CERTAIN CONVEX HARMONIC FUNCTIONS

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We define and investigate a family of complex-valued harmonic convex univalent functions related to uniformly convex analytic functions. We obtain coefficient bounds, extreme points, distortion theorems, convolution and convex combinations for this family.

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1. Introduction. A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain $\mathcal{D} \subset \mathbb{C}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. Consider the functions U and V analytic in $\mathcal{D}$ so that $u = \Re U$ and $v = \Im V$. Then the harmonic function f can be expressed by

$$f(z) = h(z) + \overline{g(z)}, \quad z \in \mathcal{D}, \quad (1.1)$$

where $h = (U + V)/2$ and $g = (U - V)/2$. We call h the analytic part and g the co-analytic part of f . If the co-analytic part of f is identically zero then f reduces to the analytic case.

The mapping $z \mapsto f(z)$ is sense-preserving and locally one-to-one in $\mathcal{D}$ if and only if the Jacobian of f is positive (see [1]), that is, if and only if

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \mathcal{D}. \quad (1.2)$$

Let $\mathcal{H}$ denote the family of functions $f = h + \bar{g}$ which are harmonic, sense-preserving, and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ with $h(0) = f(0) = f_z(0) - 1 = 0$. Thus, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (1.3)$$

Also let $\overline{\mathcal{H}}$ denote the subclass of $\mathcal{H}$ consisting of functions $f = h + \bar{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = - \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1. \quad (1.4)$$

Recently, Kanas and Wisniowska [5] (see also Kanas and Srivastava [4]), studied the class of k -uniformly convex analytic functions, denoted by $k\text{-}\mathcal{UCV}$, $0 \leq k < \infty$, so that $h \in k\text{-}\mathcal{UCV}$ if and only if

$$\Re \left\{ 1 + (z - \zeta) \frac{h''(z)}{h'(z)} \right\} \geq 0, \quad |\zeta| \leq k, \quad z \in \Delta. \quad (1.5)$$

For real ϕ we may let $\zeta = -kze^{i\phi}$. Then condition (1.5) can be written as

$$\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{zh''(z)}{h'(z)} \right\} \geq 0. \quad (1.6)$$

Now considering the harmonic functions $f = h + \bar{g}$ of the form (1.3) we define the family $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$, $0 \leq \alpha < 1$, so that $f = h + \bar{g} \in \mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ if and only if

$$\Re \left\{ 1 + (1 + ke^{i\phi}) \frac{z^2 h''(z) + \overline{2zg'(z) + z^2 g''(z)}}{zh'(z) - \overline{zg'(z)}} \right\} \geq \alpha, \quad 0 \leq \alpha < 1. \quad (1.7)$$

Finally, we let $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha) \equiv \mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha) \cap \overline{\mathcal{H}}$.

Notice that if $g \equiv 0$ and $\alpha = 0$ then the family $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ defined by (1.7) reduces to the class $k\text{-}\mathcal{UCV}$ of k -uniformly convex analytic functions defined by (1.5). If we, further, let $k = 1$ in (1.5), we obtain the class of uniformly convex analytic functions defined by Goodman [2]. A geometric characterization of the general family $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ is an open question.

In Section 2, we introduce sufficient coefficient bounds for functions to be in $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ and show that these bounds are also necessary for functions in $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$. In Section 3, the inclusion relation between the classes $k\text{-}\mathcal{UCV}$ and $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ is examined. Extreme points and distortion bounds for $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$ are given in Section 4. Finally, in Section 5, we show that the class $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ is closed under convolution and convex combinations.

Here we state a result due to Jahangiri [3], which we will use throughout this paper.

THEOREM 1.1. *Let $f = h + \bar{g}$ with h and g of the form (1.3). If*

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |b_n| \leq 1, \quad 0 \leq \alpha < 1, \quad (1.8)$$

then f is harmonic, sense-preserving, univalent in Δ , and f is convex harmonic of order α denoted by $\mathcal{H}\mathcal{H}(\alpha)$. Condition (1.8) is also necessary if $f \in \overline{\mathcal{H}\mathcal{H}}(\alpha) \equiv \mathcal{H}\mathcal{H}(\alpha) \cap \overline{\mathcal{H}}$.

2. Coefficient bounds. First we state and prove a sufficient coefficient bound for the class $\mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$.

THEOREM 2.1. *Let $f = h + \bar{g}$ be of the form (1.3). If $0 \leq k < \infty$, $0 \leq \alpha < 1$, and*

$$\sum_{n=2}^{\infty} \frac{n(n+nk-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+nk+k+\alpha)}{1-\alpha} |b_n| \leq 1, \quad (2.1)$$

then f is harmonic, sense-preserving, univalent in Δ , and $f \in \mathcal{H}\mathcal{C}\mathcal{V}(k, \alpha)$.

PROOF. Since $n - \alpha \leq n + nk - k - \alpha$ and $n + \alpha \leq n + nk + k + \alpha$ for $0 \leq k < \infty$, it follows from Theorem 1.1 that $f \in \mathcal{H}\mathcal{H}(\alpha)$ and hence f is sense-preserving and convex univalent in Δ . Now, we only need to show that if (2.1) holds then

$$\Re \left\{ \frac{zh'(z) + (1 + ke^{i\phi})z^2 h''(z) + (1 + 2ke^{i\phi})\overline{zg'(z)} + (1 + ke^{i\phi})\overline{z^2 g''(z)}}{zh'(z) - \overline{zg'(z)}} \right\} = \Re \frac{A(z)}{B(z)} \geq \alpha. \quad (2.2)$$

Using the fact that $\Re(w) \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$ it suffices to show that

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0, \quad (2.3)$$

where $A(z) = zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + 2ke^{i\phi})\overline{zg'(z)} + (1 + ke^{i\phi})\overline{z^2g''(z)}$ and $B(z) = zh'(z) - \overline{zg'(z)}$. Substituting for $A(z)$ and $B(z)$ in (2.3), we obtain

$$\begin{aligned} & |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \\ &= \left| (2 - \alpha)z + \sum_{n=2}^{\infty} n[n + 1 - \alpha + k(n - 1)e^{i\phi}]a_n z^n \right. \\ &\quad \left. + \sum_{n=1}^{\infty} n[n - 1 + \alpha + k(n + 1)e^{i\phi}]\bar{b}_n \bar{z}^n \right| \\ &\quad - \left| -\alpha z + \sum_{n=2}^{\infty} n[n - 1 - \alpha + k(n - 1)e^{i\phi}]a_n z^n \right. \\ &\quad \left. + \sum_{n=1}^{\infty} n[n + 1 + \alpha + k(n + 1)e^{i\phi}]\bar{b}_n \bar{z}^n \right| \\ &\geq (2 - \alpha)|z| - \sum_{n=2}^{\infty} n[n(k + 1) + 1 - k - \alpha]|a_n||z|^n \\ &\quad - \sum_{n=1}^{\infty} n[n(k + 1) - 1 + k + \alpha]|b_n||z|^n \\ &\quad - \alpha|z| - \sum_{n=2}^{\infty} n[n(k + 1) - 1 - k - \alpha]|a_n||z|^n \\ &\quad - \sum_{n=1}^{\infty} n[n(k + 1) + 1 + k + \alpha]|b_n||z|^n \\ &\geq 2(1 - \alpha)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{n[n(k + 1) - k - \alpha]}{1 - \alpha}|a_n| \right. \\ &\quad \left. - \sum_{n=1}^{\infty} \frac{n[n(k + 1) + k + \alpha]}{1 - \alpha}|b_n| \right\} \geq 0, \quad \text{by (2.1).} \end{aligned} \quad (2.4)$$

The harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1 - \alpha}{n(nk + n - k - \alpha)} x_n z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{n(nk + n + k + \alpha)} \bar{y}_n \bar{z}^n, \quad (2.5)$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$, show that the coefficient bound given in Theorem 2.1 is sharp.

The functions of the form (2.5) are in $\mathcal{HCV}(k, \alpha)$ because

$$\sum_{n=2}^{\infty} \frac{n(n + nk - k - \alpha)}{1 - \alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n + nk + k + \alpha)}{1 - \alpha} |b_n| = \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \quad (2.6)$$

□

Next we show that the bound (2.1) is also necessary for functions in $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$.

THEOREM 2.2. *Let $f = h + \bar{g}$ with h and g of the form (1.4). Then $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n(n+nk-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+nk+k+\alpha)}{1-\alpha} |b_n| \leq 1. \quad (2.7)$$

PROOF. In view of Theorem 2.1, we only need to show that $f \notin \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ if condition (2.7) does not hold. We note that a necessary and sufficient condition for $f = h + \bar{g}$ given by (1.4) to be in $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ is that the coefficient condition (1.7) to be satisfied. Equivalently, we must have

$$\Re \frac{(1-\alpha)zh'(z) + (1+ke^{i\phi})z^2h''(z) + (1+\alpha+2ke^{i\phi})\overline{zg'(z)} + (1+ke^{i\phi})\overline{z^2g''(z)}}{zh'(z) - \overline{zg'(z)}} \geq 0. \quad (2.8)$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$\frac{1-\alpha - \left\{ \sum_{n=2}^{\infty} n(nk+n-k-\alpha) |a_n| + \sum_{n=1}^{\infty} n(nk+n+k+\alpha) |b_n| \right\} r^{n-1}}{1 - \sum_{n=2}^{\infty} n |a_n| r^{n-1} + \sum_{n=1}^{\infty} n |b_n| r^{n-1}} \geq 0. \quad (2.9)$$

If condition (2.7) does not hold then the numerator in (2.9) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient (2.9) is negative. This contradicts the required condition for $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ and so the proof is complete. $\square$

3. Inclusion relations. As mentioned earlier in the proof of Theorem 2.1, the functions in $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ are convex harmonic in Δ . In the following example we show that this inclusion is proper.

EXAMPLE 3.1. Consider the harmonic functions

$$f_n(z) = z - \frac{1}{2}\bar{z} - \frac{1}{2n^2}\bar{z}^n, \quad z \in \Delta, \quad n = 2, 3, \dots \quad (3.1)$$

For $a_n \equiv 0$ and $b_n = -1/2n^2$, we observe that

$$\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| = \frac{1}{2} + n^2 \left(\frac{1}{2n^2} \right) = \frac{1}{2} + \frac{1}{2} = 1. \quad (3.2)$$

Therefore, by Theorem 1.1, $f_n \in \overline{\mathcal{H}\mathcal{H}}(0)$.

On the other hand,

$$\left| \frac{2k+1+\alpha}{1-\alpha} - \frac{1}{2} \right| + \left| \frac{n(nk+n+k+\alpha)}{1-\alpha} - \frac{1}{2n} \right| = \frac{2k+1+\alpha}{2(1-\alpha)} + \frac{nk+n+k+\alpha}{2n(1-\alpha)} > 1. \quad (3.3)$$

Thus, by Theorem 2.2, $f \notin \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$.

More generally, we can prove the following theorem.

THEOREM 3.2. *Let $0 \leq k < \infty$, $0 \leq \alpha < 1$, and $0 \leq \beta < 1$. If $k > \beta/(1-\beta)$ then the proper inclusion relation $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha) \subset \overline{\mathcal{H}\mathcal{H}}(\beta)$.*

PROOF. Let $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$, then, by [Theorem 2.2](#),

$$\sum_{n=2}^{\infty} \frac{n(nk+n-k-\alpha)}{1-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| \leq 1. \quad (3.4)$$

Since $(n-\beta)/(1-\beta) < (nk+n-k-\alpha)/(1-\alpha)$ and $(n+\beta)/(1-\beta) < (nk+n+k+\alpha)/(1-\alpha)$, by [Theorem 1.1](#), we conclude that $f \in \overline{\mathcal{H}\mathcal{H}}(\beta)$.

To show that the inclusion is proper, consider the harmonic functions

$$f_n(z) = z - \frac{1-\beta}{2(1+\beta)} \bar{z} - \frac{1-\beta}{2n(n+\beta)} \bar{z}^n, \quad z \in \Delta, \quad n = 2, 3, \dots \quad (3.5)$$

By [Theorem 1.1](#), $f_n \in \overline{\mathcal{H}\mathcal{H}}(\beta)$, because

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{n(n+\beta)}{1-\beta} |b_n| = \frac{1+\beta}{1-\beta} \frac{1-\beta}{2(1+\beta)} + \frac{n(n+\beta)}{1-\beta} \frac{1-\beta}{2n(n+\beta)} = 1. \quad (3.6)$$

On the contrary, by [Theorem 2.2](#), $f_n \notin \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$, because

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n(nk+n+k+\alpha)}{1-\alpha} |b_n| &= \frac{1+\alpha+2k}{1-\alpha} \frac{1-\beta}{2(1+\beta)} + \frac{n(n+\alpha+(n+1)k)}{1-\alpha} \frac{1-\beta}{2n(n+\beta)} \\ &= \frac{1-\beta}{2(1-\alpha)} \left\{ \frac{1+\alpha+2k}{1+\beta} + \frac{n+\alpha+(n+1)k}{n+\beta} \right\} \\ &> \frac{1-\beta}{2(1-\alpha)} \left\{ \frac{1+\alpha+2\beta/(1-\beta)}{1+\beta} + \frac{n+\alpha+(n+1)\beta/(1-\beta)}{n+\beta} \right\} \\ &= \frac{1}{2(1-\alpha)} \left\{ 2 + \frac{\alpha(1-\beta)(n+1+2\beta)}{(1+\beta)(n+\beta)} \right\} \geq 1. \end{aligned} \quad (3.7)$$

□

4. Extreme points and distortion bounds. Using definition (1.7), and according to the arguments given in [3], we obtain the following extreme points of the closed convex hulls of $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ denoted by $\text{clco}\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$.

THEOREM 4.1. *Let f be the form of (1.4). Then $f \in \text{clco}\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ if and only if $f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$ where $h_1(z) = z$, $h_n(z) = z - ((1-\alpha)/n(n+nk-k-\alpha))z^n$ ($n = 2, 3, \dots$), $g_n(z) = z - ((1-\alpha)/n(n+nk+k+\alpha))\bar{z}^n$ ($n = 1, 2, 3, \dots$), $\sum_{n=1}^{\infty} (X_n + Y_n) = 1$, $X_n \geq 0$ and $Y_n \geq 0$. In particular, the extreme points of $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ are $\{h_n\}$ and $\{g_n\}$.*

Similarly, follows the distortion bounds for functions in $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$.

THEOREM 4.2. *If $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ then*

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \frac{1}{2} \left(\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1, \\ |f(z)| &\geq (1 - |b_1|)r - \frac{1}{2} \left(\frac{1-\alpha}{2+k-\alpha} - \frac{1+2k+\alpha}{2+k-\alpha} |b_1| \right) r^2, \quad |z| = r < 1. \end{aligned} \quad (4.1)$$

If we let $r \rightarrow 1$ in the left-hand inequality of [Theorem 4.2](#) and collect the like terms, we obtain the following theorem.

THEOREM 4.3. *If $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ then $\{w : |w| < (3+2k-\alpha)/2(2+k-\alpha)-3(1-\alpha)/2(2+k-\alpha)|b_1|\} \subset f(\Delta)$.*

5. Convolutions and convex combinations. For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n - \sum_{n=1}^{\infty} |b_n| \bar{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n - \sum_{n=1}^{\infty} |B_n| \bar{z}^n$, we define the convolution of f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} |a_n| |A_n| z^n - \sum_{n=1}^{\infty} |b_n| |B_n| \bar{z}^n. \quad (5.1)$$

In the following theorem we examine the convolution properties of the class $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$.

THEOREM 5.1. *For $0 \leq \alpha \leq \beta < 1$, let $f \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \beta)$ and $F \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ then*

$$f * F \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \beta) \subset \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha). \quad (5.2)$$

PROOF. Express the convolution of f and F as that given by (5.1) and note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now the theorem follows upon the application of the required condition (2.7). $\square$

The convex combination properties of the class $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ is given in the following theorem.

THEOREM 5.2. *The class $\overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ is closed under convex combinations.*

PROOF. For $i = 1, 2, \dots$, suppose that $f_i \in \overline{\mathcal{H}\mathcal{C}\mathcal{V}}(k, \alpha)$ where f_i is given by $f_i(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n - \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n$. For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combinations of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n. \quad (5.3)$$

Now, the theorem follows by (2.7) upon noting that $\sum_{i=1}^{\infty} t_i = 1$. $\square$

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