RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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For analytic functions $f(z) = z^p + a_{p+1}z^{p+1} + \dots$ in the open unit disk $\mathbb{U}$ and a polynomial $Q(z)$ of degree $n > 0$, the function $F(z) = f(z)[Q(z)]^{\beta/n}$ is introduced. The object of the present paper is to determine the radius of p -valently strongly starlikeness of order γ for $F(z)$.

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1. Introduction. Let $\mathcal{A}_p$ (p is a fixed integer ≥ 1) denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let Ω denote the class of bounded functions $w(z)$ analytic in $\mathbb{U}$ and satisfying the conditions $w(0) = 0$ and $|w(z)| \leq |z|$, $z \in \mathbb{U}$. We use $\mathcal{P}$ to denote the class of functions $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ which are analytic in $\mathbb{U}$ and satisfy $\operatorname{Re} p(z) > 0$ ($z \in \mathbb{U}$).

For $0 \leq \alpha < p$ and $|\lambda| < \pi/2$, we denote by $\mathcal{P}_p^\lambda(\alpha)$, the family of functions $g(z) \in \mathcal{A}_p$ which satisfy

$$\frac{zg'(z)}{g(z)} \prec \frac{p + \{2(p - \alpha)e^{-i\lambda} \cos \lambda - p\}z}{1 - z}, \quad z \in \mathbb{U}, \quad (1.2)$$

where $\prec$ means the subordination. From the definition of subordinations, it follows that $g(z) \in \mathcal{A}_p$ has the representation

$$\frac{zg'(z)}{g(z)} = \frac{p + \{2(p - \alpha)e^{-i\lambda} \cos \lambda - p\}w(z)}{1 - w(z)}, \quad (1.3)$$

where $w(z) \in \Omega$. Clearly, $\mathcal{P}_p^\lambda(\alpha)$ is a subclass of p -valent λ -spiral functions of order α . For $\lambda = 0$, we have the class $\mathcal{P}_p^*(\alpha)$, $0 \leq \alpha < p$, of p -valent starlike functions of order α , investigated by Goluzina [5].

A function $f(z) \in \mathcal{A}_p$ is said to be p -valently strongly starlike of order γ , $0 < \gamma \leq 1$, if it satisfies

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leq \frac{\pi}{2} \gamma. \quad (1.4)$$

Başgöze [1, 2] has obtained sharp inequalities of univalence (starlikeness) for certain polynomials of the form $F(z) = f(z)[Q(z)]^{\beta/n}$, where β is real and $Q(z)$ is a polynomial of degree $n > 0$ all of whose zeros are outside or on the unit circle $\{z : |z| = 1\}$. Rajasekaran [7] extended Başgöze's results for certain classes of analytic functions of the form $F(z) = f(z)[Q(z)]^{\beta/n}$. Recently, Patel [6] generalized some of the work of Rajasekaran and Başgöze for functions belonging to the class $\mathcal{S}_p^\lambda(\alpha)$. That is, determine the radius of starlikeness for some classes of p -valent analytic functions of the polynomial form $F(z)$.

In the present paper, we extend the results of Patel [6]. Thus, we determine the radius of p -valently strongly starlike of order γ for polynomials of the form $F(z)$ in such problems.

2. Some lemmas. Before proving our next results, we need the following lemmas.

LEMMA 2.1 (see Gangadharan [4]). *For $|z| \leq r < 1$, $|z_k| = R > r$,*

$$\left| \frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2} \right| \leq \frac{Rr}{R^2 - r^2}. \quad (2.1)$$

LEMMA 2.2 (see Ratti [8]). *If $\phi(z)$ is analytic in $\mathbb{U}$ and $|\phi(z)| \leq 1$ for $z \in \mathbb{U}$, then for $|z| = r < 1$,*

$$\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{1}{1 - r}. \quad (2.2)$$

LEMMA 2.3 (see Causey and Merkes [3]). *If $p(z) = 1 + c_1z + c_2z^2 + \cdots \in \mathcal{P}$, then for $|z| = r < 1$,*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2}. \quad (2.3)$$

This estimate is sharp.

LEMMA 2.4 (see Patel [6]). *Suppose $g(z) \in \mathcal{S}_p^\lambda(\alpha)$. Then for $|z| = r < 1$,*

$$\left| \frac{zg'(z)}{g(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}. \quad (2.4)$$

This result is sharp.

LEMMA 2.5 (see Gangadharan [4]). *If $R_a \leq \operatorname{Re}(a) \sin((\pi/2)\gamma) - \operatorname{Im}(a) \cos((\pi/2)\gamma)$, $\operatorname{Im}(a) \geq 0$, then the disk $|w - a| \leq R_a$ is contained in the sector $|\arg w| \leq (\pi/2)\gamma$, $0 < \gamma \leq 1$.*

3. Main results. Our first theorem is the following one.

THEOREM 3.1. *Suppose that*

$$F(z) = f(z)[Q(z)]^{\beta/n}, \quad (3.1)$$

where β is real and $Q(z)$ is a polynomial of degree $n > 0$ with no zeros in $|z| < R$,

$R \geq 1$. If $f(z) \in \mathcal{A}_p$ satisfies

$$\operatorname{Re} \left[\left(\frac{f(z)}{g(z)} \right)^{1/\delta} \right] > 0, \quad 0 < \delta \leq 1, \quad z \in \mathbb{U}, \quad (3.2)$$

$$\operatorname{Re} \left[\frac{g(z)}{h(z)} \right] > 0, \quad z \in \mathbb{U}, \quad (3.3)$$

for some $g(z) \in \mathcal{A}_p$ and $h(z) \in \mathcal{G}_p^\lambda(\alpha)$, then $F(z)$ is p -valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} & r^4 \left[(p + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha) \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2(\delta + 1)] \\ & - r^2 \left[(p(1 + R^2) + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha) R^2 \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & - r [|\beta|R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2] + p R^2 \sin \left(\frac{\pi}{2} \gamma \right) = 0. \end{aligned} \quad (3.4)$$

PROOF. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z = 0$. Thus from (3.2) and (3.3), we have

$$F(z) = p_1^\delta(z) p_2 h(z) [Q(z)]^{\beta/n}, \quad (3.5)$$

where $p_j(z) \in \mathcal{P}$ ($j = 1, 2$).

Then

$$\frac{zF'(z)}{F(z)} = \delta \frac{zp_1'(z)}{p_1(z)} + \frac{zp_2'(z)}{p_2(z)} + \frac{zh'(z)}{h(z)} + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}. \quad (3.6)$$

Since $h(z) \in \mathcal{G}_p^\lambda(\alpha)$, by Lemma 2.4, we have

$$\left| \frac{zh'(z)}{h(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}. \quad (3.7)$$

Using (3.6) and (3.7) with Lemmas 2.1 and 2.3, we get

$$\begin{aligned} & \left| \frac{zF'(z)}{F(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \right| \\ & \leq \frac{2\{(p - \alpha)r \cos \lambda + r(\delta + 1)\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}. \end{aligned} \quad (3.8)$$

Using Lemma 2.5, we get that the above disk is contained in the sector $|\arg w| < (\pi/2)\gamma$ provided the inequality

$$\begin{aligned} & \frac{2\{(p - \alpha)r \cos \lambda + r(\delta + 1)\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2} \\ & \leq \left(p + \frac{2(p - \alpha)r^2 \cos^2 \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \sin \left(\frac{\pi}{2} \gamma \right) \\ & \quad - \frac{2(p - \alpha)r^2 \sin \lambda \cos \lambda}{1 - r^2} \cos \left(\frac{\pi}{2} \gamma \right) \end{aligned} \quad (3.9)$$

is satisfied. The above inequality is simplified to $T(r) \geq 0$, where

$$\begin{aligned} T(r) = & r^4 \left[(p - 2(p - \alpha) \cos^2 \lambda + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + (p - \alpha) \sin 2\lambda \cos \left(\frac{\pi}{2} \gamma \right) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2(\delta + 1)] \\ & + r^2 \left[(-pR^2 - p + 2(p - \alpha)R^2 \cos^2 \lambda - \beta) \sin \left(\frac{\pi}{2} \gamma \right) - (p - \alpha)R^2 \sin 2\lambda \cos \left(\frac{\pi}{2} \gamma \right) \right] \\ & - r [|\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2(\delta + 1)R^2] + pR^2 \sin \left(\frac{\pi}{2} \gamma \right). \end{aligned} \quad (3.10)$$

Since $T(0) > 0$ and $T(1) < 1$, there exists a real root of $T(r) = 0$ in $(0, 1)$. Let $R(\gamma)$ be the smallest positive root of $T(r) = 0$ in $(0, 1)$. Then $F(z)$ is p -valent strongly starlike of order γ in $|z| < R(\gamma)$. $\square$

REMARK 3.2. For $R = 1$ and $\gamma = 1$, [Theorem 3.1](#) reduces to a result by Patel [6].

THEOREM 3.3. Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies (3.2) for some $g(z) \in \mathcal{G}_p^\lambda(\alpha)$, then $F(z)$ is p -valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} & r^4 \left[(p + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha) \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + 2\delta] \\ & - r^2 \left[(p(1 + R^2) + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha)R^2 \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & - r [|\beta|R + 2(p - \alpha)R^2 \cos \lambda + 2\delta R^2] + pR^2 \sin \left(\frac{\pi}{2} \gamma \right) = 0. \end{aligned} \quad (3.11)$$

PROOF. If $f(z) \in \mathcal{A}_p$ satisfies (3.2) for some $g(z) \in \mathcal{G}_p^\lambda(\alpha)$, then

$$\frac{zF'(z)}{F(z)} = \delta \frac{zp'(z)}{p(z)} + \frac{zg'(z)}{g(z)} + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}. \quad (3.12)$$

Using [Lemma 2.4](#), we get

$$\left| \frac{zg'(z)}{g(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}. \quad (3.13)$$

By (3.12) and (3.13) with [Lemmas 2.1](#) and [2.3](#), we have

$$\begin{aligned} & \left| \frac{zF'(z)}{F(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \right| \\ & \leq \frac{2\{(p - \alpha)r \cos \lambda + r\delta\}}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}. \end{aligned} \quad (3.14)$$

The remaining parts of the proof can be proved by a method similar to the one given in the proof of [Theorem 3.1](#). $\square$

With $\lambda = 0$, $\beta = 0$, $\delta = 1$, $R = 1$, and $\gamma = 1$, [Theorem 3.3](#) gives the following corollary.

COROLLARY 3.4. Suppose that $f(z)$ is in $\mathcal{A}_p$. If $\operatorname{Re}(f(z)/g(z)) > 0$ for $z \in \mathbb{U}$ and $g(z) \in \mathcal{S}_p^*(\alpha)$, then $f(z)$ is p -valently starlike for

$$|z| < \frac{p}{(p+1-\alpha) + \sqrt{\alpha^2 - 2\alpha + 2p + 1}}. \quad (3.15)$$

THEOREM 3.5. Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies

$$\left| \left(\frac{f(z)}{g(z)} \right)^{1/\delta} - 1 \right| < 1, \quad 0 < \delta \leq 1, \quad p \sin\left(\frac{\pi}{2}\gamma\right) > \delta, \quad (3.16)$$

$$\operatorname{Re}\left(\frac{g(z)}{h(z)}\right) > 0, \quad z \in \mathbb{U} \quad (3.17)$$

for some $g(z) \in \mathcal{A}_p$ and $h(z) \in \mathcal{S}_p^\lambda(\alpha)$, then $F(z)$ is p -valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} & r^4 \left[(p+\beta) \sin\left(\frac{\pi}{2}\gamma\right) + 2(p-\alpha) \cos\lambda \sin\left(\lambda - \frac{\pi}{2}\gamma\right) \right] \\ & + r^3 [|\beta|R + 2(p-\alpha) \cos\lambda + 2 + \delta] \\ & - r^2 \left[(p(1+R^2) + \beta) \sin\left(\frac{\pi}{2}\gamma\right) + 2(p-\alpha)R^2 \cos\lambda \sin\left(\lambda - \frac{\pi}{2}\gamma\right) + \delta \right] \\ & - r [|\beta|R + 2(p-\alpha)R^2 \cos\lambda + 2(\delta+1)R^2] + pR^2 \sin\left(\frac{\pi}{2}\gamma\right) - \delta R^2 = 0. \end{aligned} \quad (3.18)$$

PROOF. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z = 0$. From (3.16), we deduce that

$$f(z) = g(z)(1+w(z))^\delta, \quad w(z) \in \Omega. \quad (3.19)$$

So that

$$F(z) = p(z)h(z)(1+z\phi(z))^\delta [Q(z)]^{\beta/n}, \quad (3.20)$$

where $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies $|\phi(z)| \leq 1$ and $p \in \mathcal{P}$ for $z \in \mathbb{U}$.

We have

$$\frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)} + \delta \left(\frac{z\phi'(z) + \phi(z)}{1+z\phi(z)} \right) + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z-z_k}. \quad (3.21)$$

Using Lemma 2.4 and (3.21), we have

$$\begin{aligned} & \left| \frac{zF'(z)}{F(z)} - \left(p + \frac{2(p-\alpha)e^{i\lambda}r^2 \cos\lambda}{1-r^2} \right) \right| \\ & \leq \frac{2\{(p-\alpha)r \cos\lambda + r\} + \delta(1+r)}{1-r^2} + \frac{|\beta|Rr}{R^2-r^2}. \end{aligned} \quad (3.22)$$

So, using Lemma 2.5 and (3.22), the result can be proved by using a method similar to the one given in the proof of Theorem 3.1. $\square$

THEOREM 3.6. Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{A}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, then $F(z)$ is p -valently strongly starlike of order γ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$\begin{aligned} & r^4 \left[(p + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha) \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) \right] \\ & + r^3 [|\beta|R + 2(p - \alpha) \cos \lambda + \delta] \\ & - r^2 \left[(p(1 + R^2) + \beta) \sin \left(\frac{\pi}{2} \gamma \right) + 2(p - \alpha)R^2 \cos \lambda \sin \left(\lambda - \frac{\pi}{2} \gamma \right) + \delta \right] \\ & - r [|\beta|R + 2(p - \alpha)R^2 \cos \lambda + \delta R^2] + pR^2 \sin \left(\frac{\pi}{2} \gamma \right) - \delta R^2 = 0. \end{aligned} \quad (3.23)$$

PROOF. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $\mathbb{U}$ and takes the value 1 at $z = 0$. Since $f(z) \in \mathcal{A}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}_p^\lambda(\alpha)$, we have

$$F(z) = g(z)(1 + z\phi(z))[Q(z)]^{\beta/n}, \quad (3.24)$$

where $\phi(z)$ is analytic in $\mathbb{U}$ and satisfies the condition $|\phi(z)| \leq 1$ for $z \in \mathbb{U}$. Thus, we have

$$\frac{zF'(z)}{F(z)} = \frac{zg'(z)}{g(z)} + \delta \left(\frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right) + \frac{\beta}{n} \sum_{k=1}^n \frac{z}{z - z_k}. \quad (3.25)$$

Using Lemma 2.4 and (3.25), we get

$$\begin{aligned} & \left| \frac{zF'(z)}{F(z)} - \left(p + \frac{2(p - \alpha)e^{i\lambda}r^2 \cos \lambda}{1 - r^2} \right) \right| \\ & \leq \frac{2(p - \alpha)r \cos \lambda + \delta(1 + r)}{1 - r^2} + \frac{|\beta|Rr}{R^2 - r^2}. \end{aligned} \quad (3.26)$$

Using Lemma 2.5 and (3.26) and a method similar to the one given in the proof of Theorem 3.1, we complete the proof of the theorem. $\square$

REMARK 3.7. Some of the results of Patel [6] can be obtained from Theorem 3.6 by taking $R = 1$ and $\gamma = 1$.

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