APPLICATION OF UNIFORM ASYMPTOTICS TO THE FIFTH PAINLEVÉ TRANSCENDANT

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We apply the uniform asymptotics method to the fifth Painlevé transcendents, find its asymptotics of the form $y = -1 + t^{-1/2}A(t)$ as $t \rightarrow \infty$ along the positive t -axis, and obtain the corresponding monodromy data.

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1. Introduction. We study the general fifth Painlevé equation

$$\begin{aligned} \frac{d^2 y}{dt^2} = & \left(\frac{1}{2y} + \frac{1}{y-1} \right) \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} \\ & + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right) + \frac{y y'}{t} + \frac{\delta y(y+1)}{y-1}, \end{aligned} \quad (1.1)$$

where α, β, γ , and δ are parameters, and its solution of the form

$$\begin{aligned} y(t) &= -1 + 4t^{-1/2}A(t), \\ y'(t) &= -2t^{-3/2}A(t) + 4t^{-1/2}A'(t) = 4t^{-1/2}A'(t) + O(t^{-3/2}), \end{aligned} \quad (1.2)$$

with $A(t) = O(1)$ as $t \rightarrow \infty$.

The fifth Painlevé equation (1.1) can be obtained as the compatibility condition of the following linear systems of equations (see [2, 3]):

$$Y'_z(z) = \begin{pmatrix} \frac{t}{2} + \frac{2v + \theta_0}{2z} - \frac{w}{z-1} & -\frac{u(v + \theta_0)}{z} + \frac{uy(2w - \theta_1)}{2(z-1)} \\ \frac{v}{uz} - \frac{2w + \theta_1}{2uy(z-1)} & -\frac{t}{2} - \frac{2v + \theta_0}{2z} + \frac{w}{z-1} \end{pmatrix} Y(z), \quad (1.3)$$

$$Y'_t(z) = \begin{pmatrix} \frac{1}{2} & \frac{u}{z} \left[v + \theta_0 - y \left(w - \frac{\theta_1}{2} \right) \right] \\ \frac{1}{uz} \left[v - \frac{1}{y} \left(w + \frac{\theta_1}{2} \right) \right] & -\frac{1}{2} \end{pmatrix} Y(z), \quad (1.4)$$

where

$$w = v + \frac{1}{2}(\theta_0 + \theta_\infty), \quad (1.5)$$

$$\begin{aligned} t \frac{dy}{dt} &= ty - 2v(y-1)^2 - \frac{1}{2}(y-1)[(\theta_0 - \theta_1 + \theta_\infty)y - (3\theta_0 + \theta_1 + \theta_\infty)], \\ t \frac{du}{dt} &= u \left\{ -2v - \theta_0 + y \left[v + \frac{1}{2}(\theta_0 - \theta_1 + \theta_\infty) \right] + \frac{1}{y} \left[v + \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty) \right] \right\}, \end{aligned} \quad (1.6)$$

with

$$\alpha = \frac{1}{2} \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right)^2, \quad \beta = -\frac{1}{2} \left(\frac{\theta_0 - \theta_1 - \theta_\infty}{2} \right)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad \delta = -\frac{1}{2}. \quad (1.7)$$

The canonical solutions of system (1.3) are defined in [2] by

$$\begin{aligned} -\frac{3\pi}{2} + k\pi &\leq \arg \lambda < -\frac{\pi}{2} + k\pi, \\ Y_k(\lambda) &\sim \hat{Y}_\infty(\lambda) e^{(t\lambda/2 - \log \lambda)\sigma_3}, \end{aligned} \quad (1.8)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the Stokes multiplier G_1 is defined in [2] by

$$Y_2(\lambda) = Y_1(\lambda)G_1, \quad (1.9)$$

where $G_1 = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ and its entry s is independent of t and γ .

2. Reduction of the problem. Generally, if $(d/dz) \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, $\phi = B^{-1/2}Y_1$, and $\psi = C^{-1/2}Y_2$, then

$$\begin{aligned} \frac{d^2\phi}{dz^2} &= \left(A^2 + BC + A' - B'B^{-1}A + \frac{3}{4}B^{-2}B'^2 - \frac{1}{2}B^{-1}B'' \right) \phi, \\ \frac{d^2\psi}{dz^2} &= \left(A^2 + BC - A' + C'C^{-1}A + \frac{3}{4}C^{-2}C'^2 - \frac{1}{2}C^{-1}C'' \right) \psi. \end{aligned} \quad (2.1)$$

We first apply the transformation

$$\hat{Y} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} u^{-(1/2)\sigma_3} Y \quad (2.2)$$

to system (1.3) to get

$$\frac{d\hat{Y}}{dz} = \frac{1}{2} \begin{pmatrix} L & N \\ M & -L \end{pmatrix} \hat{Y}, \quad (2.3)$$

where

$$\begin{aligned} L &= \frac{i(2v + \theta_0)}{z} - \frac{i(1/\gamma + \gamma)w + i(1/\gamma - \gamma)(\theta_1/2)}{z-1}, \\ M &= \frac{i(2v + \theta_0) - \theta_0}{z} + \frac{(\gamma - 1/\gamma - 2i)w - (\gamma + 1/\gamma)(\theta_1/2)}{z-1} + it, \\ N &= -\frac{i(2v + \theta_0) + \theta_0}{z} + \frac{(\gamma - 1/\gamma + 2i)w - (\gamma + 1/\gamma)(\theta_1/2)}{z-1} - it. \end{aligned} \quad (2.4)$$

Applying (2.1) to (2.3), we get

$$\begin{aligned}
 \frac{d^2\phi}{dz^2} = & \left\{ \left(\frac{t}{2} + \frac{2v+\theta_0}{2z} - \frac{w}{z-1} \right)^2 + \left[-\frac{u(v+\theta_0)}{z} + \frac{uy(2w-\theta_1)}{2(z-1)} \right] \left[\frac{v}{uz} - \frac{2w+\theta_1}{2uy(z-1)} \right] \right. \\
 & - \frac{i(2v+\theta_0)}{2z^2} + \frac{i\left(\frac{1}{y}+y\right)w + i\left(\frac{1}{y}-y\right)\frac{\theta_1}{2}}{2(z-1)^2} \\
 & - \left[\frac{2iv + (i+1)\theta_0}{z^2} - \frac{\left(y - \frac{1}{y} + 2i\right)w - \left(y + \frac{1}{y}\right)\frac{\theta_1}{2}}{(z-1)^2} \right] \frac{L}{2} \\
 & - \frac{N}{N} \\
 & + \frac{3}{4} \left[\frac{2iv + (i+1)\theta_0}{z^2} - \frac{\left(y - \frac{1}{y} + 2i\right)w - \left(y + \frac{1}{y}\right)\frac{\theta_1}{2}}{(z-1)^2} \right] \frac{L}{N^2} \\
 & \left. - \frac{-\frac{2iv + (i+1)\theta_0}{z^3} + \left(y - \frac{1}{y} + 2i\right)w - \left(y + \frac{1}{y}\right)\frac{\theta_1}{2}}{N} \right\} \phi.
 \end{aligned} \tag{2.5}$$

Now, using (1.6), the following asymptotics can be obtained:

$$\begin{aligned}
 v(t) &= -\frac{1}{8}t - \frac{1}{2}t^{1/2}A'(t) + \frac{1}{2}A^2(t) - 2A(t)A'(t) - \frac{1}{2}\theta_0 - \frac{1}{4}\theta_\infty + O(t^{-1/2}), \\
 u(t) &= Ce^{t/2}(1 + O(t^{-1/2})).
 \end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5), we get the following second-order equation:

$$\begin{aligned}
 \frac{d^2\phi}{dz^2} &= -t^2 \left\{ -\frac{(2z-1)^2}{16z(z-1)} - t^{-1} \left[-\frac{(2z-1)\theta_\infty}{4z(z-1)} - \frac{A^2 + 4A'^2}{4z(z-1)} + \frac{i(2z^2 - 2z + 1)}{8z^2(z-1)^2} \right. \right. \\
 &\quad \left. \left. + \frac{i(2z-1)[(it/4)(2z-1) + t^{1/2}Az^2]}{8z^2(z-1)^2[-it^{1/2}A' - t^{1/2}Az - it(z-1/2)^2]} \right] + O(t^{-3/2}) \right\} \phi \\
 &= -t^2 F(z, t) \phi.
 \end{aligned} \tag{2.7}$$

Equation (2.7) has two turning points

$$z_j = \frac{1}{2} \pm t^{-1/2} \sqrt{A^2 + 4A'^2 + i}(1 + o(1)), \quad j = 1, 2 \tag{2.8}$$

which merge to $1/2$ as $t \rightarrow \infty$, and Stokes directions

$$\operatorname{Re} \left(\sqrt{z(z-1)} \right) = 0. \quad (2.9)$$

Now, we define a constant α by

$$\frac{1}{2} \pi i \alpha^2 = \int_{-\alpha}^{\alpha} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{z_1}^{z_2} F^{1/2}(z, t) dz \quad (2.10)$$

and a new variable ζ by

$$\int_{\alpha}^{\zeta} (\tau^2 - \alpha^2)^{1/2} d\tau = \int_{z_1}^z F^{1/2}(s, t) ds. \quad (2.11)$$

Using [1, Theorem 1], we have the following theorem.

THEOREM 2.1. *Given any solution ϕ of (2.7), there exist constants c_1 and c_2 such that, uniformly for z on the Stokes curve, as $t \rightarrow \infty$,*

$$\left(\frac{\zeta^2 - \alpha^2}{F(z, t)} \right)^{-1/4} \phi(z, t) = \left\{ (c_1 + o(1)) D_{\nu} \left(e^{\pi i/4} \sqrt{2t} \zeta \right) + (c_2 + o(1)) D_{-\nu-1} \left(e^{-\pi i/4} \sqrt{2t} \zeta \right) \right\}, \quad (2.12)$$

where $\nu = -1/2 + (1/2)it\alpha^2$ and $D_{\nu}(z)$, $D_{-\nu-1}(z)$ are solutions of the parabolic cylinder equation.

3. Monodromy data and asymptotics

THEOREM 3.1. *For large t and z ,*

$$\begin{aligned} & \frac{1}{2} \zeta^2 - \frac{A^2 + 4A'^2 + i}{2t} \log \zeta + o(t^{-1}) \\ &= \frac{iz}{2} - \frac{i\theta_{\infty}}{2t} \log(4z) + \frac{i}{2t} \log \frac{2it^{1/2}}{2iA' + A} + \frac{1-i}{4} - \frac{\pi\theta_{\infty}}{4t} \\ & \quad - \frac{\pi i \alpha^2}{4} + o(t^{-1}) + O(z^{-1}). \end{aligned} \quad (3.1)$$

PROOF. Carrying out the integration on the left-hand side of (2.11), we have

$$\frac{1}{2} \zeta^2 - \frac{\alpha^2}{2} \log(2\zeta) - \frac{\alpha^2}{4} + \frac{\alpha^2}{2} \log \alpha + O(\alpha^4 \zeta^{-2}) = \int_{z_1}^z F^{1/2}(s, t) ds. \quad (3.2)$$

Because we are going to calculate the higher-order part of the right-hand side, we will simply ignore the lower-order part in $F(z, t)$, and split the right-hand side into two integrals

$$\int_{z_1}^z F^{1/2}(s, t) ds = \left(\int_{z_1}^{z^*} + \int_{z^*}^z \right) F^{1/2}(x, t) dx = I_1 + I_2, \quad (3.3)$$

where $z^* = 1/2 + Tt^{-1/2}$ and T is a large parameter to be specified later. Using the substitution

$$x - \frac{1}{2} = st^{-1/2}, \quad (3.4)$$

I_1 can be evaluated as follows:

$$\begin{aligned} I_1 &= \frac{1}{t} \int_{\sqrt{A^2+4A'^2}}^T \left(\sqrt{s^2 - (A^2 + 4A'^2 + i)} + o(1) \right) ds \\ &= \frac{T^2}{2t} - \frac{A^2 + 4A'^2 + i}{4t} - \frac{A^2 + 4A'^2 + i}{2t} \log(2T) \\ &\quad + \frac{A^2 + 4A'^2 + i}{4t} \log(A^2 + 4A'^2 + i) + o(t^{-1}). \end{aligned} \quad (3.5)$$

Using the formula

$$\begin{aligned} &\int \frac{2ax + b}{(ax^2 + bx + c)\sqrt{x^2 - \frac{1}{4}}} dx \\ &= 2a \frac{\left(\operatorname{arctanh} \frac{a + 2bx - 2\sqrt{(b^2 - 4ac)}x}{\sqrt{(2b^2 - 2b\sqrt{(b^2 - 4ac)} - 4ac - a^2)}\sqrt{(4x^2 - 1)}} \right)}{\sqrt{(2b^2 - 2b\sqrt{(b^2 - 4ac)} - 4ac - a^2)}} \\ &\quad + 2a \frac{\operatorname{arctanh} \frac{a + 2bx + 2\sqrt{(b^2 - 4ac)}x}{\sqrt{(2b^2 + 2b\sqrt{(b^2 - 4ac)} - 4ac - a^2)}\sqrt{(4x^2 - 1)}}}{\sqrt{(2b^2 + 2b\sqrt{(b^2 - 4ac)} - 4ac - a^2)}}, \end{aligned} \quad (3.6)$$

we find the asymptotic expression of I_2

$$\begin{aligned} I_2 &= \int_{z^*}^z \left\{ -\frac{(2s-1)^2}{16s(s-1)} - t^{-1} \left[-\frac{(2s-1)\theta_\infty}{4s(s-1)} - \frac{A^2 + 4A'^2}{4s(s-1)} + \frac{i(2s^2 - 2s + 1)}{8s^2(s-1)^2} \right. \right. \\ &\quad \left. \left. - \frac{i(2s-1)[(1/2)it^{1/2}(s-1/2) + As^2]}{8z^2(s-1)^2[iA' + As + it^{1/2}(s-1/2)^2]} \right] \right\}^{1/2} ds \\ &= \int_{z^*}^z \frac{i(2s-1)}{4\sqrt{s(s-1)}} \left\{ 1 + \frac{16s(s-1)}{t(2s-1)^2} \left[-\frac{(2s-1)\theta_\infty}{4s(s-1)} - \frac{A^2 + 4A'^2}{4s(s-1)} + \frac{i(2s^2 - 2s + 1)}{8s^2(s-1)^2} \right. \right. \\ &\quad \left. \left. + \frac{i(2s-1)[(it/4)(2s-1) + t^{1/2}As^2]}{8s^2(s-1)^2[-it/4 - t^{1/2}A' - t^{1/2}As - its(s-1)]} \right] \right\}^{1/2} ds \end{aligned}$$

$$\begin{aligned}
&= \int_{z^*}^z \left[\frac{i(2s-1)}{4\sqrt{s(s-1)}} - \frac{i\theta_\infty}{2t\sqrt{s(s-1)}} - \frac{i(A^2+4A'^2+i)}{2t(2s-1)\sqrt{s(s-1)}} \right. \\
&\quad \left. - \frac{2it^{1/2}(s-1/2)+A}{4t\sqrt{(s-1/2)^2-1/4}[iA'+(1/2)A+A(s-1/2)+it^{1/2}(s-1/2)^2]} \right] dz \\
&\quad + O\left(t^{-2} \int_{z^*}^z \frac{|ds|}{|2s-1|^3}\right) + O\left(\frac{1}{z}\right) \\
&= \frac{iz}{2} - \frac{i}{4} - \frac{i\theta_\infty}{2t} \log(4z) - \frac{i\pi(A^2+4A'^2+i)}{4t} + \frac{1}{4} - \frac{T^2}{2t} - \frac{\pi\theta_\infty}{4t} \\
&\quad - \frac{A^2+4A'^2+i}{4t} \log \frac{t}{T^2} + \frac{i}{2t} \log \frac{2it^{1/2}}{2iA'+A+O(t^{-1/4})} + O(z^{-1}) \\
&\quad + O(T^{-2}t^{-1}) + O(T^4t^{-2}).
\end{aligned} \tag{3.7}$$

Using definition (2.10) and setting $T = -\sqrt{A^2+4A'^2}$ in I_1 , we have the following expression for α :

$$\alpha^2 = -\frac{A^2+4A'^2+i}{t} + o(t^{-1}). \tag{3.8}$$

Substituting (3.8) into (3.2), setting $T < t^{1/4}$, and combining it with (3.5) and (3.7), the theorem is proved. $\square$

Knowing [4] that

$$D_\nu(z) \sim \begin{cases} z^\nu e^{-(1/4)z^2}, & \text{if } |\arg z| < \frac{3}{4}\pi, \\ z^\nu e^{-(1/4)z^2} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{(1/4)z^2}, & \text{if } \arg z = \frac{3}{4}\pi, \end{cases} \tag{3.9}$$

and $\arg(e^{\pi i/4} \sqrt{2t}\zeta) \sim \pi/4$ when $z \rightarrow -\infty$, we can choose, as $z \rightarrow -\infty$,

$$\begin{aligned}
\hat{Y}_2^{(11)}(z) &\sim z^{-\theta_\infty/2} e^{tz/2} \\
&\sim 2^{\theta_\infty} t^{1/4} \sqrt{2iA'+A} e^{-(\pi i/4)(1/2+(3it/2)\alpha^2)} e^{(1/4)t+i[t/4-(1/4)(A^2+4A'^2)\log(2t)+\theta_\infty/4]} \\
&\quad \times \zeta^{1/2} D_\nu(e^{\pi i/4} \sqrt{2t}\zeta).
\end{aligned} \tag{3.10}$$

Because $\arg(e^{\pi i/4} \sqrt{2t}\zeta) \sim 3\pi/4$ when $z \rightarrow \infty$, we have the following asymptotics for $\hat{Y}_2^{(11)}(z)$ as $z \rightarrow \infty$:

$$\begin{aligned}
\hat{Y}_2^{(11)}(z) &\sim z^{-\theta_\infty/2} e^{tz/2} \\
&\quad - z^{\theta_\infty/2} e^{-tz/2} \frac{4^{\theta_\infty} \sqrt{\pi} (A^2+2iA'^2)}{\Gamma(1/2-(ti/2)\alpha^2)} e^{(\pi i/4)(2ti\alpha^2-3)+(1/2)t+i[t/2-(1/2)(A+4A'^2)\log(2t)]}.
\end{aligned} \tag{3.11}$$

By (1.9) and (2.2), the Stokes multiplier can be defined by $\hat{Y}_2 = \hat{Y}_1 G_1$. Therefore, the monodromy data is

$$s = \frac{4^{\theta_\infty} \sqrt{\pi} (A + 2iA')}{\Gamma(1/2 - (ti/2)\alpha^2)} e^{(\pi i/4)(2ti\alpha^2 - 5) + i[t/2 - (1/2)(A + 4A'^2)\log(2t)]}. \quad (3.12)$$

Taking the square of the absolute value of both sides of this equation, we find that $A^2 + 4A'^2 \sim d^2$ where d is a constant. Solving (3.12) for $A + 2iA'$, we have

$$A + 2iA' = \left(\pm \sqrt{(A^2 + 4A'^2)} + O(t^{-1/2}) \right) e^{i[t/2 - (1/2)(A + 4A'^2)\log(2t) + \theta]}. \quad (3.13)$$

Taking the real part and the imaginary part of (3.13), we obtain the following theorem.

THEOREM 3.2. *Equation (1.1) has a solution with the following asymptotics:*

$$\begin{aligned} y(t) &\sim -1 + 4t^{-1/2}(d + O(t^{-1/2})) \cos\left(\frac{t}{2} - \frac{1}{2}d^2 \log(2t) + \theta\right), \\ y'(t) &\sim 2t^{-1/2}(d + O(t^{-1/2})) \sin\left(\frac{t}{2} - \frac{1}{2}d^2 \log(2t) + \theta\right), \quad \text{as } t \rightarrow \infty. \end{aligned} \quad (3.14)$$

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