ON INCIDENCE ALGEBRAS AND DIRECTED GRAPHS

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The incidence algebra $I(X, \mathbb{R})$ of a locally finite poset $(X, \leq)$ has been defined and studied by Spiegel and O'Donnell (1997). A poset $(V, \leq)$ has a directed graph $(G_v, \leq)$ representing it. Conversely, any directed graph G without any cycle, multiple edges, and loops is represented by a partially ordered set V_G . So in this paper, we define an incidence algebra $I(G, \mathbb{Z})$ for $(G, \leq)$ over $\mathbb{Z}$, the ring of integers, by $I(G, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}\}$ where $f_i(u, v)$ denotes the number of directed paths of length i from u to v and $f_i^*(u, v) = -f_i(u, v)$. When G is finite of order n , $I(G, \mathbb{Z})$ is isomorphic to a subring of $M_n(\mathbb{Z})$. Principal ideals I_v of $(V, \leq)$ induce the subdigraphs $\langle I_v \rangle$ which are the principal ideals $\mathcal{P}_v$ of $(G_v, \leq)$. They generate the ideals $I(\mathcal{P}_v, \mathbb{Z})$ of $I(G, \mathbb{Z})$. These results are extended to the incidence algebra of the digraph representing a locally finite weak poset both bounded and unbounded.

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1. Introduction. The incidence algebra $I(X, \mathbb{R})$ of a locally finite poset X over a commutative ring $\mathbb{R}$ with identity is discussed in much detail by Spiegel and O'Donnell [5]. Every poset $(V, \leq)$ has a directed graph $(G_v, \leq)$, or $(G, \leq)$ for short, representing it. The directed graph $(G, \leq)$ is free of cycles and multiple arcs. It is natural for $(G, \leq)$ to have an incidence algebra whose properties depend on those of the directed graph.

In this paper, we define and study an incidence algebra $I(G, \mathbb{Z})$ for $(G, \leq)$ over the ring $\mathbb{Z}$ of all integers. Section 2 contains the basic results and definitions used in this paper. Section 3 deals with $I(G, \mathbb{Z})$, the incidence algebra of $(G, \leq)$ representing a finite poset $(V, \leq)$ of cardinality n . Ideals of $I(G, \mathbb{Z})$ are given through the principal ideals of $(G, \leq)$ which are the subdigraphs induced by the principal ideals of $(V, \leq)$. An extension of the results given in Section 3 to the digraph representing a locally finite weak poset $(V, \leq)$ bounded or unbounded is the content of Section 4.

2. Definitions and basic results. We consider the directed graph $(G_v, \leq)$ representing a locally finite partially ordered set $(V, \leq)$. The terminology introduced here is used throughout the paper.

DEFINITION 2.1. For any $u \leq v$ in $(V, \leq)$, $[u, v] = \{w : u \leq w \leq v\}$ is an interval of $(V, \leq)$. The length of $[u, v]$ denoted by $l[u, v]$ is the length of the longest chain in $[u, v]$. A poset $(V, \leq)$ is locally finite if $l[u, v]$ is finite for each $[u, v]$ in $(V, \leq)$; and $(V, \leq)$ is bounded if there is a $k > 0$ such that $l[u, v] \leq k$ for all $[u, v]$ in $(V, \leq)$. Otherwise $(V, \leq)$ is unbounded.

DEFINITION 2.2. The directed graph $(G_v, \leq)$ associated with a locally finite poset $(V, \leq)$ is defined as $(G_v, \leq) = (V, E)$ where $V = (V, \leq)$ and $E = \{\text{arcs } \vec{uv}; u < v\}$; $(G_v, \leq)$ has no cycles nor multiple arcs.

LEMMA 2.3 (see [5]). *A finite poset $(V, \leq)$ can be labelled $V = \{v_1, v_2, \dots, v_n\}$ so that $v_i \leq v_j$ implies that $i \leq j$.*

PROPOSITION 2.4 (see [4]). *The vertices of a finite directed graph $G = (V, E)$ can be labelled $v_1, v_2, \dots, v_n$ such that $v_i v_j \in E$ implies that $i < j$, if and only if G has no cycles.*

NOTE 2.5. Lemma 2.3 and Proposition 2.4 have motivated the orientation of arcs in $(G_v, \leq)$.

DEFINITION 2.6 (see [5]). Let $(X, \leq)$ be a locally finite poset and $\mathbb{R}$ a commutative ring with identity. The incidence algebra $I(X, \mathbb{R})$ of X over $\mathbb{R}$ is defined by

$$I(X, \mathbb{R}) = \{f : X \times X \rightarrow \mathbb{R} : f(x, y) = 0 \text{ if } x \not\leq y\}, \quad (2.1)$$

with operations defined by

- (i) $(f + g)(x, y) = f(x, y) + g(x, y)$;
- (ii) $(f \cdot g)(x, y) = \sum_{x \leq u \leq y} f(x, u)g(u, y)$;
- (iii) $(rf)(x, y) = rf(x, y)$, for all $r \in \mathbb{R}$; $f, g \in I(X, \mathbb{R})$.

DEFINITION 2.7. For any $v \in (V, \leq)$, let $I_v = \{u \in V : u \leq v\}$; I_v is called the principal ideal generated by v .

DEFINITION 2.8. An ideal $\mathcal{I}$ of $(G_v, \leq)$ is an induced subdigraph of G_v such that all directed paths with their terminal vertex in $\mathcal{I}$ are in $\mathcal{I}$.

If I_v is a principal ideal of $(V, \leq)$, $\langle I_v \rangle$, the subdigraph induced by I_v is the principal ideal of G generated by v in $(G_v, \leq)$. Denote $\langle I_v \rangle$ by $\mathcal{I}_v$.

NOTATION 2.9. For the remaining part of the paper, the digraphs $(G, \leq)$ and $(G_\infty, \leq)$ represent the finite poset $(V, \leq)$ and the locally finite poset $(V_\infty, \leq)$, respectively.

3. An incidence algebra for $(G, \leq)$. In this section, we define an incidence algebra $I(G, \mathbb{Z})$ for the digraph $(G, \leq)$ representing the finite poset $(V, \leq)$. Subalgebras and ideals of $I(G, \mathbb{Z})$ are defined through principal ideals of $(G, \leq)$. Assume that $V = \{v_1, v_2, \dots, v_n\}$.

DEFINITION 3.1. For $u, v \in V$, let $p_k(u, v)$ denote the number of directed paths of length k from u to v and $p_k(v, u) = -p_k(u, v)$.

For $i = 0, 1, 2, \dots, n-1$, define $f_i, f_i^* : V \times V \rightarrow \mathbb{Z}$ by

$$f_i(u, v) = p_i(u, v), \quad f_i^*(u, v) = -p_i(u, v). \quad (3.1)$$

The incidence algebra $I(G, \mathbb{Z})$ of $(G, \leq)$ over the commutative ring $\mathbb{Z}$ with identity is defined by $I(G, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}, i = 0, 1, 2, \dots, n-1\}$ with operations defined as follows:

- (i) for $f \neq g$, $(f + g)(u, v) = f(u, v) + g(u, v)$;

- (ii) $(f \cdot g)(u, v) = \sum_w f(u, w)g(w, v)$;
- (iii) $(zf)(u, v) = zf(u, v)$, for all $z \in \mathbb{Z}$; $f, g \in I(G, \mathbb{Z})$.

REMARK 3.2. The function f_l is the digraph analogues of $\chi \in I(X, \mathbb{R})$ [5]. The matrix $[f_l, (v_i, v_j)]$ is the adjacency matrix of $(G, \leq)$ and $f_l^k(v_i, v_j) = f_k(v_i, v_j)$. For any interval $[u, v]$ with $l[u, v] = k$, $f_l^k(u, v) = f_k(u, v) = 0$. For every $f \in I(G, \mathbb{Z})$, there is a constant $m \in \mathbb{Z}$ such that $f^m(u, v) = 0$, for all $(u, v) \in V \times V$.

DEFINITION 3.3. With each ideal $\mathcal{I}_v = \langle I_v \rangle$ of $(G, \leq)$ we associate an incidence algebra,

$$\begin{aligned} I(\mathcal{I}_v, \mathbb{Z}) &= \{f \in I(G, \mathbb{Z}) : f : I_v \times I_v \rightarrow \mathbb{Z}\}, \\ \forall f \in I(\mathcal{I}_v, \mathbb{Z}), f(v_i, v_j) &= 0 \quad \forall (v_i, v_j) \notin I_v \times I_v. \end{aligned} \quad (3.2)$$

REMARK 3.4. If $(H, \leq)$ is a subdigraph of $(G, \leq)$, then $I(H, \mathbb{Z})$ is a subalgebra of $I(G, \mathbb{Z})$ [5]. In particular, $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G, \mathbb{Z})$; $I(\mathcal{I}_v, \mathbb{Z})$ is called the subalgebra generated by the vertex v .

REMARK 3.5 (see [5]). If S is an ideal of $\mathbb{Z}$, $I(G, S) = \{f \in I(G, \mathbb{Z}) : f(u, v) \in S\}$ is a subalgebra of $I(G, \mathbb{Z})$.

PROPOSITION 3.6. For each principal ideal $\mathcal{I}_v$ of $(G, \leq)$, $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of the ring $I(G, \mathbb{Z})$.

PROOF. Let $\mathcal{I}_v$ be a proper principal ideal of $(G, \leq)$. Denote the elements of $I(G, \mathbb{Z})$ and $I(\mathcal{I}_v, \mathbb{Z})$ by f and f' , respectively. For all $f' \in I(\mathcal{I}_v, \mathbb{Z})$, there is a unique $f \in I(G, \mathbb{Z})$ such that $f'(u, w) = f(u, w)$ for all $(u, w) \in I_v \times I_v$; and $f'(u, w) = 0$ for all $(u, w) \notin I_v \times I_v$. $\square$

Hence for any $(u, w) \in V \times V$, $f, g \in I(G, \mathbb{Z})$, $g' \in I(\mathcal{I}_v, \mathbb{Z})$

$$(fg')(u, w) = \begin{cases} (f, g)(u, w), & \text{if } (u, w) \in I_v \times I_v, \\ 0, & \text{otherwise,} \end{cases} \quad (3.3)$$

with similar values for $(g'f)(u, w)$ also.

Consequently, fg' and $g'f \in I(\mathcal{I}_v, \mathbb{Z})$, and $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G, \mathbb{Z})$.

REMARK 3.7. The incidence algebra $I(G, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over $\mathbb{Z}$ [5]. In general, every ideal of $M_n(\mathbb{Z})$ has the form $M_n(S)$ for some ideal S of $\mathbb{Z}$ [3].

PROPOSITION 3.8. Every ideal of $I(G, \mathbb{Z})$ has the form $I(\mathcal{I}_v, \mathbb{Z})$ for some principal ideal $\mathcal{I}_v$ of $(G, \leq)$.

PROOF. Let S be a proper ideal of the ring $I(G, \mathbb{Z})$. Then $S = I(H, \mathbb{Z})$ for some subdigraph $(H, \leq)$ of $(G, \leq)$. For all $f \in I(G, \mathbb{Z})$ and $g' \in S$, $fg' = g'f \in S$.

Consequently, there is an $h' \in S$ such that $fg' = h'$ satisfying $(fg')(u, v) = h'(u, v) = p_k(u, v)$ for some k and for all u, v in $V(H)$.

Then for all $(u, v) \in V(H) \times V(H)$, the number of paths of length k in H from u to v is the same as that in G .

Hence for any $v \in V(H)$, H contains all the directed paths terminating in v . Then $H = \mathcal{F}_v$ for some v . $\square$

4. The incidence algebra $I(G_\infty, \mathbb{Z})$. An extension of [Proposition 2.4](#) is obtained by the author for locally finite directed graphs [2]. This provides an isomorphism between $I(G_\infty, \mathbb{Z})$ and a subring of the ring of upper triangular matrices over $\mathbb{Z}$. Also we have extended [Propositions 3.6](#) and [3.8](#) to bounded as well as unbounded locally finite weak posets. It is assumed that the poset $(V, \leq)$ is countable.

DEFINITION 4.1. A locally finite poset $(V, \leq)$ is weak if only finitely many chains intersect at every element v in V .

DEFINITION 4.2. A directed graph $G = (V, E)$ is locally finite if every vertex is of a finite degree.

DEFINITION 4.3. A vertex v of the directed graph $(G_\infty, \leq)$, such that $d(v) \neq 0$, is called a source or atom if there is no other vertex u such that $u \leq v$.

PROPOSITION 4.4 (see [2]). *Let $G = (V, E)$ be a locally finite, acyclic digraph where V is countable; V can be labelled $\{v_1, v_2, v_3, \dots\}$ such that $v_i \vec{v}_j \in E$ implies that $i < j$ if and only if*

- (i) *the set S of its sources is nonempty and finite;*
- (ii) *for each $v \in V$ and $s \in S$ such that $s \vec{v} \in E$, any maximal directed path terminating in v is of finite length.*

DEFINITION 4.5. Let $(V, \leq)$ be a locally finite weak poset and $(G_\infty, \leq)$ the digraph representing $(V, \leq)$. The incidence algebra $I(G_\infty, \mathbb{Z})$ of $(G_\infty, \leq)$ over the ring $\mathbb{Z}$ of integers is given by

$$I(G_\infty, \mathbb{Z}) = \{f_i, f_i^* : V \times V \rightarrow \mathbb{Z}\} \quad (4.1)$$

satisfying the operations given in [Definition 3.1](#).

DEFINITION 4.6. An infinite matrix $A = [a_{ij}]$ is row (column) finite if $a_{ij} \neq 0$ for finitely many $j(i)$.

NOTE 4.7. (i) Each $f \in I(G_\infty, \mathbb{Z})$ is represented by a matrix $[f]$ where $[f]_{i,j} = f(v_i, v_j)$ and $[f]$ is both row and column finite, $M_\infty(\mathbb{Z})$ denotes the ring of row and column finite matrices over $\mathbb{Z}$. The incidence algebra $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of $M_\infty(\mathbb{Z})$.

(ii) A characterization of infinite directed graphs for which $[f]$ is nilpotent is obtained by the author in [1].

PROPOSITION 4.8. *Let $(V, \leq)$ be a weak, locally finite poset satisfying the following:*

- (i) *the set S of its atoms is nonempty and finite;*
- (ii) *for each $v \in V$, $s \in S$ such that $s \leq v$, any chain with v as upperbound is of finite length. Then $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over $\mathbb{Z}$.*

PROOF. The digraph $(G_\infty, \leq)$ is locally finite. By Proposition 4.4, $f(v_r, v_s) \geq 0$ for all $r \leq s$ and for all $f \in I(G_\infty, \mathbb{Z})$. Hence every f has a representation as an upper triangular matrix over $\mathbb{Z}$. $\square$

DEFINITION 4.9. In a bounded weak, locally finite poset $(V, \leq)$ a principal ideal $I_v = \{u : u \leq v\}$. The corresponding principal ideal of $(G_\infty, \leq)$ is given by $\mathcal{I}_v = \langle I_v \rangle$.

PROPOSITION 4.10. Let $(V, \leq)$ be any weak, locally finite bounded poset and $\mathcal{I}_v$ a principal ideal of $(G_\infty, \leq)$. Then

- (1) $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;
- (2) $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.

NOTE 4.11. This follows from Propositions 3.6 and 4.4.

PROPOSITION 4.12 (see [5]). Let $(V, \leq)$ be an unbounded weak poset. Then V contains a subpartially ordered set isomorphic to $\mathbb{Z}^+$, $\mathbb{Z}^-$, or UCn where Cn denotes a chain of length n .

REMARK 4.13. When $(V, \leq)$ is unbounded, the principal ideals of $(V, \leq)$ and $(G_\infty, \leq)$ are not well defined. Hence Proposition 4.4 is not true for unbounded posets, in general.

But there are unbounded, locally finite posets satisfying Proposition 4.4. As an example we have $(V, \leq) = (\mathbb{Z}^+, \text{usual ordering})$.

Let $(V, \leq)$ be an unbounded, locally finite, weak poset such that $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices. Then

- (i) principal ideals of $(G_\infty, \leq)$ are defined as $\mathcal{I}_v = \langle I_v \rangle$;
- (ii) for each $\mathcal{I}_v$, $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;
- (iii) for every $\mathcal{I}_v$, $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.

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