CHARACTERIZATIONS OF PROJECTIVE AND k -PROJECTIVE SEMIMODULES

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To my late father

This paper deals with projective and k -projective semimodules. The results for projective semimodules are generalization of corresponding results for projective modules.

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1. Introduction. Throughout this paper, R denotes a semiring with identity 1, all semimodules M are left R -semimodules and in all cases are unitary semimodules, that is, $1 \cdot m = m$ for all $m \in M$ all left R -semimodule ${}_R M$.

We recall here (cf. [1, 2, 3, 4, 5]) the following facts:

- (a) let $\alpha : M \rightarrow N$ be a homomorphism of semimodules. The subsemimodule $\text{Im } \alpha$ of N is defined as follows: $\text{Im } \alpha = \{n \in N : n + \alpha(m') = \alpha(m) \text{ for some } m, m' \in M\}$. The homomorphism α is said to be an isomorphism if α is injective and surjective; to be i -regular if $\alpha(M) = \text{Im } \alpha$; to be k -regular if for $m, m' \in M$, $\alpha(m) = \alpha(m')$ implying that $m + k = m' + k'$ for some $k, k' \in \text{Ker } \alpha$; and to be regular if it is both i -regular and k -regular;
- (b) the sequence $K \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is called an exact sequence if $\text{Ker } \beta = \text{Im } \alpha$, and proper exact if $\text{Ker } \beta = \alpha(K)$;
- (c) for any two R -semimodules N, M , $\text{Hom}_R(N, M) := \{\alpha : N \rightarrow M \mid \alpha \text{ is an } R\text{-homomorphism of semimodules}\}$ is a semigroup under addition. If M, N, U , are R -semimodules and $\alpha : M \rightarrow N$ is a homomorphism, then $\text{Hom}(I_U, \alpha) : \text{Hom}_R(U, M) \rightarrow \text{Hom}_R(U, N)$ is given by $\text{Hom}(I_U, \alpha)\gamma = \alpha\gamma$ where I_U is the identity;
- (d) P is a projective semimodule if and only if for each surjective R -homomorphism $\alpha : M \rightarrow N$, the induced homomorphism

$$\bar{\alpha} : \text{Hom}(P, M) \rightarrow \text{Hom}(P, N) \tag{1.1}$$

is surjective;

- (e) a left R -semimodule P is Mk -projective if and only if it is projective with respect to every surjective k -regular homomorphism $\varphi : M \rightarrow N$.

In Section 2, we study the structure of k -projective semimodules. Proposition 2.2 shows that for a semimodule P , the class of all semimodules M such that P is Mk -projective is closed under subtractive subsemimodules, factor semimodules, and gives a sufficient condition for the class to be closed undertaking homomorphic images. Example 2.3 sheds light upon one difference between the structure of projectivity in

module theory and semimodule theory. In [Section 3](#), we characterize projective and k -projective semimodules via the Hom functor. Theorems 3.5 and 3.7 assert that P is M -projective (Mk -projective) if and only if $\text{Hom}_R(P, -)$ preserves the exactness of all proper exact sequences $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$, with β k -regular (both α and β k -regular).

2. k -projective semimodules. We study the structure of k -projective semimodules via the Hom function. We show that the class of all semimodules M , such that P is Mk -projective, is closed under subtractive subsemimodules, factor semimodule and undertaking homomorphic image for a k -regular homomorphism.

For proving [Proposition 2.2](#) we need the following proposition, which is modified from [5, Theorem 2.6].

PROPOSITION 2.1. *Let R be a semiring,*

- (i) *if $0 \rightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is a proper exact sequence of R -semimodules and α is k -regular, then for every R -semimodule E ,*

$$0 \rightarrow \text{Hom}_R(E, M) \xrightarrow{\tilde{\alpha}} \text{Hom}_R(E, M') \xrightarrow{\tilde{\beta}} \text{Hom}(E, M'') \quad (2.1)$$

is a proper exact sequence of Abelian semigroups and $\tilde{\alpha}$ is regular, where $\tilde{\alpha}(\xi) = \alpha\xi$ for $\xi \in \text{Hom}(E, M)$ and $\tilde{\beta}(\gamma) = \beta\gamma$ for $\gamma \in \text{Hom}(E, M')$;

- (ii) *if $M \xrightarrow{\alpha} M' \xrightarrow{\beta} M'' \rightarrow 0$ is a proper exact sequence of R -semimodules and β is k -regular, then for every R -semimodule E ,*

$$0 \rightarrow \text{Hom}(M'', E) \xrightarrow{\tilde{\beta}} \text{Hom}(M', E) \xrightarrow{\tilde{\alpha}} \text{Hom}(M, E) \quad (2.2)$$

is a proper exact sequence of Abelian semigroups and $\tilde{\beta}$ is regular, where $\tilde{\beta}(\xi) = \beta\xi$.

PROOF. (i) Since the sequence $0 \rightarrow M \xrightarrow{\alpha} M' \xrightarrow{\beta} M''$ is proper exact, then the sequence is exact with α being i -regular.

Using [5, Theorem 2.6], the sequence

$$0 \rightarrow \text{Hom}(E, M) \xrightarrow{\tilde{\alpha}} \text{Hom}(E, M') \xrightarrow{\tilde{\beta}} \text{Hom}(E, M'') \quad (2.3)$$

is exact with $\tilde{\alpha}$ being regular. This means that the sequence is proper exact. (ii) can be proved by the same argument. $\square$

PROPOSITION 2.2. *Let P be a left R -semimodule. If $0 \rightarrow M' \xrightarrow{\theta} M \xrightarrow{\eta} M'' \rightarrow 0$ is a proper exact sequence with θ being regular, η being k -regular, and P is Mk -projective, then P is k -projective relative to both M' and M'' .*

PROOF. Let $\Psi : M'' \rightarrow N$ be surjective k -regular homomorphism and $\alpha : P \rightarrow N$ be homomorphism. Since η is surjective k -regular, then $\Psi\eta$ is k -regular. Since P is Mk -projective, then there exists a homomorphism $\varphi : P \rightarrow M$ such that the following diagram commutative:

$$\begin{array}{ccccccc} & & & P & & & \\ & & \nearrow \varphi & \downarrow \alpha & & & \\ M & \xrightarrow{\eta} & M'' & \xrightarrow{\Psi} & N & \longrightarrow & 0 \end{array} \quad (2.4)$$

Therefore P is M'' - k -projective.

To prove that P is $M'k$ -projective. Let $\Psi : M' \rightarrow N$ be a surjective k -regular homomorphism and set $K = \text{Ker} \Psi$. Since Ψ is surjective k -regular homomorphism, then $M'/K \simeq N$. Define $\bar{\theta} : M'/K \rightarrow M/\theta(K)$ by the rule $\bar{\theta}(m'/K) = \theta(m')/\theta(K)$, and $\bar{\eta} : M/\theta(K) \rightarrow M''$ by the rule $\bar{\eta}(m/\theta(K)) = \eta(m)$. Clearly, both $\bar{\theta}$ and $\bar{\eta}$ are well defined homomorphisms. Consider the sequence $0 \rightarrow M'/K \xrightarrow{\bar{\theta}} M/\theta(K) \xrightarrow{\bar{\eta}} M'' \rightarrow 0$. Let $m/\theta(K) \in \text{Ker} \bar{\eta}$, then $\eta(m) = 0$, hence $m \in \text{Ker} \eta = \theta(M')$. Hence $\text{Ker} \bar{\eta} = \bar{\theta}(M'/K)$. Clearly $\bar{\eta}$ is surjective, and $\bar{\theta}$ is injective. Since θ is i -regular, then $\bar{\theta}$ is i -regular. Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & M' & \xrightarrow{\theta} & M & \xrightarrow{\eta} & M'' \longrightarrow 0 \\
 & & \downarrow \pi_k & & \downarrow \pi_{h(k)} & & \downarrow \\
 0 & \longrightarrow & M'/K & \xrightarrow{\bar{\theta}} & M/\theta(K) & \xrightarrow{\bar{\eta}} & M'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \quad (2.5)$$

Applying $\text{Hom}_R(P, -)$ to this diagram we have the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & \text{Hom}(P, M') & \xrightarrow{\theta_*} & \text{Hom}(P, M) & \xrightarrow{\eta_*} & \text{Hom}(P, M'') \\
 & & \downarrow (\pi_k)_* & & \downarrow (\pi_{\theta(k)})_* & & \downarrow I_* \\
 0 & \longrightarrow & \text{Hom}(P, M'/K) & \xrightarrow{\bar{\theta}_*} & \text{Hom}(P, M/\theta(K)) & \xrightarrow{\bar{\eta}_*} & \text{Hom}(P, M'') \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array} \quad (2.6)$$

Using [Proposition 2.1](#), and since P is Mk -projective, then all rows and columns are proper exact sequence. We should show that $(\pi_K)_*$ is surjective. Let $\alpha \in \text{Hom}(P, M'/K)$. Since $(\pi_{\theta(K)})_*$ is surjective, then there exists $\beta \in \text{Hom}(P, M)$ such that $(\pi_{\theta(K)})_*(\beta) = \bar{\theta}_*(\alpha)$. Now $\bar{\eta}_* \bar{\theta}_*(\alpha) = \bar{\eta}_*((\pi_{\theta(K)})_*(\beta)) = I_* \eta_*(\beta) = 0$. Hence $\eta_*(\beta) \subset \text{Ker} I_* = 0$. Hence $\beta = \theta_*(\gamma)$ where $\gamma \in \text{Hom}(P, M')$. Thus $\bar{\theta}_*(\alpha) = (\pi_{\theta(K)})_*(\beta) = (\pi_{\theta(K)})_* \theta_*(\gamma) = \bar{\theta}_*(\pi_K)_*(\gamma)$. Again by [Proposition 2.1](#), $\bar{\theta}_*$ is injective, hence $\alpha = (\pi_K)_*(\gamma)$. Thus $(\pi_K)_*$ is surjective. Therefore P is $M'k$ -projective. $\square$

Let $\Omega(P)$ be the collection of all semimodules M such that P is Mk -projective. The above results show that this class is closed under subtractive subsemimodules and

give us a sufficient condition to be closed undertaking a homomorphic image. Since for every subsemimodule K of M , the canonical surjection $\pi_K : M \rightarrow M/K$ is k -regular surjective, then the class $\Omega(P)$ is closed under factor semimodules.

We know that in module theory any projective module is a direct summand of a free module. However, for arbitrary semirings this is not true.

EXAMPLE 2.3. Let R be the field $\mathbb{Z}/\langle p \rangle$ for any prime integer. Let $S = \{\{\bar{0}\}, \mathbb{Z}/\langle p \rangle\}$ set $R' = \{(\bar{a}, I) : \bar{a} \in I \in S\}$, that is

$$R' = \{(\bar{0}, \{\bar{0}\}), (\bar{a}, \mathbb{Z}/\langle p \rangle), \bar{a} \in \mathbb{Z}/\langle p \rangle\}. \quad (2.7)$$

Define operations $\oplus$ and $\otimes$ on R' by setting

$$\begin{aligned} (\bar{a}, I) \oplus (\bar{b}, H) &= (\bar{a} + \bar{b}, I + H), \\ (\bar{a}, \bar{I}) \otimes (\bar{b}, H) &= (\bar{a}\bar{b}, IH). \end{aligned} \quad (2.8)$$

Clearly R' is semiring. Let $I^+(R')$ be the set of all additively idempotent elements of $R' : I^+(R') = \{(\bar{0}, \{\bar{0}\}), (\bar{0}, \mathbb{Z}/\langle p \rangle)\}$. We note that the function $\alpha : R' \rightarrow I^+(R')$, defined by $\alpha(\bar{0}, \{\bar{0}\}) = (\bar{0}, \{\bar{0}\})$ and $\alpha(\bar{a}, \mathbb{Z}/\langle p \rangle) = (\bar{0}, \mathbb{Z}/\langle p \rangle)$, is a surjective R' -homomorphism of left R' -semimodules. Furthermore, the restriction of α to $I^+(R')$ is the identity map. Therefore $I^+(R')$ is a retract of R' . Since R' is projective, as a left semimodule over itself, by [5, Corollary 15.13] we see that $I^+(R')$ is also projective. If $I^+(R')$ is a free semimodule, then $(\bar{0}, \mathbb{Z}/\langle p \rangle)$ is a basis, but $(\bar{a}, \mathbb{Z}/\langle p \rangle)(\bar{0}, \mathbb{Z}/\langle p \rangle) = (\bar{0}, \mathbb{Z}/\langle p \rangle)$ for every $\bar{a} \in \mathbb{Z}/\langle p \rangle$. Hence $I^+(R')$ is not free. Now, suppose that $I^+(R')$ is a direct summand of a free R' -semimodule, say F . Then $F \cong K \oplus I^+(R')$ for an R' -semimodule K . Let $\varphi : K \oplus I^+(R') \rightarrow F = \oplus_{\alpha} R'_{\alpha}$ be an isomorphism, where $R'_{\alpha} = R'$ for all α . Since $(0, (\bar{0}, \mathbb{Z}/\langle p \rangle))$ is an idempotent element where $0 \in K$, then $\varphi(0, (\bar{0}, \mathbb{Z}/\langle p \rangle))$ is an idempotent element in F . Since the only idempotent elements of R' are $(\bar{0}, \{\bar{0}\})$ and $(\bar{0}, \mathbb{Z}/\langle p \rangle)$, then $\varphi(0, (\bar{0}, \mathbb{Z}/\langle p \rangle)) = (x_{\alpha})$, where only finite numbers of x_{α} are nonzeros and $x_{\alpha} = (\bar{0}, \mathbb{Z}/\langle p \rangle)$. Let (y_{α}) have components $y_{\alpha} = (\bar{1}, \mathbb{Z}/\langle p \rangle)$ for α , with $x_{\alpha} = (\bar{0}, \mathbb{Z}/\langle p \rangle)$, and otherwise $y_{\alpha} = (\bar{0}, \{\bar{0}\})$. Suppose $\varphi(k, (\bar{0}, \mathbb{Z}/\langle p \rangle)) = (y_a)$, where $0 \neq k \in K$. Clearly, $p(y_{\alpha}) = (x_a)$, $p\varphi(k, (\bar{0}, \mathbb{Z}/\langle p \rangle)) = \varphi(0, (\bar{0}, \mathbb{Z}/\langle p \rangle))$. Hence $p(k, (\bar{0}, \mathbb{Z}/\langle p \rangle)) = (0, (\bar{0}, \mathbb{Z}/\langle p \rangle))$. Hence $pk = 0$. Therefore $p(k, (\bar{0}, \{\bar{0}\})) = 0$. Hence $(k, (\bar{0}, \{\bar{0}\}))$ has an additive inverse. Thus, $\varphi(k, (\bar{0}, \{\bar{0}\}))$ also has an additive inverse. Now, every element of F is of the form (u_{α}) , $u_{\alpha} \in R'$. Since every nonzero element of R' has no additive inverse, then every nonzero element of F has no additive inverse. Thus we have a contradiction. Therefore, $I^+(R')$ is not a direct summand of a free R' -semimodule.

3. Characterizations of projective and k -projective semimodules. We characterize projective and k -projective semimodules via the Hom functor.

We state and prove the following lemma and corollaries which are needed in the proof of Theorem 3.5.

LEMMA 3.1. *let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R -semimodules. Then, the sequence is exact if there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & N & & \\
 & & \uparrow & \searrow \eta & \\
 & \varphi & & & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow & & \\
 & & K & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \quad (3.1)$$

of R -semimodule in which the nonhorizontal sequences are all exact.

PROOF. Let $x \in \text{Ker } \beta$, then $\eta\varphi(x) = \beta(x) = 0$, hence $\varphi(x) \in \text{Ker } \eta$. Since $\text{Ker } \eta = 0$, then $x \in \text{Ker } \varphi = \text{Im } \theta$. Hence $x + \theta(k_1) = \theta(k_2)$, therefore $x + \theta(\Psi(m_1)) = \theta(\Psi(m_2))$. Since $\theta\Psi = \alpha$, then $x + \alpha(m_1) = \alpha(m_2)$. Thus $x \in \text{Im } \alpha$. Conversely, let $x \in \text{Im } \alpha$, then $x + \alpha(m_1) = \alpha(m_2)$ for some $m_1, m_2 \in M$. Again since $\theta\Psi = \alpha$, then $x + \theta\Psi(m_1) = \theta\Psi(m_2)$, hence $x \in \text{Im } \theta$. But $\text{Im } \theta = \text{Ker } \varphi$, hence $\beta(x) = \eta\varphi(x) = 0$. Thus $\text{Im } \alpha = \text{Ker } \beta$. $\square$

Since every proper exact sequence is exact sequence. Then we have the following corollary.

COROLLARY 3.2. *Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R -semimodules. Then the sequence is proper exact if there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & N & & \\
 & & \uparrow & \searrow \eta & \\
 & \varphi & & & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow & & \\
 & & K & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \quad (3.2)$$

of R -semimodules in which the nonhorizontal sequences are all proper exact.

PROOF. Since every proper exact sequence is exact sequence, then using [Lemma 3.1](#), we have $\text{Ker } \beta = \text{Im } \alpha$, hence $\alpha(M') \subset \text{Ker } \beta$. Now let $x \in \text{Ker } \beta$, then $\beta(x) = \eta\varphi(x) = 0$, hence $\varphi(x) \in \text{Ker } \eta$. But $\text{Ker } \eta = 0$, therefore $x \in \text{Ker } \varphi = \theta(K)$. Hence $x = \theta(k) = \theta(\Psi(m')) = \alpha(m')$. Thus $\alpha(M') = \text{Ker } \beta$. $\square$

COROLLARY 3.3. Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R -semimodules with β being k -regular. Then the sequence is proper exact if and only if there exists a commutative diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & N & & \\
 & & \uparrow & \searrow \eta & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow \theta & & \\
 & & K & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \quad (3.3)$$

of R -semimodules in which the nonhorizontal sequences are all proper exact.

PROOF. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a proper exact sequence with β being k -regular. Consider the following diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & M/\text{Ker } \beta & & \\
 & & \uparrow \varphi & \searrow \eta & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow i & & \\
 & & \text{Ker } \beta & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \quad (3.4)$$

where $\Psi(m') = \alpha(m')$ for all $m' \in M'$, $i(x) = x$ for all $x \in \text{Ker } \beta$, $\varphi(m) = m/\text{Ker } \beta$ for all $m \in M$, and $\eta(m/\text{Ker } \beta) = \beta(m)$ for all $m/\text{Ker } \beta \in M/\text{Ker } \beta$. Let $m_1/\text{Ker } \beta = m_2/\text{Ker } \beta$, then $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \text{Ker } \beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore η is well defined. Now if $\beta(m_1) = \beta(m_2)$, and since β is k -regular, we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \text{Ker } \beta$.

Hence $m_1/\text{Ker } \beta = m_2/\text{Ker } \beta$, therefore η is injective. Clearly the sequence $0 \rightarrow \text{Ker } \beta \xrightarrow{i} M \xrightarrow{\varphi} M/\text{Ker } \beta \rightarrow 0$ is proper exact sequence. Thus diagram (3.4) is a commutative diagram in which the nonhorizontal sequences are all proper exact.

Conversely, see [Corollary 3.2](#). $\square$

COROLLARY 3.4. *Let R be a semiring and let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be a sequence of R -semimodules with β being k -regular. Then the sequence is exact if and only if there exists a commutative diagram*

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & N & & \\
 & & \uparrow & \searrow \eta & \\
 & & \varphi & & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow & & \\
 & & K & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \tag{3.5}$$

of R -semimodules in which the nonhorizontal sequences are all exact.

PROOF. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be an exact sequence with β being k -regular. Consider the following diagram:

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & M/\text{Im } \alpha & & \\
 & & \uparrow & \searrow \eta & \\
 & & \varphi & & \\
 M' & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & M'' \\
 & \searrow \Psi & \uparrow & & \\
 & & \text{Im } \alpha & & \\
 & & \uparrow & \searrow & \\
 & & 0 & & 0
 \end{array} \tag{3.6}$$

where $\Psi(m') = \alpha(m') \in \text{Im } \alpha$ for all $m' \in M'$, $i(x) = x$ for all $x \in \text{Im } \alpha$, $\varphi(m) = m/\text{Im } \alpha$ for all $m \in M$, and $\eta(m/\text{Im } \alpha) = \beta(m)$ for all $m/\text{Im } \alpha \in M/\text{Im } \alpha$. Let $m_1/\text{Im } \alpha = m_2/\text{Im } \alpha$, then $m_1 + t_1 = m_2 + t_2$, $t_1, t_2 \in \text{Im } \alpha = \text{Ker } \beta$. Hence $\beta(m_1) = \beta(m_2)$, therefore η is well defined. Now if $\beta(m_1) = \beta(m_2)$, then since β is k -regular we have $m_1 + k_1 = m_2 + k_2$, $k_1, k_2 \in \text{Ker } \beta = \text{Im } \alpha$. Hence $m_1/\text{Im } \alpha = m_2/\text{Im } \alpha$, therefore η is

injective. Clearly, the sequence $0 \rightarrow \text{Im } \alpha \xrightarrow{i} M \xrightarrow{\varphi} M/\text{Im } \alpha \rightarrow 0$ is exact. Thus diagram (3.6) is a commutative diagram in which the nonhorizontal sequences are all exact.

Conversely, see Lemma 3.1. $\square$

THEOREM 3.5. *The following statements about left R -semimodule P are equivalent:*

- (i) P is projective;
- (ii) for every proper exact sequence of left R -semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with β being k -regular the sequence

$$\text{Hom}(P, M') \xrightarrow{\tilde{\alpha}} \text{Hom}(P, M) \xrightarrow{\tilde{\beta}} \text{Hom}(P, M'') \quad (3.7)$$

is proper exact.

PROOF. (ii) $\Rightarrow$ (i). Let $\alpha : N \rightarrow M$ be surjective homomorphism. Since $N \xrightarrow{\alpha} M \rightarrow 0$ is proper exact sequence with $M \rightarrow 0$ being regular, then by (ii) $\text{Hom}(P, N) \rightarrow \text{Hom}(P, M) \rightarrow 0$ is proper exact. Therefore P is projective.

(i) $\Rightarrow$ (ii). Let P be a projective semimodule. Suppose that $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is proper exact with β being k -regular. Consider the sequence $0 \rightarrow \text{Ker } \beta \xrightarrow{i} M \xrightarrow{\pi} M/\text{Ker } \beta \rightarrow 0$, where $\pi(m) = m/\text{Ker } \beta$ is the canonical surjection. Clearly i is injective. Since P is projective, and using [1, Theorem 10], the sequence

$$0 \rightarrow \text{Hom}_R(P, \text{Ker } \beta) \xrightarrow{\tilde{i}} \text{Hom}(P, M) \xrightarrow{\tilde{\pi}} \text{Hom}(P, M/\text{Ker } \beta) \rightarrow 0 \quad (3.8)$$

is proper exact sequence. Define $\Psi : M' \rightarrow \text{Ker } \beta$ and $\eta : M/\text{Ker } \beta \rightarrow M''$, where $\Psi(m') = \alpha(m')$ and $\eta(m/\text{Ker } \beta) = \beta(m)$. Let $\eta(m/\text{Ker } \beta) = \eta(m'/\text{Ker } \beta)$, then $\beta(m) = \beta(m')$. Since β is k -regular, then $m + k = m' + k'$, where $k, k' \in \text{Ker } \beta$. Hence $m/\text{Ker } \beta = m'/\text{Ker } \beta$. Therefore η is injective. Now consider the commutative diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ & & \text{Hom}_R(P, M/\text{Ker } \beta) & & \\ & \nearrow & \uparrow \tilde{\pi} & \searrow \tilde{\eta} & \\ \text{Hom}_R(P, M') & \xrightarrow{\tilde{\alpha}} & \text{Hom}_R(P, M) & \xrightarrow{\tilde{\beta}} & \text{Hom}_R(P, M'') \\ & \searrow \tilde{\Psi} & \uparrow \tilde{i} & & \\ & & \text{Hom}_R(P, \text{Ker } \beta) & & \\ & & \uparrow & & \\ & & 0 & & \end{array} \quad (3.9)$$

where $\tilde{\Psi}(\xi) = \Psi\xi$ and $\tilde{\eta}(y) = \eta y$, $\xi \in \text{Hom}_R(P, M')$ and $y \in \text{Hom}_R(P, M/\text{Ker } \beta)$. Now let $\xi, y \in \text{Hom}_R(P, M/\text{Ker } \beta)$ such that $\tilde{\eta}(\xi) = \tilde{\eta}(y)$. Since η is injective, then $\xi = y$. Let $\xi \in \text{Hom}(P, \text{Ker } \beta)$. Since P is projective, then there exist $\theta : P \rightarrow M'$ such that the

following diagram is commutative:

$$\begin{array}{ccccc}
 & & P & & \\
 & \swarrow \theta & \downarrow \xi & & \\
 M' & \xrightarrow{\Psi} & \text{Ker } \beta & \longrightarrow & 0
 \end{array} \quad (3.10)$$

Thus $\tilde{\Psi}$ is surjective. Thus the nonhorizontal sequences are all proper exact. Using [Corollary 3.2](#), the sequence

$$\text{Hom}_R(P, M') \xrightarrow{\tilde{\alpha}} \text{Hom}_R(P, M) \xrightarrow{\tilde{\beta}} \text{Hom}_R(P, M'') \quad (3.11)$$

is proper exact. $\square$

COROLLARY 3.6. *The following statement about left R -semimodule P are equivalent:*

- (i) P is projective;
- (ii) for every proper exact sequence of left R -semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with β being regular, the sequence

$$\text{Hom}(P, M') \xrightarrow{\tilde{\alpha}} \text{Hom}(P, M) \xrightarrow{\tilde{\beta}} \text{Hom}(P, M'') \quad (3.12)$$

is proper exact.

PROOF. It is a consequence of [Theorem 3.5](#). $\square$

THEOREM 3.7. *The following statements about left R -semimodule P are equivalent:*

- (i) P is k -projective;
- (ii) for every proper exact sequence of left R -semimodules $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ with both α and β being k -regular, the sequence

$$\text{Hom}(P, M') \xrightarrow{\tilde{\alpha}} \text{Hom}(P, M) \xrightarrow{\tilde{\beta}} \text{Hom}(P, M'') \quad (3.13)$$

is proper exact.

PROOF. (ii) $\Rightarrow$ (i). Let $\alpha : N \rightarrow M \rightarrow 0$ be k -surjective homomorphism. Since $N \rightarrow M \rightarrow 0$ is proper exact sequence with α being k -regular, then by (ii) $\text{Hom}(P, N) \rightarrow \text{Hom}(P, M) \rightarrow 0$ is proper exact. Therefore P is k -projective.

(i) $\Rightarrow$ (ii). It is similar to the proof of [Theorem 3.5](#) and using [\[2, Theorem 8\]](#). $\square$

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REFERENCES

- [1] H. M. J. Al-Thani, *A note on projective semimodules*, Kobe J. Math. **12** (1995), no. 2, 89–94.
- [2] ———, *k -projective semimodules*, Kobe J. Math. **13** (1996), no. 1, 49–59.
- [3] J. S. Golan, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 54, Longman Scientific & Technical, Harlow, 1992.

- [4] M. Takahashi, *On the bordism categories. II. Elementary properties of semimodules*, Math. Sem. Notes Kobe Univ. **9** (1981), no. 2, 495–530.
- [5] ———, *On the bordism categories. III. Functors Hom and $\otimes$ for semimodules*, Math. Sem. Notes Kobe Univ. **10** (1982), no. 1, 211–236.

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