Let $R$ be a commutative ring with nonzero identity. Our objective is to investigate representable modules and to examine in particular when submodules of such modules are representable. Moreover, we establish a connection between the secondary modules and the pure-injective, the $\Sigma$-pure-injective, and the prime modules.

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1. Introduction. In this paper, all rings are commutative rings with identity and all modules are unital. The notion of associated prime ideals and the related one of primary decomposition are classical. In a dual way, we define the attached prime ideals and the secondary representation. This theory is developed in the appendix to Section 6 in Matsumura [6] and in Macdonald [5]. Now we define the concepts that we will need.

Let $R$ be a ring and let $0 \neq M$ be an $R$-module. Then $M$ is called a secondary module (second module) provided that for every element $r$ of $R$ the homothety $M \xrightarrow{r} M$ is either surjective or nilpotent (either surjective or zero). This implies that $\text{nilrad}(M) = P$ $(\text{Ann}(M) = P')$ is a prime ideal of $R$, and $M$ is said to be $P$-secondary ($P'$-second), so every second module is secondary (the concept of second module is introduced by Yassemi [14]). A secondary representation for an $R$-module $M$ is an expression for $M$ as a finite sum of secondary modules (see [5]). If such a representation exists, we will say that $M$ is representable.

If $R$ is a ring and $N$ is a submodule of an $R$-module $M$, the ideal $\{ r \in R : rM \subseteq N \}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of $M$, $\text{Ann}(M)$. A proper submodule $N$ of a module $M$ over a ring $R$ is said to be prime submodule (primary submodule) if for each $r \in R$ the homothety $M/N \xrightarrow{r} M/N$ is either injective or nilpotent (either injective or zero), so $(0 : M/N) = P$ $(\text{nilrad}(M/N) = P')$ is a prime ideal of $R$, and $N$ is said to be $P$-prime submodule ($P'$-primary submodule). So $N$ is prime in $M$ if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. We say that $M$ is a prime module (primary module) if zero submodule of $M$ is prime (primary) submodule of $M$, so $N$ is a prime submodule of $M$ if and only if $M/N$ is a prime module. Moreover, every prime module is primary.

Let $R$ be a ring, and let $N$ be an $R$-submodule of $M$. Then $N$ is pure in $M$ if for any finite system of equations over $N$ which is solvable in $M$, the system is also solvable in $N$. A module is said to be absolutely pure if every embedding of it into any other modules is pure embedding. A submodule $N$ of an $R$-module $M$ is called relatively divisible (or an RD-submodule) if $rN = N \cap rM$ for all $r \in R$. Every RD-submodule of a $P$-secondary module over a commutative ring $R$ is $P$-secondary (see [2, Lemma 2.1]).
A module $M$ is pure-injective if and only if any system of equations in $M$ which is finitely solvable in $M$, has a global solution in $M$ [7, Theorem 2.8]. The module $N$ is a pure-essential extension of $M$ if $M$ is pure in $N$ and for all nonzero submodules $L$ of $N$, if $M \cap L = 0$, then $(M \oplus L)/L$ is not pure in $N/L$. A pure-injective hull $H(M)$ of a module $M$ is a pure essential extension of $M$ which is pure-injective. Every module $M$ has a pure-injective hull which is unique to isomorphism over $M$ [12].

Given an $R$-module $M$ and index set $I$, the direct sum of the family $\{M_i : i \in I\}$ where $M_i = M$ for each $i \in I$ will be denoted by $M^{(I)}$. Given a module property $\mathcal{P}$, we will say that a module $M$ is $\sum_{i \in I} \mathcal{P}$ if $M^{(I)}$ satisfies $\mathcal{P}$ for every index set $I$.

Let $R$ be a commutative ring. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = a^2b$, and $R$ is said to be regular if each of its elements is regular. An important property of regular rings is that every module is absolutely pure (see [13, Theorem 37.6]).

Let $R$ be a ring and $M$ an $R$-module. A prime ideal $P$ of $R$ is called an associated prime ideal of $M$ if $P$ is the annihilator $\text{Ann}(x)$ of some $x \in M$. The set of associated primes of $M$ is written $\text{Ass}(M)$. For undefined terms, we refer to [6, 7].

### 2. Secondary submodules

In general, a nonzero submodule of a representable (even secondary) $R$-module is not representable (secondary), but we have the following results.

**Lemma 2.1.** Let $R$ be a commutative ring and let $0 \neq N$ be an RD-submodule of $R$-module $M$. Then $M$ is $P$-secondary if and only if $N$ and $M/N$ are $P$-secondary.

**Proof.** If $M$ is $P$-secondary, then $N$ and $M/N$ are $P$-secondary by [2, Lemma 2.1] and [5, Theorem 2.4], respectively. Conversely, suppose that $r \in R$. If $r \in P$, then $r^n(M/N) = 0$ and $r^nN = 0$ for some $n$, hence $r^nM \subseteq N$ and $0 = r^nN = r^nM \cap N = r^nM$. If $r \notin P$, then $rM + N = M$, $rN = N$, and $N = rN = rM \cap N$, so we have $rM = M$, as required.

**Corollary 2.2.** Let $R$ be a commutative regular ring, and let $0 \neq N$ be a submodule of $R$-module $M$. Then $M$ is $P$-secondary if and only if $N$ and $M/N$ are $P$-secondary.

**Proof.** This follows from Lemma 2.1.

**Theorem 2.3.** Let $R$ be a commutative regular ring. Then every nonzero submodule of a representable $R$-module is representable.

**Proof.** Let $M$ be a representable $R$-module and let $M = \sum_{i=1}^n M_i$ be a minimal secondary representation with $\text{nilrad}(M_i) = P_i$. There is an element $r_1 \in P_1$ such that $r_1 \notin \cup_{i=2}^n P_i$. Otherwise $P_1 \subseteq \cup_{i=2}^n P_i$, so by [10, Theorem 3.61], $P_1 \subseteq P_j$ for some $j$, and hence $P_1 = P_j$, a contradiction. Thus there exists a positive integer $m_1$ such that $r_1^{m_1} \in \text{Ann}(M_1)$ and the module $r_1^{m_1}M = \sum_{i=2}^n r_1^{m_1}M_i$ is representable. By using this process for the ideals $P_2, \ldots, P_{n-1}$, there are integers $m_2, \ldots, m_{n-1}$ and elements $r_2 \in P_2, \ldots, r_{n-1} \in P_{n-1}$ such that $s_nM = M_n$, where $0 \neq s_n = r_1^{m_1}r_2^{m_2} \cdots r_{n-1}^{m_{n-1}}$, $s_n \in \cap_{i=1}^{n-1} P_i$ and $s_n \notin P_n$. Therefore by a similar argument, there are elements $s_1, \ldots, s_{n-1}$
such that $M = \sum_{i=1}^{n}s_iM$, where for each $i$, where $i = 1, \ldots, n$, $s_i \notin P_i$, $s_iM = M_i$, and $s_i \in \cap_{i=1}^{n} \operatorname{Ann}(M_i)$.

Let $N$ be a nonzero submodule of $M$ and $0 \neq a \in N$. Then $a = s_1b_1 + \cdots + s_nb_n$ for some $b_i \in M$, $i = 1, \ldots, n$. By assumption, there exists $t_1, \ldots, t_n \in R$ such that for each $i$, $s_i = s_i^2t_i$. As $0 \neq a$, $s_i^2b_i \neq 0$ for some $i$ and $s_it_ia = s_i^2t_ib_i = s_is_ib_i$, so $s_iN \neq 0$. We can assume that $s_1N \neq 0, \ldots, s_nN \neq 0$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$. By a similar argument as above, if $a \in N$, then $a = \sum_{j=1}^{k}s_jtija \in \sum_{j=1}^{k}s_jN$, and hence $N = \sum_{j=1}^{k}s_jN$. Since for each $j$, where $j = 1, \ldots, k$, $s_jN$ is pure in the $P_{ij}$-secondary module $M_{ij}$, it is $P_{ij}$-secondary by [2, Lemma 2.1], as required.

**Theorem 2.4.** Let $R$ be a commutative ring and let $N$ be a prime submodule of secondary $R$-module of $M$. Then $N$ is $(N : M)$-secondary.

**Proof.** Suppose that $M$ is a $P$-secondary module over $R$. Let $r \in R$. If $r \notin P$, then $r^nN \subseteq r^nM = 0$ for some $n$. If $r \notin P$, then $rM = M$. Suppose that $n \in N$, so there is an element $m \in M$ such that $n = rm$. As $N$ is a prime submodule of $M$ and $N \neq rM = M$, $m \in N$, so $rN = N$, hence $N$ is $P$-secondary.

By [4, Lemma 1], the ideal $P' = (N : M) = \{r \in R : rM \subseteq N\}$ is prime. Clearly, $P' \subseteq P$. Let $s \in P$. Then $s^nN = s^nM = 0$ for some $n$. There is an element $m \in M$ such that $m \notin N$ and $s^nm = 0 \in N$, so $s^n \in P'$, hence $s \in P'$. Thus $P = P'$, as required.

**Proposition 2.5.** Let $R$ be a commutative ring and let $N$ be a prime submodule of $P$-second $R$-module of $M$. Then $N$ is an RD-submodule of $M$.

**Proof.** Let $r \in R$. If $r \notin P$, then $rN \subseteq rM = 0$, so $rN = N \cap rM = 0$. If $r \notin P$, then $rM = M$, so the homothety $M/N \cong M/N$ is not zero since $N$ is prime. It follows that the above homothety is injective. If $a \in N \cap rM$, then there is $b \in M$ such that $a = rb$. Since $r(b + N) = 0$, so $b \in N$, hence $rN = N \cap rM$, as required.

**Theorem 2.6.** Let $M$ be a $P$-second module over a commutative ring $R$, and let $N$ be a prime submodule of $M$. Then every submodule of $M$ properly containing $N$ is an RD-submodule. In particular, it is $P$-second.

**Proof.** Let $K$ be a submodule of $M$ properly containing $N$. Then $K/N$ is a prime submodule of prime and $P$-second module $M/N$, so by Proposition 2.5, $K/N$ is an RD-submodule of $M/N$. Now the assertion follows from [3, Consequences 18-2.2(c)] and Proposition 2.5.

**Lemma 2.7.** Let $M$ be a nonzero module over a commutative domain $R$. Then $M$ is $(0)$-second if and only if $M$ is $(0)$-secondary.

**Proof.** The proof is completely straightforward.

By [3, Proposition 11-3.11] and [11, Proposition 12, page 506] (see also [14]), and the definitions of secondary and primary modules, we obtain the following corollary.

**Corollary 2.8.** Let $R$ be a commutative ring.

(i) Every Artinian primary module over $R$ is secondary.

(ii) Every Noetherian primary module over $R$ is primary.

(iii) Every finitely generated secondary module is primary.
**Lemma 2.9.** Let $R$ be a commutative ring. Let $K$ and $N$ be submodules of an $R$-module $M$ such that $N$ is prime and $K$ is $P$-secondary. Then $N \cap K$ is $P$-secondary.

**Proof.** Let $r \in R$. If $r \notin P$, then $r^n(N \cap K) \subseteq r^nK = 0$ for some $n$. Suppose $r \notin P$ and $t \in N \cap K$. Then $t = rs$ for some $s \in K$ since $K$ is $P$-secondary. As $N$ is prime, we have $s \notin N$, and hence $t \in r(N \cap K)$. This gives, $N \cap K = r(N \cap K)$. \hfill \Box

**Theorem 2.10.** Let $M$ be a representable module over a commutative ring $R$, and let $N$ be a prime submodule of $M$ with $(N : M) = P$. Then the following hold:

(i) $N$ is representable;

(ii) $M/N$ is $P$-secondary.

**Proof.** (i) Let $M$ be a representable $R$-module and let $M = \sum_{i=1}^m M_i$ be a minimal secondary representation with $\text{nilrad}(M_i) = P_i$. For each $i$, $i = 1, 2, \ldots, m$, let $m_i \in M_i$ and $r_i \in P_i$. Then $r_i^{n_i}m_i = 0$ for some $n_i$, and we have $(r_i^{n_i} + P)(m_i + M_i) = 0$ and hence either $P_i \subseteq P$ or $M_i \subseteq N$ ($i = 1, 2, \ldots, m$). It follows that $M_i \not\subseteq N$ for some $i$ (otherwise $M = N$). If $M_i \not\subseteq N$ and $M_j \not\subseteq N$ for $i \neq j$, then $P = P_i = P_j$, a contradiction (for if $t \in P - P_i$ then $M_i = tM_i \subseteq tM \subseteq N$). Therefore, without loss of generality, we can assume that $M_1 \not\subseteq N$ and $M_i \subseteq N$, so $P_1 = P$ and $P_i \not\subseteq P$ ($i = 2, 3, \ldots, m$). Then\[ N = N \cap M = N \cap (M_1 + \cdots + M_m) = M_2 + \cdots + M_m + (N \cap M_1). \tag{2.1}\]

Now the assertion follows from Lemma 2.9.

(ii) Since $M = M_1 + N$, we have $M/N = (M_1 + N)/N \cong M_1/(M_1 \cap N)$, as required. \hfill \Box

**Proposition 2.11.** Let $R$ be a Dedekind domain, and let $M$ be a $0 \neq P$-secondary $R$-module. Then $M$ is a $P$-primary module.

**Proof.** Let $r \in R$. If $r \in P$, then the homothety $M \overset{r}{\rightarrow} M$ is nilpotent since $M$ is secondary. Suppose that $r \notin P$. If $ra = 0$ for some $0 \neq a \in M$, then by [6, Theorem 6.1], there exists $0 \neq b \in M$ and $Q \in \text{Ass}(M)$ such that $r \in Q$ and $Q = (0 :_R b)$. As $(0 : M) \subseteq (0 : b) = Q$, we have $P = Q$, a contradiction. So the homothety $M \overset{r}{\rightarrow} M$ is injective, as required. \hfill \Box

**Remarks.** (i) Let $R$ be a domain which is not a field. Then $R$ is a prime $R$-module (since $R$ is torsion-free) but it is not secondary (even if it is not pure-injective).

(ii) Let $R$ be a local Dedekind domain with maximal ideal $P = Rp$. We show that the module $E(R/P)$ is not prime (but it is $(0)$-secondary). Set $E = E(R/P)$ and $A_n = (0 :_E P^n)$ $(n \geq 1)$. Then by [2, Lemma 2.6], $Pa_{n+1} = A_n$, $A_n \subseteq E$ is a cyclic $R$-module with $A_n = Ra_n$ such that $pa_{n+1} = a_n$, every nonzero proper submodule $L$ of $E$ is of the form $L = A_m$ for some $m$ and $E$ is Artinian module with a strictly increasing sequence of submodules\[ A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots \tag{2.2}\]

We claim that $(A_n :_R E) = 0$ for every $n$. Suppose that $r \in (A_n :_R E)$ with $r \neq 0$. Then $rE \subseteq A_n$ and for all $a \in M$, we have $a = rb$ for some $b \in M$ since $E$ is injective (= divisible). Thus $a = rb \in A_n$, so $E = A_n$, a contradiction. Therefore $(A_n :_R E) = 0$ for
Let \( R \) be a Dedekind domain, and let \( M \) be an \( R \)-module. Then \( M \) is 0 \( \neq P \)-second if and only if \( M \) is \( P \)-prime.

**Proof.** By Proposition 2.11, it is enough to show that if \( M \) is \( P \)-prime, then \( M \) is 0 \( \neq P \)-second. Since \((0: M) = P\) is a maximal ideal in \( R \), so \( M \) is a vector space over \( R/P \), hence \( M \) is \( P \)-second.

**Proposition 2.13.** Let \( R \) be a Dedekind domain. Then any \( 0 \neq P \)-prime \( R \)-module is a direct sum of copies of \( R/P \).

**Proof.** By the proof of Proposition 2.11, every element of \( R-P \) acts invertibly on \( M \), so the \( R \)-module structure of \( M \) extends naturally to a structure of \( M \) as a module over the localisation \( R_P \) of \( R \) at \( P \). Therefore, we can assume that \( R \) is a commutative local Dedekind domain with maximal ideal \( P = R_P \).

Let \( M \) denote the indecomposable summand of \( M \), so \( M \) is \( P \)-prime. Let \( m_j \) be a nonzero element of \( M \), hence \((0 : m_j) = (0 : M) = P\). Then \( Rm_j \cong R/P \) is pure in \( M \) since \( m_j \) is not divisible by \( p \) in \( M \), but by [1, Proposition 1.3], the module \( R/P \) is itself pure-injective, so \( Rm_j \) is a direct summand of \( M \), and hence \( M \cong Rm_j \), as required.

### 3. Pure-injective modules

**Proposition 3.1.** Let \( M \) be a \( P \)-secondary module over a commutative ring \( R \). Then \( H = H(M) \), the pure-injective hull, is \( P \)-secondary.

**Proof.** Let \( r \in R \). If \( r \notin P \), then \( rM = M \), so \( M \) satisfies the sentence for all \( x \) there exists \( y \ (x = ry) \), and hence so does \( H \) (because any module and its pure-injective hull satisfy the same sentences [7, Chapter 4]). If \( r \in R \), then \( r^n M = 0 \), so \( M \) satisfies the sentence for all \( x \ (r^n x = 0) \), hence so does in \( H \), as required.

**Theorem 3.2.** The following conditions are equivalent for a Prufer domain \( R 

(i) the ring \( R \) is a Dedekind domain;

(ii) every secondary \( R \)-module is pure-injective.

**Proof.** Let \( R \) be a Dedekind domain and \( M \) a secondary \( R \)-module. If \( \text{Ann}(M) = 0 \), then \( M \) is divisible, hence injective. If \( \text{Ann}(M) \neq 0 \), then \( M \) is a torsion \( R \)-module of bounded order, so that \( M \) is \( \Sigma \)-pure-injective (see [15]). In both cases, \( M \) is \( \Sigma \)-pure-injective (so pure-injective).

Conversely, let \( R \) be a Prufer domain with the property that every secondary module is pure-injective. In order to prove that \( R \) is Dedekind domain, it suffices to show that every divisible \( R \)-module is injective. Let \( M \) be a divisible \( R \)-module. Then \( M \) is secondary, Hence pure-injective. Since \( R \) is Prufer, pure-injective modules are RD-injective (see [7]). The embedding of \( M \) in its injective envelope \( E(M) \) is an RD-pure monomorphism, because for every nonzero \( r \in R \) we have that \( M = rM \), so that \( rE(M) \cap M \subseteq M \subseteq rM \). Since \( M \) is the RD-injective, \( M \) is a direct summand of \( E(M) \). Thus \( M \) is injective. This shows that \( R \) is a Dedekind domain.
**Remarks.** (i) There is a module over a commutative regular ring which is injective but not secondary (see [9, Theorem 2.3]). The commutative regular ring \( R = F \times F \), \( F \) a field, is an Artinian Gorenstein, that is, \( R \) is injective (so pure-injective) as an \( R \)-module. But \( R \) is not secondary, because multiplication by \((1,0)\) is neither nilpotent nor surjective.

(ii) The above consideration thus leads us to the following question: are secondary modules pure-injective? The answer is yes because of the following reason. Every non-Noetherian Prufer domain has secondary modules that are not pure-injective. For instance, every non-Noetherian valuation domain has secondary modules that are not pure-injective.

**Proposition 3.3.** Let \( M \) be an \( R \)-module.

(i) \( M \) is \( \Sigma \)-secondary if and only if \( M \) is secondary.

(ii) Let \( M \) be a direct sum of modules \( M_i \) (\( i \in I \)) where for each \( i, M_i \) is secondary and \( \text{Ann}(M_i) = \text{Ann}(M_j) \) for all \( i,j \in I \). Then \( M \) is secondary.

**Proof.** (i) The necessity is immediate by the definition. Conversely, suppose that \( M \) is \( P \)-secondary. Given an index set \( J \), and let \( r \in R \). If \( r \in P \), then \( r^n M = 0 \) for some \( n \), so \( r^n M(J) = 0 \). If \( r \notin P \) then \( rM = M \), so \( rM(J) = M(J) \), as required.

(ii) Since the annihilators of all direct summands coincide, we can assume that \( M_i \) is \( P \)-secondary (say) for all \( i \in I \). Now the proof of (ii) is similar to that (i) and we omit it.

**Corollary 3.4.** Let \( M \) be an indecomposable \( \Sigma \)-pure-injective module over a commutative Prufer ring \( R \). Then \( M \) is secondary.

**Proof.** Set \( P = \{ r \in R : \text{Ann}_M r \neq 0 \} \) and \( P' = \cap_{n} P^n \). Then \( P \) and \( P' \) are prime ideals in \( R \) by [8, Fact 3.1 and Lemma 2.1]. By [8, Fact 3.2], \( M \) is either \( P \)-secondary or \( P' \)-secondary, as required.

**Corollary 3.5.** Every \( \Sigma \)-pure-injective module over a Prufer ring is representable.

**Proof.** Suppose \( M \) is a \( \Sigma \)-pure-injective module over a commutative Prufer ring \( R \). By [8, page 967], we can write \( M = M_1 \oplus \cdots \oplus M_m \) where \( M_i \) is secondary for all \( i \) by Proposition 3.3 and Corollary 3.4, as required.

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**References**


SHAHABADDIN EBRAHIMI ATANI: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GUilan,
P.O. BOX 1914, RASHT, IRAN

E-mail address: ebrahimi@cd.gu.ac.ir
Submit your manuscripts at
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