the condition that the map γ is everywhere locally nondegenerate.

Now, we extend the above notion of everywhere locally nondegenerate map to the context of foliations, which we call transversely $\mathbb{CP}^n$ -structure.

DEFINITION 2.3. A transversely $\mathbb{CP}^n$ -structure on $\mathcal{F}$ is defined by giving a data $\{U_i, \phi_i\}_{i \in I}$ exactly as in Definition 2.1 satisfying conditions (1) and (2) and the following stronger version of (3): for every $i \in I$, there is an everywhere locally nondegenerate map

$$\gamma_i: D_i := \text{image}(\phi_i) \rightarrow \mathbb{CP}^n \quad (2.10)$$

such that, for every pair $i, j \in I$, there is a commutative diagram of maps

$$\begin{array}{ccc} U_i \cap U_j & \xrightarrow{\text{Id}} & U_i \cap U_j \\ \downarrow \phi_i & & \downarrow \phi_j \\ \phi_i(U_i \cap U_j) & \xrightarrow{f_{i,j}} & \phi_j(U_i \cap U_j) \\ \downarrow \gamma_i & & \downarrow \gamma_j \\ \mathbb{CP}^n & \xrightarrow{T} & \mathbb{CP}^n, \end{array} \quad (2.11)$$

where T is an automorphism of $\mathbb{CP}^n$, that is, $T \in \text{GL}(n+1, \mathbb{C})$. As before, two such data $\{U_i, \phi_i, \gamma_i\}_{i \in I}$ and $\{U_i, \phi_i, \gamma_i\}_{i \in J}$ are called *equivalent* if their union $\{U_i, \phi_i, \gamma_i\}_{i \in I \cup J}$ is

also a data for a transversely $\mathbb{CP}^n$ -structure. A *transversely $\mathbb{CP}^n$ -structure* on $\mathcal{F}$ will mean an equivalence class of such data.

The above condition forces the map $f_{i,j}$ to be holomorphic. So, a transversely $\mathbb{CP}^n$ -structure on $\mathcal{F}$ defines a transversely holomorphic structure on $\mathcal{F}$. If $\tilde{\mathcal{F}}$ is a transversely holomorphic structure on $\mathcal{F}$, then a transversely $\mathbb{CP}^n$ -structure on $\tilde{\mathcal{F}}$ is a transversely $\mathbb{CP}^n$ -structure on $\mathcal{F}$ such that the underlying transversely holomorphic structure coincides with $\tilde{\mathcal{F}}$.

Note that, a transversely $\mathbb{CP}^1$ -structure on $\mathcal{F}$ is by definition a transversely projective structure on $\mathcal{F}$.

We fix a transversely holomorphic structure $\tilde{\mathcal{F}}$ on $\mathcal{F}$.

The normal bundle

$$N := \frac{TM}{\mathcal{F}} \quad (2.12)$$

is a complex line bundle. Therefore, for every integer $k \in \mathbb{Z}$, we have a complex line bundle $N^{\otimes k}$ obtained by taking the k th tensor power of the complex line bundle N . By $N^{\otimes -1}$ we mean the dual line bundle N^* .

Any such line bundle $N^{\otimes k}$ has a natural transversely holomorphic structure. This means that, there is a Dolbeault operator

$$\tilde{\partial}_{N^{\otimes k}} : N^{\otimes k} \longrightarrow N^* \otimes N^{\otimes k} = N^{\otimes k-1} \quad (2.13)$$

satisfying the Leibniz identity. The operator $\tilde{\partial}_{N^{\otimes k}}$ is simply the Dolbeault operator on the holomorphic tangent bundle $T_{\mathbb{C}}^{\otimes k}$ of the complex line $\mathbb{C}$ transported to M using the projections ϕ_i . It may be noted that, the condition in [Definition 2.1\(3\)](#) that every $f_{i,j}$ is holomorphic ensures that these locally defined operators patch compatibly to define the global differential operator $\tilde{\partial}_{N^{\otimes k}}$.

Also, the line bundle N , and hence any $N^{\otimes k}$, has the Bott partial connection (see [\[8\]](#)). Recall that, the Lie bracket operation on the sheaf of sections of the tangent bundle TM defines the Bott partial connection

$$N \longrightarrow \mathcal{F}^* \otimes N \quad (2.14)$$

along the foliation $\mathcal{F}$. The Jacobi identity for Lie bracket ensures that this partial connection is flat.

It is easy to see that both the complex structure of N and the transversely holomorphic structure of N are compatible with respect to the Bott partial connection. In other words, both the complex vector space structure of the fibers of N and the Dolbeault operator $\tilde{\partial}_N$ defined in (2.13) commute with the differential operator in (2.14) defining the Bott connection. Equivalently, parallel translation (for the Bott connection) along the leaves of the foliation $\tilde{\mathcal{F}}$ of holomorphic sections of N remain holomorphic. Also, parallel translations for the Bott connection commute with multiplication by $\sqrt{-1}$ of the fibers of N .

The Bott partial connection on N induces a flat partial connection on any $N^{\otimes k}$. All the above compatibility properties of the Bott connection on N evidently remain valid for any $N^{\otimes k}$.

Let $\mathcal{V}_{\tilde{\mathcal{F}}}(k)$ denote the space of all globally defined smooth sections s of the complex line bundle $N^{\otimes k}$ such that s is transversely holomorphic for the transversely holomorphic foliation $\tilde{\mathcal{F}}$ and it is flat with respect to the Bott partial connection for $\tilde{\mathcal{F}}$. So $\mathcal{V}_{\tilde{\mathcal{F}}}(k)$ is a complex vector space; it need not be of finite dimension. However, in the situation where M is compact, it was proved by Duchamp and Kalka [4, Theorem 1.27, page 323], and also independently by Gómez-Mont [6, Theorem 1, page 169], that the dimension of $\mathcal{V}_{\tilde{\mathcal{F}}}(k)$ is finite.

Let $\mathcal{P}(\tilde{\mathcal{F}})$ denote the space of all equivalence classes of transversely projective structures on the transversely holomorphic foliation $\tilde{\mathcal{F}}$. Transversely projective structures were defined in Definition 2.2 and transversely projective structures on $\tilde{\mathcal{F}}$ were defined in the paragraph following Definition 2.2. The space $\mathcal{P}(\tilde{\mathcal{F}})$ may be empty.

The following theorem is the main result of this section.

THEOREM 2.4. *There is a canonical bijective map from the space of all transversely $\mathbb{C}\mathbb{P}^n$ -structures on $\tilde{\mathcal{F}}$ and the Cartesian product*

$$\mathcal{P}(\tilde{\mathcal{F}}) \times \left(\bigoplus_{k=3}^{n+1} \mathcal{V}_{\tilde{\mathcal{F}}}(-k) \right). \quad (2.15)$$

In particular, a transversely $\mathbb{C}\mathbb{P}^n$ -structure gives a transversely projective structure on $\tilde{\mathcal{F}}$ by simply taking the zero section in $\mathcal{V}_{\tilde{\mathcal{F}}}(-k)$ for all $k \in [3, n+1]$.

The theorem will be proved after establishing a few lemmas. We start with the definition of jet bundles and differential operators.

Let E be a holomorphic vector bundle on a Riemann surface X , and let n be a positive integer. The n th-order jet bundle of E , denoted by $J^n(E)$, is defined to be the following direct image on X :

$$J^n(E) := p_{1*} \left(\frac{p_2^* E}{p_2^* E \otimes \mathbb{O}_{X \times X}(-(n+1)\Delta)} \right), \quad (2.16)$$

where $p_i : X \times X \rightarrow X$, $i = 1, 2$, is the projection onto the i th factor, and Δ is the diagonal divisor on $X \times X$. Therefore, for any $x \in X$, the fiber $J^n(E)_x$ is the space of all sections of E over the n th-order infinitesimal neighborhood of x .

Let K_X denote the holomorphic cotangent bundle of X . There is a natural exact sequence

$$0 \rightarrow K_X^{\otimes n} \otimes E \rightarrow J^n(E) \rightarrow J^{n-1}(E) \rightarrow 0 \quad (2.17)$$

constructed using the obvious inclusion of $\mathbb{O}_{X \times X}(-(n+1)\Delta)$ in $\mathbb{O}_{X \times X}(-n\Delta)$. The inclusion map $K_X^{\otimes n} \otimes E \rightarrow J^n(E)$ is constructed by using the homomorphism

$$K_X^{\otimes n} \rightarrow J^n(\mathbb{O}_X), \quad (2.18)$$

which is defined at any $x \in X$ by sending $(df)^{\otimes n}$, where f is any holomorphic function with $f(x) = 0$, to the jet of the function $f^n/n!$ at x .

The sheaf of differential operators $\text{Diff}_X^n(E, F)$ is defined to be $\text{Hom}(J^n(E), F)$. The homomorphism

$$\sigma : \text{Diff}_X^n(E, F) \rightarrow \text{Hom}(K_X^{\otimes n} \otimes E, F), \quad (2.19)$$

obtained by restricting a homomorphism from $J^n(E)$ to F to the subsheaf $K_X^{\otimes n} \otimes E$ in (2.17), is known as the *symbol map*.

Let X denote a simply connected open subset of $\mathbb{CP}^1$. Take a holomorphic map $\gamma : X \rightarrow \mathbb{CP}^n$. Let ζ denote the line bundle $\gamma^* \mathcal{O}_{\mathbb{CP}^n}(1)$ over X . In the notation of the exact sequence (2.4), the line bundle $\mathcal{O}_{\mathbb{CP}^n}(1)$ is S^* . Pulling back the universal exact sequence (2.4) to X and then taking the dual, we have

$$0 \rightarrow \gamma^* Q^* \rightarrow W \xrightarrow{p} \zeta \rightarrow 0, \quad (2.20)$$

where W is the trivial vector bundle of rank $n+1$ over X with fiber $(\mathbb{C}^{n+1})^*$. Of course, $(\mathbb{C}^{n+1})^* = \mathbb{C}^{n+1}$.

The trivialization of W induces a homomorphism

$$\bar{p} : W \rightarrow J^n(\zeta) \quad (2.21)$$

which can be defined as follows: for any point $x \in X$ and vector $w \in W_x$ in the fiber, let $\bar{w}$ denote the unique flat section of W such that $\bar{w}(x) = w$. Now, $\bar{p}(w)$ is the restriction of the section $p(\bar{w})$ of ζ to the n th-order infinitesimal neighborhood of x . Recall that, the fiber $J^n(\zeta)_x$ is the space of sections of ζ over the n th-order infinitesimal neighborhood of x .

LEMMA 2.5. *The map γ is everywhere locally nondegenerate if and only if the homomorphism $\bar{p}$ in (2.21) is an isomorphism.*

PROOF. This is a straightforward consequence of the condition of everywhere locally nondegeneracy. For some point $x \in X$, if $\bar{p}_x : W_x \rightarrow J^n(\zeta)_x$ is not an isomorphism, then take a nonzero vector w in the kernel of $\bar{p}_x$, since $W_x = (\mathbb{C}^{n+1})^*$, the vector w defines a hyperplane H in $\mathbb{CP}^n$. Clearly, H contains $\gamma(x)$. The given condition $\bar{p}_x(w) = 0$ can be seen to be equivalent to the condition that the order of contact of H with $\gamma(X)$ at $\gamma(x)$ is at least n . In other words, γ is degenerate at x .

Conversely, if γ is degenerate at a point $x \in X$, take a hyperplane H in $\mathbb{CP}^n$ containing $\gamma(x)$ such that the order of contact between $\gamma(X)$ and H at $\gamma(x)$ is at least n . Let $w \in (\mathbb{C}^{n+1})^*$ be a functional defining the hyperplane H . It is easy to see that $\bar{p}_x(w) = 0$. This completes the proof. $\square$

Assume that γ is everywhere locally nondegenerate. So the homomorphism $\bar{p}$ in (2.21) gives a trivialization of the jet bundle $J^n(\zeta)$. Now, from (2.17) it follows that $\wedge^{n+1} J^n(\zeta)$ is canonically isomorphic to $K_X^{n(n+1)/2} \otimes \zeta^{n+1}$. The trivialization of $J^n(\zeta)$ induces a trivialization of $K_X^{n(n+1)/2} \otimes \zeta^{n+1}$. Fix a square-root ξ of the holomorphic tangent bundle T_X . In other words, ξ is a holomorphic line bundle and an isomorphism between T_X and $\xi^{\otimes 2}$ is chosen. The above trivialization of $K_X^{n(n+1)/2} \otimes \zeta^{n+1}$ induces an isomorphism

$$J^i(\zeta^j) = J^i(\xi^{nj}) \otimes (\xi^{nj})^* \otimes \zeta^j \quad (2.22)$$

for every i and j . Indeed, this is an immediate consequence of the fact that ζ and ξ^n differ by tensoring with a finite-order line bundle. By a finite-order line bundle we mean a line bundle some tensor power of which has a canonical trivialization.

Consider the homomorphism

$$\hat{p}: W \longrightarrow J^{n+1}(\zeta) \quad (2.23)$$

which sends any $w \in W_x$ to the restriction of the section $p(\bar{w})$ of ζ to the $(n+1)$ th-order infinitesimal neighborhood of x . Here p as in (2.20) and $\bar{w}$ as in the definition of the map $\bar{p}$ in (2.21). From its definition it is immediate that the composition $f_n \circ \hat{p} \circ \bar{p}^{-1}$ is the identity map of $J^n(\zeta)$, where f_n is the projection $J^{n+1}(\zeta) \rightarrow J^n(\zeta)$ defined in (2.17). In other words, $\hat{p} \circ \bar{p}^{-1}$ is a splitting of the jet sequence

$$0 \longrightarrow K_X^{n+1} \otimes \zeta \longrightarrow J^{n+1}(\zeta) \longrightarrow J^n(\zeta) \longrightarrow 0 \quad (2.24)$$

defined in (2.17).

There is a unique homomorphism $J^{n+1}(\zeta) \rightarrow K_X^{n+1} \otimes \zeta$ satisfying the two conditions that its kernel is the image of $\hat{p} \circ \bar{p}^{-1}$ and the composition of the natural inclusion of $K_X^{n+1} \otimes \zeta$ in $J^{n+1}(\zeta)$ (as in (2.17)) with it is the identity map of $K_X^{n+1} \otimes \zeta$. By the earlier definition of differential operators given in terms of jet bundles, this homomorphism defines a differential operator

$$D_Y \in H^0(X, \text{Diff}_X^{n+1}(\zeta, K_X^{n+1} \otimes \zeta)). \quad (2.25)$$

Since D_Y is defined by a splitting of a jet sequence, its symbol is the constant function 1 (the symbol of a differential operator is defined in (2.19)). Now, using (2.22), the differential operator D_Y gives a differential operator

$$D(Y) \in H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})) \quad (2.26)$$

of symbol 1.

It can be deduced from the definition of jet bundles that, for any holomorphic vector bundle E , there is a natural injective homomorphism $J^{i+j}(E) \rightarrow J^i(J^j(E))$ for any $i, j \geq 0$. Therefore, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \xi^{\otimes(-n-2)} & \longrightarrow & J^{n+1}(\xi^{\otimes n}) & \longrightarrow & J^n(\xi^{\otimes n}) \longrightarrow 0 \\ & & \downarrow & & \downarrow \tau & & \parallel \\ 0 & \longrightarrow & K_X \otimes J^n(\xi^{\otimes n}) & \longrightarrow & J^1(J^n(\xi^{\otimes n})) & \longrightarrow & J^n(\xi^{\otimes n}) \longrightarrow 0, \end{array} \quad (2.27)$$

where the injective homomorphism τ is obtained from the above remark.

If

$$f: J^n(\xi^{\otimes n}) \longrightarrow J^{n+1}(\xi^{\otimes n}) \quad (2.28)$$

is a splitting of the top exact sequence in (2.27), then the composition $\tau \circ f$ defines a splitting of the bottom exact sequence in (2.27). But a splitting of the exact sequence

$$0 \longrightarrow K_X \otimes E \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0 \quad (2.29)$$

is a holomorphic connection on E (see [1]). Furthermore, any holomorphic connection on a Riemann surface is flat. Therefore, $\tau \circ f$ defines a flat connection on $J^n(\xi^{\otimes n})$. Let ∇^f denote this flat connection on $J^n(\xi^{\otimes n})$ obtained from a splitting f .

Since X is simply connected, ∇^f gives a trivialization of $J^n(\xi^{\otimes n})$. In other words, if we choose a point $z \in X$, using parallel translations, $J^n(\xi^{\otimes n})$ gets identified with the trivial vector bundle over X with $J^n(\xi^{\otimes n})_z$ as the fiber.

Fix an isomorphism of the fiber $J^n(\xi^{\otimes n})_z$ with $\mathbb{C}^{n+1}$. As before, let W denote the trivial vector bundle over X with $\mathbb{C}^{n+1}$ as the fiber. So we have $J^n(\xi^{\otimes n}) = W$.

For any point $y \in X$, consider the one-dimensional subspace $(\xi^{\otimes n} \otimes K_X^n)_y$ of the fiber $J^n(\xi^{\otimes n})_y$ given in (2.17). Let

$$\gamma : X \rightarrow \mathbb{CP}^n \quad (2.30)$$

denote the map that sends any point $y \in X$ to the line in $\mathbb{C}^{n+1}$ that corresponds to the line $(\xi^{\otimes n} \otimes K_X^n)_y$ by the isomorphism between the fibers $J^n(\xi^{\otimes n})_y$ and W_y .

If we change the isomorphism between $J^n(\xi^{\otimes n})_z$ and $\mathbb{C}^{n+1}$ by an automorphism $A \in \text{GL}(n+1, \mathbb{C})$, then the map γ is altered by the automorphism A of $\mathbb{CP}^n$.

LEMMA 2.6. *Let $f : J^n(\xi^{\otimes n}) \rightarrow J^{n+1}(\xi^{\otimes n})$ be a splitting of the top exact sequence in (2.27). Then the map γ constructed in (2.30) from f is everywhere locally nondegenerate.*

PROOF. The lemma follows from Lemma 2.5 and the fact that the connection ∇^f , from which γ is constructed, is given by a splitting f (as in (2.28)). In [3], a different but equivalent formulation of the lemma can be found. $\square$

Two everywhere locally nondegenerate maps f_1 and f_2 of X into $\mathbb{CP}^n$ are called equivalent if there is an automorphism $A \in \text{Aut}(\mathbb{CP}^n) = \text{PGL}(n+1, \mathbb{C})$ such that $A \circ f_1 = f_2$.

Let $\mathcal{A}$ denote the space of all equivalence classes of everywhere locally nondegenerate maps of X into $\mathbb{CP}^n$.

Take a differential operator $D \in H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2}))$ of symbol 1. Since the symbol of D is 1, it gives a splitting of the top exact sequence in (2.27). Denoting this splitting $J^n(\xi^{\otimes n}) \rightarrow J^{n+1}(\xi^{\otimes n})$ by $\bar{D}$, consider $\tau \circ \bar{D}$, which, as we already noted, is a flat connection on $J^n(\xi^{\otimes n})$. It may be noted that since $\xi^{\otimes 2} = T_X$, the line bundle $\wedge^{n+1} J^n(\xi^{\otimes n})$ is canonically trivialized.

Let $\mathcal{B}$ denote the space of global differential operators

$$D \in H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})) \quad (2.31)$$

of symbol 1 and satisfying the condition that the connection on $\wedge^{n+1} J^n(\xi^{\otimes n})$ induced by the connection $\tau \circ \bar{D}$ on $J^n(\xi^{\otimes n})$ preserves the trivialization of $\wedge^{n+1} J^n(\xi^{\otimes n})$.

From the construction of the differential operator $D(\gamma)$ in (2.26) it follows that $D(\gamma) \in \mathcal{B}$.

Let

$$F : \mathcal{A} \rightarrow \mathcal{B} \quad (2.32)$$

be the map that sends any everywhere locally nondegenerate map γ to the differential operator $D(\gamma)$ constructed in (2.26).

As above, for a differential operator $D \in \mathcal{B}$, the corresponding splitting is denoted by $\bar{D}$. Let

$$G : \mathcal{B} \rightarrow \mathcal{A} \quad (2.33)$$

be the map that sends any operator D to the map γ constructed in (2.30) using the splitting $f = \bar{D}$ as in (2.28).

LEMMA 2.7. *The map F defined in (2.32) is one-to-one and onto.*

PROOF. In fact, unraveling the definitions of the maps F and G , defined in (2.32) and (2.33), respectively, yields that they are inverses of each other. We omit the details; it can be found in [3]. $\square$

Let $\mathcal{P}(X)$ denote the space of all projective structures on the Riemann surface X . It is known that $\mathcal{P}(X)$ is an affine space for the space of quadratic differentials, namely, $H^0(X, K_X^2)$ (see [7]).

LEMMA 2.8. *There is a natural bijective map between $\mathcal{B}$ and the Cartesian product*

$$\mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \quad (2.34)$$

if $n \geq 2$. If $n = 1$, then $\mathcal{B}$ is in bijective correspondence with $\mathcal{P}(X)$.

PROOF. The key input in the proof is [2, Theorem 6.3, page 19]. Now we recall its statement.

Let Y be a Riemann surface equipped with a projective structure. Let $k, l \in \mathbb{Z}$ and let $n \in \mathbb{N}$ be such that $k \notin [-n+1, 0]$ and $l - k - j \notin \{0, 1\}$ for any integer $j \in [1, n]$. Then,

$$H^0(Y, \text{Diff}_Y^n(\mathcal{L}^k, \mathcal{L}^l)) = \bigoplus_{i=0}^n H^0(Y, \mathcal{L}^{l-k-2n+2i}), \quad (2.35)$$

where $\mathcal{L}$ is the square-root of the canonical bundle defined by the projective structure.

A clarification of the above statement is needed. In [2], a projective structure means an $\text{SL}(2, \mathbb{C})$ structure. But here projective structure means a $\text{PGL}(2, \mathbb{C})$ structure. But we know that a $\text{PGL}(2, \mathbb{C})$ structure on a Riemann surface always lifts to an $\text{SL}(2, \mathbb{C})$ structure [7]. Furthermore, the space of such lifts is in bijective correspondence with the space of theta-characteristics (square-root of the holomorphic cotangent bundle) of Y .

Therefore, given a $\text{PGL}(2, \mathbb{C})$ structure P on X , the pair (P, ξ) determines a unique $\text{SL}(2, \mathbb{C})$ structure.

Now, set $k = -n$ and $l = n+2$ in (2.35). This yields an isomorphism

$$F : H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})) \rightarrow \bigoplus_{i=0}^n H^0(X, K_X^{\otimes i}). \quad (2.36)$$

For any $D \in H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2}))$, the component of $F(D)$ in

$$H^0(X, K_X^{\otimes 0}) = H^0(X, \mathbb{O}_X) \quad (2.37)$$

is the symbol of D . Furthermore, the condition in the definition of $\mathcal{B}$ that, the connection on $\bigwedge^{n+1} J^n(\xi^{\otimes n})$ induced by the connection $\tau \circ \bar{D}$ on $J^n(\xi^{\otimes n})$ preserves the trivialization of $\bigwedge^{n+1} J^n(\xi^{\otimes n})$, is actually equivalent to the condition that the component of $F(D)$ in $H^0(X, K_X)$ vanishes (see [3]). Therefore, using F , the space $\mathcal{B}$ gets

identified with the direct sum

$$\bigoplus_{i=2}^{n+1} H^0(X, K_X^{\otimes i}), \quad (2.38)$$

if X is equipped with a projective structure.

Using the fact that the space of projective structures on X , namely $\mathcal{P}(X)$, is an affine space for $H^0(X, K_X^2)$, it is easy to deduce that given any

$$D \in H^0(X, \text{Diff}_X^{n+1}(\xi^n, \xi^{-n-2})), \quad (2.39)$$

there is a unique projective structure $P \in \mathcal{P}(X)$ such that, for the map F in (2.36) corresponding to P , the component of $F(D)$ in $H^0(X, K_X^{\otimes 2})$ vanishes identically. Let $\overline{F(D)}$ denote the projection of $F(D)$ in $\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i})$; F corresponds to this unique projective structure. Now, we have a bijective map

$$\tilde{F} : \mathcal{B} \longrightarrow \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right), \quad (2.40)$$

that sends any D to the pair $(P, \overline{F(D)})$ constructed above. (See [3, Section 4] for the details.)

If $n = 1$, then using [2, Theorem 6.3] and the fact that $\mathcal{P}(X)$ is an affine space for $H^0(X, K_X^{\otimes 2})$, it follows immediately that $\mathcal{B} = \mathcal{P}(X)$. This completes the proof of the lemma. $\square$

For the first part of the proof of Lemma 2.8, we should have directly used [2, Corollary 6.6] instead of deriving it using [2, Theorem 6.3]. Unfortunately, in the statement of [2, Corollary 6.6], the word “compact” is used which technically makes it useless for our purpose. But, of course, compactness is not used in the proof of [2, Corollary 6.6]. When [2, 3] were written, we had primarily compact Riemann surfaces in mind.

Combining Lemmas 2.7 and 2.8, we have the following corollary.

COROLLARY 2.9. *There is a natural bijective map*

$$\Gamma : \mathcal{A} \longrightarrow \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \quad (2.41)$$

for $n \geq 2$. If $n = 1$ then $\mathcal{A}$ is in bijective correspondence with $\mathcal{P}(X)$.

When X is a compact Riemann surface, the above corollary is [3, Theorem 5.5]. Again since “compactness” condition is thrown in [3] indiscriminately, a vast part of it is technically useless for our present purpose. Nevertheless, the ideas of [3] have been borrowed here.

Let $Y \subset X$ be a simply connected open subset. Let $\mathcal{A}_Y$ denote the space of all equivalence classes of everywhere locally nondegenerate maps of Y into $\mathbb{CP}^n$. In other words, $\mathcal{A}_Y$ is obtained by substituting Y in place of X in the definition of $\mathcal{A}$. The space of all projective structures on Y is denoted by $\mathcal{P}(Y)$.

The restriction of ξ to Y defines a square-root of the tangent bundle T_Y . There is a natural restriction map $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ and also there are homomorphisms

$$H^0(X, K_X^{\otimes i}) \rightarrow H^0(Y, K_Y^{\otimes i}) \quad (2.42)$$

for every $i \in \mathbb{Z}$ defined by restriction of sections. Similarly, we have a map $\mathcal{A} \rightarrow \mathcal{A}_Y$, which sends a map γ of X to $\mathbb{C}\mathbb{P}^n$ to the restriction of γ to Y .

Let

$$\Gamma_Y : \mathcal{A}_Y \rightarrow \mathcal{P}(Y) \times \left(\bigoplus_{i=3}^{n+1} H^0(Y, K_Y^{\otimes i}) \right) \quad (2.43)$$

be the isomorphism for Y obtained in [Corollary 2.9](#). The map Γ in [Corollary 2.9](#) has the property that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Gamma} & \mathcal{P}(X) \times \left(\bigoplus_{i=3}^{n+1} H^0(X, K_X^{\otimes i}) \right) \\ \downarrow & & \downarrow \\ \mathcal{A}_Y & \xrightarrow{\Gamma_Y} & \mathcal{P}(Y) \times \left(\bigoplus_{i=3}^{n+1} H^0(Y, K_Y^{\otimes i}) \right). \end{array} \quad (2.44)$$

The vertical maps are defined by restriction. The commutativity of this diagram is indeed easy to see from the construction of Γ .

Now that we have [Corollary 2.9](#) and (2.44), we are ready to prove [Theorem 2.4](#).

PROOF OF [Theorem 2.4](#). Assume that $n \geq 2$, since the theorem is obvious in the case of $n = 1$.

Suppose we are given a transversely $\mathbb{C}\mathbb{P}^n$ -structure, as defined in [Definition 2.3](#). We assume that all the subsets $D_i := \text{image}(\phi_i)$ of $\mathbb{C}$ in [Definition 2.1](#) are simply connected. Clearly, this is a harmless assumption.

Consider a triplet (U_i, ϕ_i, γ_i) as in [Definition 2.3](#). Now, using the map Γ in [Corollary 2.9](#), from the everywhere locally nondegenerate map γ_i we have a projective structure on $D_i = \text{image}(\phi_i)$ together with a holomorphic section of $T_{D_i}^{\otimes -l}$ for all $l \in [3, n+1]$. This projective structure on D_i is denoted by $\mathcal{P}_i$, and the holomorphic section of $T_{D_i}^{\otimes -l}$ obtained above is denoted by ω_i^l . The projective structure $\mathcal{P}_i$ induces a transversely projective structure on the open subset U_i of M . We denote this transversely projective structure on U_i by $\tilde{\mathcal{P}}_i$. The pullback, using the map ϕ_i , of the holomorphic section ω_i^l of $T_{D_i}^{\otimes -l}$ defines a section of $N^{\otimes -l}$ over U_i . This section of $N^{\otimes -l}$ over U_i is denoted by $\tilde{\omega}_i^l$. Since ω_i^l is holomorphic, we have the section $\tilde{\omega}_i^l$ over U_i to be transversely holomorphic. Furthermore, $\tilde{\omega}_i^l$ is obviously flat with respect to the Bott partial connection. The proof of the theorem is completed by showing that all these locally defined transversely projective structures $\tilde{\mathcal{P}}_i$ (resp., transversely holomorphic flat sections $\tilde{\omega}_i^l$) patch compatibly to define globally on M a transversely projective structure (resp., transversely holomorphic flat section of $N^{\otimes -l}$).

If we take another triplet (U_j, ϕ_j, γ_j) , $j \in I$, as in [Definition 2.3](#), then the two projective structures on $D_i \cap D_j$, namely $\mathcal{P}_i$ and $\mathcal{P}_j$, coincide. This is an immediate consequence of the commutativity of the diagram (2.44). Therefore, we have a projective

structure on the union $D_i \cup D_j$, and hence the two transversely projective structures, namely $\bar{\mathcal{P}}_i$ and $\bar{\mathcal{P}}_j$, coincide over $U_i \cap U_j$. Consequently, the transversely projective structures $\{\bar{\mathcal{P}}_i\}_{i \in I}$ patch together compatibly to define a transversely projective structure on $\bar{\mathcal{F}}$. Similarly, from the commutativity of the diagram (2.44), it follows that the two sections $\bar{\omega}_i^l$ and $\bar{\omega}_j^l$ coincide over $U_i \cap U_j$. In other words, these local sections $\bar{\omega}_i^l$ of $N^{\otimes -l}$ patch together to give an element of $\mathcal{V}_{\bar{\mathcal{F}}}(-l)$. This completes the proof of the theorem. $\square$

[Theorem 2.4](#) can be considered as a generalization of [10, Theorem 6.1].

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