NONCOMPLETE AFFINE STRUCTURES ON LIE ALGEBRAS OF MAXIMAL CLASS

E. REMM and MICHEL GOZE

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Every affine structure on Lie algebra $\mathfrak{g}$ defines a representation of $\mathfrak{g}$ in $\text{aff}(\mathbb{R}^n)$. If $\mathfrak{g}$ is a nilpotent Lie algebra provided with a complete affine structure then the corresponding representation is nilpotent. We describe noncomplete affine structures on the filiform Lie algebra L_n . As a consequence we give a nonnilpotent faithful linear representation of the 3-dimensional Heisenberg algebra.

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1. Affine structure on a nilpotent Lie algebra

1.1. Affine structure on nilpotent Lie algebras

DEFINITION 1.1. Let $\mathfrak{g}$ be an n -dimensional Lie algebra over $\mathbb{R}$. An affine structure is given by a bilinear mapping

$$\nabla : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \tag{1.1}$$

satisfying

$$\begin{aligned} \nabla(X, Y) - \nabla(Y, X) &= [X, Y], \\ \nabla(X, \nabla(Y, Z)) - \nabla(Y, \nabla(X, Z)) &= \nabla([X, Y], Z), \end{aligned} \tag{1.2}$$

for all $X, Y, Z \in \mathfrak{g}$.

If $\mathfrak{g}$ is provided with an affine structure, then the corresponding connected Lie group G is an affine manifold such that every left translation is an affine isomorphism of G . In this case, the operator ∇ is nothing but the connection operator of the affine connection on G .

Let $\mathfrak{g}$ be a Lie algebra with an affine structure ∇ . Then the mapping

$$f : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g}), \tag{1.3}$$

defined by

$$f(X)(Y) = \nabla(X, Y), \tag{1.4}$$

is a linear representation (non faithful) of $\mathfrak{g}$ satisfying

$$f(X)(Y) - f(Y)(X) = [X, Y]. \tag{1.5}$$

REMARK 1.2. The adjoint representation $\tilde{f}$ of $\mathfrak{g}$ satisfies

$$\tilde{f}(X)(Y) - \tilde{f}(Y)(X) = 2[X, Y] \quad (1.6)$$

and cannot correspond to an affine structure.

1.2. Classical examples of affine structures. (i) Let $\mathfrak{g}$ be the n -dimensional abelian Lie algebra. Then the representation

$$f : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}), \quad X \mapsto f(X) = 0 \quad (1.7)$$

defines an affine structure.

(ii) Let $\mathfrak{g}$ be a $2p$ -dimensional Lie algebra endowed with a symplectic form

$$\theta \in \Lambda^2 \mathfrak{g}^* \quad \text{such that } d\theta = 0 \quad (1.8)$$

with

$$d\theta(X, Y, Z) = \theta(X, [Y, Z]) + \theta(Y, [Z, X]) + \theta(Z, [X, Y]). \quad (1.9)$$

For every $X \in \mathfrak{g}$ we can define a unique endomorphism ∇_X by

$$\theta(\text{ad } X(Y), Z) = -\theta(Y, \nabla_X(Z)). \quad (1.10)$$

Then $\nabla(X, Y) = \nabla_X(Y)$ is an affine structure on $\mathfrak{g}$.

(iii) Following the work of Benoist [1] and Burde [2, 3, 4], we know that there exists a nilpotent Lie algebra without affine structures.

1.3. Faithful representations associated to an affine structure. Let ∇ be an affine structure on an n -dimensional Lie algebra $\mathfrak{g}$. We consider the $(n + 1)$ -dimensional linear representation given by

$$\rho : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g} \oplus \mathbb{R}) \quad (1.11)$$

given by

$$\rho(X) : (Y, t) \mapsto (\nabla(X, Y) + tX, 0). \quad (1.12)$$

It is easy to verify that ρ is a faithful representation of dimension $n + 1$.

We can note that this representation gives also an affine representation of $\mathfrak{g}$

$$\psi : \mathfrak{g} \rightarrow \text{aff}(\mathbb{R}^n), \quad X \mapsto \begin{pmatrix} A(X) & X \\ 0 & 0 \end{pmatrix}, \quad (1.13)$$

where $A(X)$ is the matrix of the endomorphisms $\nabla_X : Y \mapsto \nabla(X, Y)$ in a given basis.

DEFINITION 1.3. We say that the representation ρ is nilpotent if the endomorphisms $\rho(X)$ are nilpotent for every X in $\mathfrak{g}$.

PROPOSITION 1.4. Suppose that $\mathfrak{g}$ is a complex non-abelian indecomposable nilpotent Lie algebra and let ρ be a faithful representation of $\mathfrak{g}$. Then there exists a faithful nilpotent representation of the same dimension.

PROOF. Consider the $\mathfrak{g}$ -module M associated to ρ . Then, as $\mathfrak{g}$ is nilpotent, M can be decomposed as

$$M = \bigoplus_{i=1}^k M_{\lambda_i}, \quad (1.14)$$

where M_{λ_i} is a $\mathfrak{g}$ -submodule, and the λ_i are linear forms on $\mathfrak{g}$. For all $X \in \mathfrak{g}$, the restriction of $\rho(X)$ to M_i is in the following form:

$$\begin{pmatrix} \lambda_i(X) & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_i(X) \end{pmatrix}. \quad (1.15)$$

Let $\mathbb{K}_{\lambda_i}$ be the one-dimensional $\mathfrak{g}$ -module defined by

$$\mu : X \in \mathfrak{g} \longrightarrow \mu(X) \in \text{End } \mathbb{K} \quad (1.16)$$

with

$$\mu(X)(a) = \rho(X)(a) = \lambda_i(X)a. \quad (1.17)$$

The tensor product $M_{\lambda_i} \otimes \mathbb{K}_{-\lambda_i}$ is the $\mathfrak{g}$ -module associated to

$$X \cdot (Y \otimes a) = \rho(X)(Y) \otimes a - Y \otimes \lambda_i(X)a. \quad (1.18)$$

Then $\widetilde{M} = \bigoplus (M_{\lambda_i} \otimes \mathbb{K}_{-\lambda_i})$ is a nilpotent $\mathfrak{g}$ -module. We prove that $\widetilde{M}$ is faithful. Recall that a representation ρ of $\mathfrak{g}$ is faithful if and only if $\rho(Z) \neq 0$ for every $Z \neq 0 \in Z(\mathfrak{g})$. Consider $X \neq 0 \in Z(\mathfrak{g})$. If $\tilde{\rho}(X) = 0$, then $\rho(X)$ is a diagonal endomorphism. By hypothesis $\mathfrak{g} \neq Z(\mathfrak{g})$ and there is $i \geq 1$ such that $X \in \mathcal{C}^i(\mathfrak{g})$, we have

$$X = \sum_j a_j [Y_j, Z_j] \quad (1.19)$$

with $Y_j \in \mathcal{C}^{i-1}(\mathfrak{g})$ and $Z_j \in \mathfrak{g}$. The endomorphisms $\rho(Y_j)\rho(Z_j) - \rho(Z_j)\rho(Y_j)$ are nilpotent and the eigenvalues of $\rho(X)$ are 0. Thus $\rho(X) = 0$ and ρ is not faithful. Then $\tilde{\rho}(X) \neq 0$ and $\tilde{\rho}$ is a faithful representation.

2. Affine structures on Lie algebra of maximal class

2.1. Definition

DEFINITION 2.1. An n -dimensional nilpotent Lie algebra $\mathfrak{g}$ is called of maximal class if the smallest k such that $\mathcal{C}^k \mathfrak{g} = \{0\}$ is equal to $n-1$.

In this case the descending sequence is

$$\mathfrak{g} \supset \mathcal{C}^1 \mathfrak{g} \supset \cdots \supset \mathcal{C}^{n-2} \mathfrak{g} \supset \{0\} = \mathcal{C}^{n-1} \mathfrak{g} \quad (2.1)$$

and we have

$$\begin{aligned} \dim \mathcal{C}^1 \mathfrak{g} &= n-2, \\ \dim \mathcal{C}^i \mathfrak{g} &= n-i-1, \quad \text{for } i = 1, \dots, n-1. \end{aligned} \quad (2.2)$$

EXAMPLE 2.2. The n -dimensional nilpotent Lie algebra L_n defined by

$$[X_1, X_i] = X_{i+1} \quad \text{for } i \in \{2, \dots, n-1\} \quad (2.3)$$

is of maximal class.

We can note that any Lie algebra of maximal class is a linear deformation of L_n [5].

□

2.2. On non-nilpotent affine structure. Let $\mathfrak{g}$ be an n -dimensional Lie algebra of maximal class provided with an affine structure ∇ . Let ρ be the $(n+1)$ -dimensional faithful representation associated to ∇ and we note that $M = \mathfrak{g} \oplus \mathbb{C}$ is the corresponding complex $\mathfrak{g}$ -module. As $\mathfrak{g}$ is of maximal class, its decomposition has one of the following forms

$$M = M_0, \quad M \text{ is irreducible}, \quad (2.4)$$

or

$$M = M_0 \oplus M_\lambda, \quad \lambda \neq 0. \quad (2.5)$$

For a general faithful representation, we call characteristic the ordered sequence of the dimensions of the irreducible submodules. In the case of maximal class we have $c(\rho) = (n+1)$ or $(n, 1)$ or $(n-1, 1, 1)$ or $(n-1, 2)$. In fact, the maximal class of $\mathfrak{g}$ implies that there exists an irreducible submodule of dimension greater than or equal to $n-1$. More generally, if the characteristic sequence of a nilpotent Lie algebra is equal to $(c_1, \dots, c_p, 1)$ (see [5]) then for every faithful representation ρ we have $c(\rho) = (d_1, \dots, d_q)$ with $d_1 \geq c_1$.

THEOREM 2.3. *Let $\mathfrak{g}$ be the Lie algebra of the maximal class L_n . Then there are faithful $\mathfrak{g}$ -modules which are not nilpotent.*

PROOF. Consider the following representation given by the matrices $\rho(X_i)$ where $\{X_1, \dots, X_n\}$ is a basis of $\mathfrak{g}$

$$\rho(X_1) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & & 0 & 1 \\ a & a & 0 & & & & \vdots & 0 \\ 0 & 0 & 0 & & & & 0 & 0 \\ \vdots & \ddots & \frac{1}{2} & \ddots & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots & 0 \\ \vdots & & & \ddots & \frac{i-3}{i-2} & \ddots & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & 0 \\ \alpha & \beta & 0 & \cdots & \cdots & 0 & \frac{n-3}{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho(X_2) = \begin{pmatrix} a & a & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ a & a & 0 & & & & \vdots & 1 \\ -1 & 1 & 0 & & & & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \ddots & & & \vdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots & 0 \\ \vdots & & & \ddots & \frac{1}{i-2} & \ddots & \vdots & 0 \\ 0 & 0 & & & \ddots & \ddots & \ddots & 0 \\ \beta & \alpha & 0 & \cdots & \cdots & \cdots & \frac{1}{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.6)$$

and for $j \geq 3$ the endomorphisms $\rho(X_j)$ satisfy

$$\begin{aligned} \rho(X_j)(e_1) &= -\frac{1}{j-1}e_{j+1}, \\ \rho(X_j)(e_2) &= \frac{1}{j-1}e_{j+1}, \\ \rho(X_j)(e_3) &= \frac{1}{j(j-1)}e_{j+2}, \\ &\vdots \\ \rho(X_j)(e_{i-j+1}) &= \frac{(j-2)!(i-j-1)!}{(i-2)!}e_i, \quad i = j-2, \dots, n, \\ \rho(X_j)(e_{i-j+1}) &= 0, \quad i = n+1, \dots, n+j-1, \\ \rho(X_j)(e_{n+1}) &= e_j, \end{aligned} \quad (2.7)$$

where $\{e_1, \dots, e_n, e_{n+1}\}$ is the basis given by $e_i = (X_i, 0)$ and $e_{n+1} = (0, 1)$. We easily verify that these matrices describe a nonnilpotent faithful representation. $\square$

2.3. Noncomplete affine structure on L_n . The previous representation is associated to an affine structure on the Lie algebra L_n given by

$$\nabla(X_i, Y) = \rho(X_i)(Y, 0), \quad (2.8)$$

where L_n is identified to the n -dimensional first factor of the $(n+1)$ -dimensional faithful module. This affine structure is complete if and only if the endomorphisms $R_X \in \text{End}(\mathfrak{g})$ defined by

$$R_X(Y) = \nabla(Y, X) \quad (2.9)$$

are nilpotent for all $X \in \mathfrak{g}$ (see [6]). But the matrix of R_{X_1} has the form

$$\begin{pmatrix} a & a & 0 & \cdots & 0 & \cdots & 0 & 0 \\ a & a & & & \vdots & & \vdots & 0 \\ 0 & -1 & & & \vdots & & \vdots & 0 \\ 0 & 0 & -\frac{1}{2} & \cdots & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & 0 & \ddots & & \cdots & \vdots & 0 \\ 0 & 0 & \vdots & \ddots & -\frac{1}{j-1} & & \vdots & 0 \\ \alpha & \beta & \vdots & \cdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{n-2} & 0 \end{pmatrix}. \quad (2.10)$$

Its trace is $2a$ and for $a \neq 0$ it is not nilpotent. We have proved the following proposition.

PROPOSITION 2.4. *There exist affine structures on the Lie algebra of maximal class L_n which are noncomplete.*

REMARK 2.5. The most simple example is on $\dim 3$ and concerns the Heisenberg algebra. We find a nonnilpotent faithful representation associated to the noncomplete affine structure given by

$$\nabla_{X_1} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \alpha & \beta & 0 \end{pmatrix}, \quad \nabla_{X_2} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ \beta-1 & \alpha+1 & 0 \end{pmatrix}, \quad \nabla_{X_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2.11)$$

where X_1, X_2 , and X_3 are a basis of H_3 satisfying $[X_1, X_2] = X_3$ and ∇_{X_i} the endomorphisms of $\mathfrak{g}$ given by

$$\nabla_{X_i}(X_j) = \nabla(X_i, X_j). \quad (2.12)$$

The affine representation is written as

$$\begin{pmatrix} a(x_1+x_2) & a(x_1+x_2) & 0 & x_1 \\ a(x_1+x_2) & a(x_1+x_2) & 0 & x_2 \\ \alpha x_1 + (\beta-1)x_2 & \beta x_1 + (\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2.13)$$

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E. REMM: FACULTÉ DES SCIENCES ET TECHNIQUES, 4, RUE DES FRÈRES LUMIÈRE, F. 68093
MULHOUSE CEDEX, FRANCE

E-mail address: e.remm@univ-mulhouse.fr

MICHEL GOZE: FACULTÉ DES SCIENCES ET TECHNIQUES, 4, RUE DES FRÈRES LUMIÈRE, F. 68093
MULHOUSE CEDEX, FRANCE

E-mail address: m.goze@univ-mulhouse.fr