RADIUS OF STRONGLY STARLIKENESS FOR CERTAIN ANALYTIC FUNCTIONS

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For analytic functions \( f(z) = z^p + a_{p+1} z^{p+1} + \cdots \) in the open unit disk \( U \) and a polynomial \( Q(z) \) of degree \( n > 0 \), the function \( F(z) = f(z)[Q(z)]^{\beta/n} \) is introduced. The object of the present paper is to determine the radius of \( p \)-valently strongly starlikeness of order \( \gamma \) for \( F(z) \).

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1. Introduction. Let \( \mathcal{A}_p \) (\( p \) is a fixed integer \( \geq 1 \)) denote the class of functions \( f(z) \) of the form

\[
f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k
\]

which are analytic in the open unit disk \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \). Let \( \Omega \) denote the class of bounded functions \( w(z) \) analytic in \( \mathbb{U} \) and satisfying the conditions \( w(0) = 0 \) and \( |w(z)| \leq |z| \), \( z \in \mathbb{U} \). We use \( \mathcal{P} \) to denote the class of functions \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) which are analytic in \( \mathbb{U} \) and satisfy \( \text{Re} \, p(z) > 0 \) (\( z \in \mathbb{U} \)).

For \( 0 \leq \alpha < p \) and \( |\lambda| < \pi/2 \), we denote by \( \mathcal{S}_p^\lambda(\alpha) \), the family of functions \( g(z) \in \mathcal{A}_p \) which satisfy

\[
\frac{z g'(z)}{g(z)} < \frac{p + [2(p-\alpha)e^{-i\lambda}\cos\lambda - p]}{1 - z}, \quad z \in \mathbb{U},
\]

where \( < \) means the subordination. From the definition of subordinations, it follows that \( g(z) \in \mathcal{A}_p \) has the representation

\[
\frac{z g'(z)}{g(z)} = \frac{p + [2(p-\alpha)e^{-i\lambda}\cos\lambda - p]}{1 - w(z)} w(z),
\]

where \( w(z) \in \Omega \). Clearly, \( \mathcal{S}_p^\lambda(\alpha) \) is a subclass of \( p \)-valent \( \lambda \)-spiral functions of order \( \alpha \). For \( \lambda = 0 \), we have the class \( \mathcal{S}_p^\alpha(\alpha) \), \( 0 \leq \alpha < p \), of \( p \)-valent starlike functions of order \( \alpha \), investigated by Goluzina [5].

A function \( f(z) \in \mathcal{A}_p \) is said to be \( p \)-valently strongly starlike of order \( \gamma \), \( 0 < \gamma \leq 1 \), if it satisfies

\[
\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| \leq \frac{\pi}{2} \gamma.
\]
Başgöze [1, 2] has obtained sharp inequalities of univalence (starlikeness) for certain polynomials of the form \( F(z) = f(z)[Q(z)]^{\beta/n} \), where \( \beta \) is real and \( Q(z) \) is a polynomial of degree \( n > 0 \) all of whose zeros are outside or on the unit circle \( \{ z : |z| = 1 \} \). Rajasekaran [7] extended Başgöze’s results for certain classes of analytic functions of the form \( F(z) = f(z)[Q(z)]^{\beta/n} \). Recently, Patel [6] generalized some of the work of Rajasekaran and Başgöze for functions belonging to the class \( \mathcal{H}^p_5(\lambda, \alpha) \). That is, determine the radius of starlikeness for some classes of \( p \)-valent analytic functions of the polynomial form \( F(z) \).

In the present paper, we extend the results of Patel [6]. Thus, we determine the radius of \( p \)-valently strongly starlike of order \( \gamma \) for polynomials of the form \( F(z) \) in such problems.

2. Some lemmas. Before proving our next results, we need the following lemmas.

**Lemma 2.1** (see Gangadharan [4]). For \( |z| \leq r < 1, |z_k| = R > r \),

\[
\left| \frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2} \right| \leq \frac{Rr}{R^2 - r^2}. \tag{2.1}
\]

**Lemma 2.2** (see Ratti [8]). If \( \phi(z) \) is analytic in \( U \) and \( |\phi(z)| \leq 1 \) for \( z \in U \), then for \( |z| = r < 1 \),

\[
\left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{1}{1 - r}. \tag{2.2}
\]

**Lemma 2.3** (see Causey and Merkes [3]). If \( p(z) = 1 + c_1z + c_2z + \cdots \in \mathcal{H}^p_5 \), then for \( |z| = r < 1 \),

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2r}{1 - r^2}. \tag{2.3}
\]

This estimate is sharp.

**Lemma 2.4** (see Patel [6]). Suppose \( g(z) \in \mathcal{H}^p_5(\alpha) \). Then for \( |z| = r < 1 \),

\[
\left| \frac{zg'(z)}{g(z)} - \left( p + \frac{2(p - \alpha)e^{i\lambda}r^2\cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha)r \cos \lambda}{1 - r^2}. \tag{2.4}
\]

This result is sharp.

**Lemma 2.5** (see Gangadharan [4]). If \( R_a \leq \text{Re}(a) \sin((\pi/2)\gamma) - \text{Im}(a) \cos((\pi/2)\gamma) \), \( \text{Im}(a) \geq 0 \), then the disk \( |w - a| \leq R_a \) is contained in the sector \( |\arg w| \leq (\pi/2)\gamma \), \( 0 < \gamma \leq 1 \).

3. Main results. Our first theorem is the following one.

**Theorem 3.1.** Suppose that

\[
F(z) = f(z)[Q(z)]^{\beta/n}, \tag{3.1}
\]

where \( \beta \) is real and \( Q(z) \) is a polynomial of degree \( n > 0 \) with no zeros in \( |z| < R \),
If \( f(z) \in \mathcal{S}_p \) satisfies
\[
\Re \left[ \left( \frac{f(z)}{g(z)} \right)^{1/\delta} \right] > 0, \quad 0 < \delta \leq 1, \; z \in \mathbb{U},
\]
(3.2)
and
\[
\Re \left[ \frac{g(z)}{h(z)} \right] > 0, \quad z \in \mathbb{U},
\]
(3.3)
for some \( g(z) \in \mathcal{S}_p \) and \( h(z) \in \mathcal{F}_p^\lambda(\alpha) \), then \( F(z) \) is \( p \)-valently strongly starlike of order \( \gamma \) in \( |z| < R(\gamma) \), where \( R(\gamma) \) is the smallest positive root of the equation
\[
r^4 \left[ (p + \beta) \sin \left( \frac{\pi}{2} y \right) + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} y \right) \right] + r^3 \left[ |\beta| R + 2(p - \alpha) \cos \lambda + 2(\delta + 1) \right] \\
- r^2 \left[ (p(1 + R^2) + \beta) \sin \left( \frac{\pi}{2} y \right) + 2(p - \alpha) R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} y \right) \right] \\
- r \left[ |\beta| R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2 \right] + p R^2 \sin \left( \frac{\pi}{2} y \right) = 0.
\]
(3.4)

**Proof.** We choose a suitable branch of \( (f(z)/g(z))^{1/\delta} \) so that \( (f(z)/g(z))^{1/\delta} \) is analytic in \( U \) and takes the value 1 at \( z = 0 \). Thus from (3.2) and (3.3), we have
\[
F(z) = p_1^\delta(z) p_2 h(z) \left[ Q(z) \right]^{\beta/n},
\]
(3.5)
where \( p_j(z) \in \mathcal{F}_p \) \( (j = 1, 2) \).

Then
\[
\frac{z F'(z)}{F(z)} = \delta \frac{z p_1'(z)}{p_1(z)} + \frac{z p_2'(z)}{p_2(z)} + \frac{z h'(z)}{h(z)} + \frac{\beta}{n} \sum_{k=1}^{n} \frac{z}{z - z_k}.
\]
(3.6)
Since \( h(z) \in \mathcal{F}_p^\lambda(\alpha) \), by Lemma 2.4, we have
\[
\left| \frac{z h'(z)}{h(z)} - \left( p + \frac{2(p - \alpha) e^{i\lambda} r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha) r \cos \lambda}{1 - r^2}.
\]
(3.7)
Using (3.6) and (3.7) with Lemmas 2.1 and 2.3, we get
\[
\left| \frac{z F'(z)}{F(z)} - \left( p + \frac{2(p - \alpha) e^{i\lambda} r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \right| \\
\leq \frac{2 \left((p - \alpha) r \cos \lambda + r(\delta + 1) \right)}{1 - r^2} + \frac{|\beta| R r}{R^2 - r^2}.
\]
(3.8)
Using Lemma 2.5, we get that the above disk is contained in the sector \( |\arg w| < \frac{\pi}{2}y \) provided the inequality
\[
\frac{2 \left((p - \alpha) r \cos \lambda + r(\delta + 1) \right)}{1 - r^2} + \frac{|\beta| R r}{R^2 - r^2} \\
\leq \left( p + \frac{2(p - \alpha) r^2 \cos^2 \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \sin \left( \frac{\pi}{2} y \right) \\
- \frac{2(p - \alpha) r^2 \sin \lambda \cos \lambda}{1 - r^2} \cos \left( \frac{\pi}{2} y \right)
\]
(3.9)
is satisfied. The above inequality is simplified to \(T(r) \geq 0\), where

\[
T(r) = r^4 \left[ (p - 2(p - \alpha) \cos^2 \lambda + \beta \sin \left( \frac{\pi}{2} y \right) + (p - \alpha) \sin 2\lambda \cos \left( \frac{\pi}{2} y \right) \right]
+ r^3 \left[ |\beta| R + 2(p - \alpha) \cos \lambda + 2(\delta + 1) \right]
+ r^2 \left[ ( - p R^2 - p + 2(p - \alpha) R^2 \cos^2 \lambda - \beta ) \sin \left( \frac{\pi}{2} y \right) - (p - \alpha) R^2 \sin 2\lambda \cos \left( \frac{\pi}{2} y \right) \right]
- r \left[ |\beta| R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2 \right] + p R^2 \sin \left( \frac{\pi}{2} y \right).
\]

(3.10)

Since \(T(0) > 0\) and \(T(1) < 1\), there exists a real root of \(T(r) = 0\) in \((0, 1)\). Let \(R(\gamma)\) be the smallest positive root of \(T(r) = 0\) in \((0, 1)\). Then \(F(z)\) is \(p\)-valently strongly starlike of order \(\gamma\) in \(|z| < R(\gamma)\).

**Remark 3.2.** For \(R = 1\) and \(\gamma = 1\), Theorem 3.1 reduces to a result by Patel [6].

**Theorem 3.3.** Suppose that \(F(z)\) is given by (3.1). If \(f(z) \in \mathcal{A}_p\) satisfies (3.2) for some \(g(z) \in \mathcal{F}_p^\alpha(\alpha)\), then \(F(z)\) is \(p\)-valently strongly starlike of order \(\gamma\) in \(|z| < R(\gamma)\), where \(R(\gamma)\) is the smallest positive root of the equation

\[
r^4 \left[ (p + \beta) \sin \left( \frac{\pi}{2} y \right) + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} y \right) \right]
+ r^3 \left[ |\beta| R + 2(p - \alpha) \cos \lambda + 2\delta \right]
- r^2 \left[ (p(1 + R^2) + \beta) \sin \left( \frac{\pi}{2} y \right) + 2(p - \alpha) R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} y \right) \right]
- r \left[ |\beta| R + 2(p - \alpha) R^2 \cos \lambda + 2\delta R^2 \right] + p R^2 \sin \left( \frac{\pi}{2} y \right) = 0.
\]

(3.11)

**Proof.** If \(f(z) \in \mathcal{A}_p\) satisfies (3.2) for some \(g(z) \in \mathcal{F}_p^\alpha(\alpha)\), then

\[
\frac{z F'(z)}{F(z)} = \delta \frac{z p'(z)}{p(z)} + n \frac{z g'(z)}{g(z)} + \beta \sum_{k=1}^{n} \frac{z}{z - z_k}.
\]

(3.12)

Using Lemma 2.4, we get

\[
\left| \frac{z g'(z)}{g(z)} - \left( p + \frac{2(p - \alpha) e^{i\lambda} r^2 \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha) r \cos \lambda}{1 - r^2}.
\]

(3.13)

By (3.12) and (3.13) with Lemmas 2.1 and 2.3, we have

\[
\left| \frac{z F'(z)}{F(z)} - \left( p + \frac{2(p - \alpha) e^{i\lambda} r^2 \cos \lambda}{1 - r^2} - \frac{\beta r^2}{R^2 - r^2} \right) \right| \leq \frac{2 \left( (p - \alpha) r \cos \lambda + r \delta \right)}{1 - r^2} + \frac{|\beta| R r}{R^2 - r^2}.
\]

(3.14)

The remaining parts of the proof can be proved by a method similar to the one given in the proof of Theorem 3.1.

With \(\lambda = 0, \beta = 0, \delta = 1, R = 1, \) and \(\gamma = 1\), Theorem 3.3 gives the following corollary.
**Corollary 3.4.** Suppose that \( f(z) \) is in \( A_p \). If \( \Re(f(z)/g(z)) > 0 \) for \( z \in U \) and \( g(z) \in \mathcal{H}_p^\alpha \), then \( f(z) \) is \( p \)-valently starlike for 
\[
|z| < \frac{p}{(p + 1 - \alpha) + \sqrt{\alpha^2 - 2\alpha + 2p + 1}}.
\]

**Theorem 3.5.** Suppose that \( F(z) \) is given by (3.1). If \( f(z) \in A_p \) satisfies 
\[
\left| \left( \frac{f(z)}{g(z)} \right)^{1/\delta} - 1 \right| < 1, \quad 0 < \delta \leq 1, \quad p \sin \left( \frac{\pi}{2} \gamma \right) > \delta,
\]
\[
\Re \left( \frac{g(z)}{h(z)} \right) > 0, \quad z \in \mathbb{U}
\]
for some \( g(z) \in A_p \) and \( h(z) \in \mathcal{H}_p^\lambda \), then \( F(z) \) is \( p \)-valently strongly starlike of order \( \gamma \) in \( |z| < R(\gamma) \), where \( R(\gamma) \) is the smallest positive root of the equation 
\[
r^4 \left[ (p + \beta) \sin \left( \frac{\pi}{2} \gamma \right) + 2(p - \alpha) \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) \right] \\
+ r^3 \left[ \beta |R + 2(p - \alpha) \cos \lambda + 2 + \delta \right] \\
- r^2 \left[ (p(1 + R^2) + \beta) \sin \left( \frac{\pi}{2} \gamma \right) + 2(p - \alpha) R^2 \cos \lambda \sin \left( \lambda - \frac{\pi}{2} \gamma \right) + \delta \right] \\
- r \left[ |R + 2(p - \alpha) R^2 \cos \lambda + 2(\delta + 1) R^2 + p R^2 \sin \left( \frac{\pi}{2} \gamma \right) \right] - \delta R^2 = 0.
\]

**Proof.** We choose a suitable branch of \( (f(z)/g(z))^{1/\delta} \) so that \( (f(z)/g(z))^{1/\delta} \) is analytic in \( U \) and takes the value 1 at \( z = 0 \). From (3.16), we deduce that 
\[
f(z) = g(z)(1 + w(z))^\delta, \quad w(z) \in \Omega.
\]
So that 
\[
F(z) = p(z) h(z) (1 + z \phi(z))^\delta [Q(z)]^{\beta/n},
\]
where \( \phi(z) \) is analytic in \( U \) and satisfies \( |\phi(z)| \leq 1 \) and \( p \in \mathbb{P} \) for \( z \in \mathbb{U} \). We have 
\[
\frac{zF'(z)}{F(z)} = \frac{zh'(z)}{h(z)} + \frac{zp'(z)}{p(z)} + \frac{\delta(\phi'(z) + \phi(z))}{1 + z \phi(z)} + \frac{\beta}{n} \sum_{k=1}^{n} \frac{z}{z - z_k}.
\]
Using Lemma 2.4 and (3.21), we have 
\[
\left| \frac{zF'(z)}{F(z)} - \left( p + \frac{2(p - \alpha)e^{i\lambda} r^2 \cos \lambda}{1 - r^2} \right) \right| \\
\leq \frac{2(\beta |R + 2(p - \alpha) \cos \lambda + r|) + \delta(1 + r)}{1 - r^2} + \frac{|\beta| R r}{R^2 - r^2}.
\]
So, using Lemma 2.5 and (3.22), the result can be proved by using a method similar to the one given in the proof of Theorem 3.1.
THEOREM 3.6. Suppose that $F(z)$ is given by (3.1). If $f(z) \in \mathcal{S}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}^\lambda_p(\alpha)$, then $F(z)$ is $p$-valently strongly starlike of order $\gamma$ in $|z| < R(\gamma)$, where $R(\gamma)$ is the smallest positive root of the equation

$$r^4 \left[ (p + \beta) \sin \left( \frac{\pi}{2} \gamma \right) + 2(p - \alpha) \cos \lambda \sin \left( \frac{\pi}{2} \gamma \right) \right]$$
$$+ r^3 \left[ \beta R + 2(p - \alpha) \cos \lambda + \delta \right]$$
$$- r^2 \left[ (p + R^2 + \beta) \sin \left( \frac{\pi}{2} \gamma \right) + 2(p - \alpha) R^2 \cos \lambda \sin \left( \frac{\pi}{2} \gamma \right) + \delta \right]$$
$$- r \left[ \beta R + 2(p - \alpha) R^2 \cos \lambda + \delta R^2 \right] + p R^2 \sin \left( \frac{\pi}{2} \gamma \right) - \delta R^2 = 0. \tag{3.23}$$

PROOF. We choose a suitable branch of $(f(z)/g(z))^{1/\delta}$ so that $(f(z)/g(z))^{1/\delta}$ is analytic in $U$ and takes the value 1 at $z = 0$. Since $f(z) \in \mathcal{S}_p$ satisfies (3.16) for some $g(z) \in \mathcal{S}^\lambda_p(\alpha)$, we have

$$F(z) = g(z) \left( 1 + z \phi(z) \right) \left[ Q(z) \right]^{\beta/n}, \tag{3.24}$$

where $\phi(z)$ is analytic in $U$ and satisfies the condition $|\phi(z)| \leq 1$ for $z \in U$. Thus, we have

$$z F'(z) F(z) = z g'(z) \frac{g(z)}{g(z)} + \delta \left( \frac{z \phi'(z) + \phi(z)}{1 + z \phi(z)} \right) + \beta \sum_{k=1}^{n} \frac{z}{z - z_k}. \tag{3.25}$$

Using Lemma 2.4 and (3.25), we get

$$\left| \frac{z F'(z)}{F(z)} - \left( p + \frac{2(p - \alpha) e^{i \lambda} r \cos \lambda}{1 - r^2} \right) \right| \leq \frac{2(p - \alpha) r \cos \lambda + \delta (1 + r)}{1 - r^2} + \frac{\beta |\beta| R r}{R^2 - r^2}. \tag{3.26}$$

Using Lemma 2.5 and (3.26) and a method similar to the one given in the proof of Theorem 3.1, we complete the proof of the theorem. \qed

REMARK 3.7. Some of the results of Patel [6] can be obtained from Theorem 3.6 by taking $R = 1$ and $\gamma = 1$.

REFERENCES


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