ON NEW GENERALIZATIONS OF HARDY'S INTEGRAL INEQUALITIES

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We give some new generalizations of Hardy's integral inequalities.

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1. Introduction. The classical Hardy inequality [3] states that: for $f(x) \geq 0$, $p > 1$, $1/p + 1/q = 1$, and $0 < \int_0^\infty f^p(x) dx < \infty$,

$$\int_0^\infty \left[\frac{1}{x} \int_0^x f(t) dt \right]^p dx < q^p \int_0^\infty f^p(t) dt, \quad (1.1)$$

where $q = p/(p-1)$ is the best possible constant.

The dual form of (1.1) is as follows: if $0 < \int_0^\infty (xf(x))^p dx < \infty$, then

$$\int_0^\infty \left(\int_x^\infty f(t) dt \right)^p dx < p^p \int_0^\infty (tf(t))^p dt, \quad (1.2)$$

where the constant p^p in (1.2) is still best possible.

Bicheng et al. [2] gave some new generalizations of (1.1) which can be stated as follows:

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t) dt \right)^p dx < q^p \left[1 - \left(\frac{a}{b} \right)^{1/q} \right]^p \int_a^b f^p(t) dt; \quad (1.3)$$

$$\int_a^\infty \left(\frac{1}{x} \int_a^x f(t) dt \right)^p dx < q^p \int_a^\infty [1 - \theta_p(t)] f^p(t) dt \quad (0 < \theta_p(t) < 1); \quad (1.4)$$

$$\int_0^b \left(\frac{1}{x} \int_0^x f(t) dt \right)^p dx < q^p \int_0^b \left[1 - \left(\frac{t}{b} \right)^{1/q} \right] f^p(t) dt, \quad (1.5)$$

where $\theta_p(t) = (1/p) \sum_{k=1}^\infty \binom{p}{k+1} (-1)^{k-1} (a/t)^{k/q} > 0$ for $t > a > 0$, and $\theta_p(a) = 1/q$.

Recently, Becheng and Debnath [1] gave improvement of (1.3) and some generalizations of (1.2):

$$\begin{aligned} \int_a^b \left(\frac{1}{x} \int_a^x f(t) dt \right)^p dx &< q^p \eta_p(a, b) \int_a^b f^p(t) dt; \\ \int_a^\infty \left(\int_x^\infty f(t) dt \right)^p dx &< p^p \int_a^\infty \left[1 - \left(\frac{a}{t} \right)^{1/p} \right] (tf(t))^p dt; \\ \int_0^b \left(\int_x^b f(t) dt \right)^p dx &< p^p \int_0^b \mu_p(t) (tf(t))^p dt, \end{aligned} \quad (1.6)$$

where the constants $\eta_p(a, b) = \max_{a \leq t \leq b} \{(1/q)t^{1/q} \int_t^b x^{-1-1/q} [1 - (a/x)^{1/q}]^{p-1} dx\}$, $\mu_p(t) = (1/p) \{1 - (t/b)^{1/p}\}^p (b/t)^{1/p}$.

In this paper, we show some new improvements and generalizations of the inequalities (1.1) and (1.2).

2. Main results

LEMMA 2.1. *Let $a \geq 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty f^p(t) dt < \infty$. Then, there exists a real number $x_0 \in (a, \infty)$ such that, for any $x > x_0$, the following inequality is true:*

$$\begin{aligned} \left(\int_a^x f(t) dt \right)^p &< \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \left(1 - \frac{1}{r} \right)^{p-1} \\ &\times \left(x^{1-1/(1-1/r)q(p-1)} - a^{1-1/(1-1/r)q(p-1)} \right)^{p-1} \int_a^x t^{1/(1-1/r)q} f^p(t) dt. \end{aligned} \quad (2.1)$$

PROOF. By Hölder's inequality, we have

$$\begin{aligned} \left(\int_a^x f(t) dt \right)^p &= \left(\int_a^x t^{1/(1-1/r)pq} f(t) t^{-1/(1-1/r)pq} dt \right)^p \\ &\leq \int_a^x t^{1/(1-1/r)q} f^p(t) dt \left(\int_a^x \left(t^{-1/(1-1/r)pq} \right)^{p/(p-1)} dt \right)^{p-1} \\ &= \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \left(1 - \frac{1}{r} \right)^{p-1} \\ &\times \left(x^{1-1/(1-1/r)q(p-1)} - a^{1-1/(1-1/r)q(p-1)} \right)^{p-1} \int_a^x t^{1/(1-1/r)q} f^p(t) dt. \end{aligned} \quad (2.2)$$

We need to show that there exists a real number $x_0 \in (a, \infty)$, such that (2.2) does not assume equality for any $x > x_0$. Otherwise, there exists $x = x_n \in (a, \infty)$, where $n = 1, 2, 3, \dots, x_n \uparrow \infty$, such that (2.2) becomes an equality. By the same argument, there exists a real number $c > 0$, and an integer N , such that for $n > N$,

$$\left(t^{1/(1-1/r)pq} f(t) \right)^p = c \left(t^{-1/(1-1/r)pq} \right)^{p/(p-1)} \quad \text{a.e. in } [a, x_n]. \quad (2.3)$$

Hence

$$\begin{aligned} \int_a^{x_n} f^p(t) dt &= \int_a^{x_n} c \frac{t^{-1/(1-1/r)q(p-1)}}{t^{1/(1-1/r)q}} dt \\ &= \int_a^{x_n} c t^{-p/(1-1/r)q(p-1)} dt \rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (2.4)$$

This is a contradiction to the fact that $0 < \int_a^\infty f^p(t) dt < \infty$. Hence, (2.1) holds true and the proof is complete. $\square$

LEMMA 2.2. Let $b > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and let $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t) dt < \infty$. Then, there exists a real number $x_0 \in (0, b)$ such that, for any $x \in (0, x_0)$, the following inequality is true:

$$\left(\int_x^b f(t) dt \right)^p < \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \left(x^{-1/(1-1/r)p} - b^{-1/(1-1/r)p} \right)^{p-1} \times \int_x^b t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt. \quad (2.5)$$

PROOF. For any $x \in (0, b)$, by Hölder's inequality, we have

$$\begin{aligned} \left(\int_x^b f(t) dt \right)^p &= \left[\int_x^b t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t) t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2} dt \right]^p \\ &\leq \int_x^b t^{(1+(1-1/r)p)(p-1)/(1-1/r)p} f^p(t) dt \left(\int_x^b t^{-(1+(1-1/r)p)/(1-1/r)p} dt \right)^{p-1} \\ &= \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \left(x^{-1/(1-1/r)p} - b^{-1/(1-1/r)p} \right)^{p-1} \\ &\quad \times \int_x^b t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt. \end{aligned} \quad (2.6)$$

We need to show that there exists a real number $x_0 \in (0, b)$, such that (2.6) does not assume equality for any $x \in (0, x_0)$. Otherwise, there exists $x = x_n \in (0, b)$, where $n = 1, 2, 3, \dots, x_n \downarrow 0$, such that (2.6) becomes an equality. Then there exist c_n and d_n which are not always zero, such that (see [4, page 29])

$$\begin{aligned} c_n \left[t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t) \right]^p \\ = d_n \left[t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2} \right]^{p/(p-1)} \quad \text{a.e. in } [x_n, b]. \end{aligned} \quad (2.7)$$

Since $f(t) \neq 0$ a.e. in $(0, b)$, there exists an integer N such that, for $n > N$, $f(t) \neq 0$ a.e. in $(0, x_n)$. Thus, for both $c_n = c \neq 0$ and $d_n = d \neq 0$ for $n > N$,

$$\int_0^b t^{p-1+1/(1-1/r)} f^p(t) dt = \lim_{n \rightarrow \infty} \int_{x_n}^b \frac{t^{-(1+1/(1-1/r)p)}}{t^{1-(1+1/(1-1/r)p)}} dt = \frac{d}{c} \lim_{n \rightarrow \infty} \int_{x_n}^b \frac{dt}{t} = \infty. \quad (2.8)$$

This contradicts the fact that $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t) dt < \infty$. Hence, (2.5) is valid and this completes the proof of the lemma. $\square$

LEMMA 2.3. Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty t^{p-1+1/(1-1/r)} f^p(t) dt < \infty$. Then, there exists a real number $x_0 \in (a, \infty)$ such that, for any $x \in (a, x_0)$, the following inequality is true:

$$\left(\int_x^\infty f(t) dt \right)^p < \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} x^{-(p-1)/(1-1/r)p} \int_x^\infty t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt. \quad (2.9)$$

PROOF. For any $x \in (a, \infty)$, by Hölder's inequality, we have

$$\left(\int_x^\infty f(t) dt \right)^p \leq \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} x^{-(p-1)/(1-1/r)p} \int_x^\infty t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt. \quad (2.10)$$

We show that there exists a real number $x_0 \in (a, \infty)$, such that (2.10) does not assume equality for any $x \in (a, x_0)$. Otherwise, there exists $x = x_n \in (a, \infty)$, where $n = 1, 2, 3, \dots, x_n \downarrow a$, such that (2.10) becomes an equality. By the same argument there exist a real number $c > 0$, and an integer N , such that for $n > N$,

$$\begin{aligned} & \left[t^{(1+(1-1/r)p)(p-1)/(1-1/r)p^2} f(t) \right]^p \\ &= c \left[t^{-(1+(1-1/r)p)(p-1)/(1-1/r)p^2} \right]^{p/(p-1)} \quad \text{a.e. in } [x_n, \infty), \end{aligned} \quad (2.11)$$

and hence $\int_a^\infty t^{p-1+(1-1/r)p} f^p(t) dt = c \lim_{n \rightarrow \infty} \int_{x_n}^\infty (dt/t) = \infty$. This contradicts the fact that $0 < \int_a^\infty t^{p-1+(1-1/r)p} f^p(t) dt < \infty$. Hence (2.9) is valid and this completes the proof of the lemma. $\square$

THEOREM 2.4. Let $0 < a < b$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty f^p(t) dt < \infty$. Then

$$\int_a^b \left(\frac{1}{x} \int_a^x f(t) dt \right)^p dx < \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^p \left(1 - \frac{1}{r} \right)^p \eta(a, b) \int_a^b f^p(t) dt, \quad (2.12)$$

where the constant

$$\begin{aligned} \eta(a, b) &= \max_{a \leq t \leq b} \left\{ \frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q} \right. \\ &\quad \times \left. \int_t^b x^{-1-1/(1-1/r)q} \left[1 - \left(\frac{a}{x} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} dx \right\}, \quad (2.13) \\ \eta(a, b) &< \frac{(p+q)(p-1)-p}{p(p-1)} \left[1 - \left(\frac{a}{b} \right)^{1-1/(1-1/r)q(p-1)} \right]^p. \end{aligned}$$

PROOF. In view of the proof of Lemma 2.1, we obtain

$$\begin{aligned} & \int_a^b \left(\frac{1}{x} \int_a^x f(t) dt \right)^p dx \\ &< \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^{p-1} \left(1 - \frac{1}{r} \right)^{p-1} \\ &\quad \times \int_a^b \left\{ \int_t^b x^{-1-1/(1-1/r)q} \left[1 - \left(\frac{a}{x} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} dx \right\} t^{1/(1-1/r)q} f^p(t) dt \\ &= \left(\frac{pq(p-1)}{(p+q)(p-1)-p} \right)^p \left(1 - \frac{1}{r} \right)^p \int_a^b g(t) f^p(t) dt, \end{aligned} \quad (2.14)$$

where the weight function $g(t)$ is defined by

$$g(t) := \frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q} \times \int_t^b x^{-1-1/(1-1/r)q} \left[1 - \left(\frac{a}{x} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} dx, \quad t \in [a, b]. \quad (2.15)$$

Setting $\eta(a, b) := \max_{a \leq t \leq b} g(t)$, since $g(t)$ is a nonconstant continuous function, then by (2.14) we have (2.12). Since $g(b) = 0$, and for any $t \in [a, b]$,

$$\begin{aligned} g(t) &< \frac{(p+q)(p-1)-p}{pq(p-1)(1-1/r)} t^{1/(1-1/r)q} \int_t^b x^{-1-1/(1-1/r)q} \left[1 - \left(\frac{a}{b} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} dx \\ &= \frac{(p+q)(p-1)-p}{pq(p-1)} \left[1 - \left(\frac{a}{b} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} \left[1 - \left(\frac{t}{b} \right)^{1/(1-1/r)q} \right] \\ &\leq \frac{(p+q)(p-1)-p}{pq(p-1)} \left[1 - \left(\frac{a}{b} \right)^{1-1/(1-1/r)q(p-1)} \right]^{p-1} \left[1 - \left(\frac{a}{b} \right)^{1/(1-1/r)q} \right] \\ &< \frac{(p+q)(p-1)-p}{pq(p-1)} \left[1 - \left(\frac{a}{b} \right)^{1-1/(1-1/r)q(p-1)} \right]^p. \end{aligned} \quad (2.16)$$

This completes the proof. $\square$

THEOREM 2.5. Let $a > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_a^\infty (tf(t))^p dt < \infty$, $0 < \int_a^\infty t^{p-1+1/(1+1/r)} f^p(t) dt < \infty$. Then

$$\int_a^\infty \left(\int_x^\infty f(t) dt \right)^p dx < \left(\left(1 - \frac{1}{r} \right) p \right)^p \frac{r}{r-p} \int_a^\infty \left[1 - \left(\frac{a}{t} \right)^{(r-p)/(r-1)p} \right] (tf(t))^p dt. \quad (2.17)$$

PROOF. Applying (2.9), we have

$$\begin{aligned} &\int_a^\infty \left(\int_x^\infty f(t) dt \right)^p dx \\ &< \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \int_a^\infty x^{-(p-1)/(1-1/r)p} \int_x^\infty t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt dx \\ &= \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \int_a^\infty \left(\int_a^t x^{-(p-1)/(1-1/r)p} dx \right) t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt \\ &= \left(\left(1 - \frac{1}{r} \right) p \right)^p \frac{r}{r-p} \int_a^\infty \left[1 - \left(\frac{a}{t} \right)^{(r-p)/(r-1)p} \right] (tf(t))^p dt. \end{aligned} \quad (2.18)$$

Hence, (2.17) is valid. This completes the proof of the theorem. $\square$

THEOREM 2.6. Let $b > 0$, $p > 1$, $1/p + 1/q = 1 - 1/r$, $f \geq 0$, $r > 1$, and $0 < \int_0^b (tf(t))^p dt < \infty$, $0 < \int_0^b t^{p-1+1/(1-1/r)} f^p(t) dt < \infty$. Then

$$\int_0^b \left(\int_x^b f(t) dt \right)^p dx < \left(\left(1 - \frac{1}{r} \right) p \right)^p \int_0^b \mu(t) (tf(t))^p dt, \quad (2.19)$$

where $\mu(t) := 1/(1-1/r)p \{ \int_0^t x^{-(p-1)/(1-1/r)p} [1 - (x/b)^{1/(1-1/r)p}]^{p-1} dx \} t^{(p-r)/(r-1)p}$, $t \in (0, b]$.

PROOF. Applying (2.5), we have

$$\begin{aligned} \int_0^b \left(\int_x^b f(t) dt \right)^p dx &< \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \int_0^b \left(x^{-1/(1-1/r)p} - b^{-1/(1-1/r)p} \right)^{p-1} \\ &\quad \times \int_x^b t^{p-1+(p-1)/(1-1/r)p} f^p(t) dt dx \\ &= \left(\left(1 - \frac{1}{r} \right) p \right)^{p-1} \int_0^b \left(\int_0^t x^{-(p-1)(1-1/r)p} \left[1 - \left(\frac{x}{b} \right)^{1/(1-1/r)p} \right]^{p-1} dx \right) \\ &\quad \times t^{(p-r)/(r-1)p} (t f(t))^p dt \\ &= \left(\left(1 - \frac{1}{r} \right) p \right)^p \int_0^b \mu(t) (t f(t))^p dt, \end{aligned} \tag{2.20}$$

where $\mu(t) := 1/(1-1/r)p \{ \int_0^t x^{-(p-1)/(1-1/r)p} [1 - (x/b)^{1/(1-1/r)p}]^{p-1} dx \} t^{(p-r)/(r-1)p}$, $t \in (0, b]$. This proves (2.19) and the proof of the theorem is complete. $\square$

REMARK 2.7. Let $r \rightarrow \infty$, (2.1) changes into [2, (2.3)]. Hence, (2.1) is a generalization of [2, (2.3)].

REMARK 2.8. Let $r \rightarrow \infty$, (2.5) and (2.9) change into [1, (3.1) and (3.5)], respectively. Hence (2.5) and (2.9) is generalization of [1, (3.1) and (3.5)], respectively.

REMARK 2.9. Let $r \rightarrow \infty$, (2.12), (2.17), and (2.19) change into [1, (3), (4), and (5)], respectively. Hence, (2.12), (2.17), and (2.19) is generalization of [1, (3), (4), and (5)], respectively.

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