COMPUTATIONAL PROOFS OF CONGRUENCES FOR 2-COLORED FROBENIUS PARTITIONS

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In 1994, the following infinite family of congruences was conjectured for the partition function $c\phi_2(n)$ which counts the number of 2-colored Frobenius partitions of n : for all $n \geq 0$ and $\alpha \geq 1$, $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$, where λ_α is the least positive reciprocal of 12 modulo 5^α . In this paper, the first four cases of this family are proved.

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1. Background and introduction. In his 1984 Memoir of the American Mathematical Society, Andrews [2] introduced two families of partition functions, $\phi_k(m)$ and $c\phi_k(m)$, which he called generalized Frobenius partition functions. In this paper, we will focus our attention on one of these functions, namely $c\phi_2(m)$, which denotes the number of generalized Frobenius partitions of m with 2 colors. In [2], Andrews gives the generating function for $c\phi_2(m)$:

$$\sum_{m \geq 0} c\phi_2(m)q^m = \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty^4 (q^4; q^4)_\infty}, \quad (1.1)$$

where $(a; b)_\infty = (1-a)(1-ab)(1-ab^2)(1-ab^3) \cdots$. Andrews then proves the following: for all $n \geq 0$,

$$c\phi_2(5n+3) \equiv 0 \pmod{5}, \quad (1.2)$$

$$c\phi_2(2n+1) \equiv 0 \pmod{4}. \quad (1.3)$$

More recently, Sellers [9] conjectured the following infinite family of congruences satisfied by $c\phi_2$.

CONJECTURE 1.1. *For all $n \geq 0$ and $\alpha \geq 1$,*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \quad (1.4)$$

where λ_α is the least positive reciprocal of 12 modulo 5^α .

The case $\alpha = 1$ is (1.2).

The reader will note the similarity of this conjecture to the well-known family of congruences for $p(m)$, the classical partition function of m : for all $n \geq 0$,

$$p(5^\alpha n + \gamma_\alpha) \equiv 0 \pmod{5^\alpha}, \quad (1.5)$$

where y_α is the least positive reciprocal of 24 modulo 5^α . (For two different proofs of (1.5), see [1, 6].) Unfortunately, (1.4) has proven to be much more difficult to prove than (1.5).

The goal of this paper is to prove the following theorem.

THEOREM 1.2. *For all $n \geq 0$ and $\alpha = 1, 2, 3, 4$,*

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \quad (1.6)$$

where λ_α is the least positive reciprocal of 12 modulo 5^α .

In order to prove this theorem, we implement a finitization technique developed recently (cf. [3]). In essence, we prove that, for fixed α ,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \forall n \quad (1.7)$$

if and only if

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \forall n \leq C(\alpha), \quad (1.8)$$

where $C(\alpha)$ is an explicit constant dependent on α . We then compute all values of $c\phi_2$ needed to utilize the equivalence above. The development of $C(\alpha)$ requires the theory of modular forms as outlined below.

2. Determination of $C(\alpha)$. In this section, we use the theory of modular forms to determine the constant $C(\alpha)$. We do so by constructing a modular form whose Fourier coefficients inherit the congruence properties modulo 5^α of $c\phi_2$ in the desired arithmetic progression. Then, thanks to a theorem of Sturm [10], we will be able to provide explicitly a constant $C(\alpha)$ such that if a congruence for the Fourier coefficients of our modular form (or equivalently, for $c\phi_2$) holds for all $n \leq C(\alpha)$, the congruence must hold for all n .

For a general introduction to the theory of modular forms, see [7]. For an exposition focused on the results we use below, see [3].

We now state Sturm's theorem [10].

THEOREM 2.1 (Sturm). *If $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=0}^{\infty} b(n)q^n$ are holomorphic modular forms of weight k with respect to some congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients, then $f(z) \equiv g(z) \pmod{l}$ where l is prime if and only if*

$$\mathrm{Ord}_l(f(z) - g(z)) > \frac{k}{12} [\mathrm{SL}_2(\mathbb{Z}) : \Gamma], \quad (2.1)$$

where $\mathrm{Ord}_l(F(q)) := \min\{n \mid A(n) \not\equiv 0 \pmod{l}\}$.

Sturm's theorem also holds when the prime l is replaced by 5^α , or in fact by any positive integer. Thus, when we let $g(z) = 0$, Sturm's theorem allows us to determine when the coefficients $a(n)$ of a holomorphic modular form have the property that $a(n) \equiv 0 \pmod{5^\alpha}$ for all n .

We are now ready to state the main result needed to prove Theorem 1.2.

THEOREM 2.2. *Suppose that α is a positive integer, and let*

$$C(\alpha) := 6(b-1+4\varepsilon \cdot 5^{\alpha-1})5^{\alpha-1} - \left\lfloor \frac{b}{12} \right\rfloor, \quad (2.2)$$

where $b = b(\alpha)$ is the smallest integer greater than $4 \cdot 5^{\alpha-2}$ with $b \equiv 5^\alpha \pmod{12}$, $\varepsilon = \varepsilon(\alpha) = 1$ if α is odd, and $\varepsilon = \varepsilon(\alpha) = 2$ if α is even. Then

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \forall n \quad (2.3)$$

if and only if

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha} \quad \forall n \leq C(\alpha), \quad (2.4)$$

where λ_α is the least positive reciprocal of 12 modulo 5^α .

PROOF. Let

$$f(z) = \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)} \eta^b(2 \cdot 5^\alpha z) \left(\frac{\eta^5(z)}{\eta(5z)} \right)^{\varepsilon \cdot 5^{\alpha-1}} = \sum_{n=0}^{\infty} a(n)q^n, \quad (2.5)$$

where $\eta(z)$ is the Dedekind eta-function, defined by $\eta(z) = q^{1/24}(q; q)_\infty$, $q = e^{2\pi iz}$, $b = b(\alpha)$ is the smallest integer greater than $4 \cdot 5^{\alpha-2}$ with $b \equiv 5^\alpha \pmod{12}$, $\varepsilon = \varepsilon(\alpha) = 1$ if α is odd, and $\varepsilon = \varepsilon(\alpha) = 2$ if α is even.

Using results from [4, Theorems 3 and 5] on the properties of η -products, we find that $f(z)$ is a holomorphic modular form of weight $(b-1)/2 + 2\varepsilon \cdot 5^{\alpha-1}$ and character χ_0 , the trivial character, with respect to $\Gamma_0(16 \cdot 5^\alpha)$.

Notice that

$$\left(\frac{\eta^5(z)}{\eta(5z)} \right)^{\varepsilon \cdot 5^{\alpha-1}} = 1 + 5^\alpha \sum_{n=1}^{\infty} h(n)q^n, \quad (2.6)$$

where the $h(n)$ are integers, and thus the Fourier coefficients of $f(z)$ are congruent to the Fourier coefficients of

$$\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)} \eta^b(2 \cdot 5^\alpha z) \pmod{5^\alpha}. \quad (2.7)$$

Next, note that in terms of eta-functions,

$$\sum_{n \geq 0} c\phi_2(n)q^n = q^{1/12} \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}. \quad (2.8)$$

Thus, if we let

$$q^{-2b \cdot 5^\alpha / 24} \eta^b(2 \cdot 5^\alpha z) = \sum_{n=0}^{\infty} d(2 \cdot 5^\alpha n) q^{2 \cdot 5^\alpha n}, \quad (2.9)$$

then

$$a\left(5^\alpha n + \lambda_\alpha + \frac{2b \cdot 5^\alpha - 2}{24}\right) \equiv \sum_{m=0}^{\infty} d(2 \cdot 5^\alpha m) c\phi_2(5^\alpha n + \lambda_\alpha - 2 \cdot 5^\alpha m) \pmod{5^\alpha}. \quad (2.10)$$

Since $d(0) = 1$, this becomes

$$a\left(5^\alpha n + \lambda_\alpha + \frac{2b \cdot 5^\alpha - 2}{24}\right) \equiv c\phi_2(5^\alpha n + \lambda_\alpha) + \sum_{m=1}^{\infty} d(2 \cdot 5^\alpha m) c\phi_2(5^\alpha n + \lambda_\alpha - 2 \cdot 5^\alpha m) \pmod{5^\alpha}. \quad (2.11)$$

By induction, it is easy to see that $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all $n \leq C(\alpha)$ if and only if $a(5^\alpha n + \lambda_\alpha + (2b \cdot 5^\alpha - 2)/24) \equiv 0 \pmod{5^\alpha}$ for all $n \leq C(\alpha)$. Hence, we also have that $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all n if and only if $a(5^\alpha n + \lambda_\alpha + (2b \cdot 5^\alpha - 2)/24) \equiv 0 \pmod{5^\alpha}$ for all n .

Now notice that $\lambda_\alpha + (2b \cdot 5^\alpha - 2)/24 \equiv 0 \pmod{5^\alpha}$ by hypothesis, so consider

$$f_1(z) = f(z) \mid T_{5^\alpha} = \sum_{n=0}^{\infty} a(5^\alpha n) q^n, \quad (2.12)$$

which is also a holomorphic modular form of weight $(b-1)/2 + 2\varepsilon \cdot 5^{\alpha-1}$ and character χ_0 with respect to $\Gamma_0(16 \cdot 5^\alpha)$. (See [7, pages 153–175] for a full explanation of the action of the Hecke operators T_p .) We find by Sturm's theorem that $a(5^\alpha n) \equiv 0 \pmod{5^\alpha}$ for all n if and only if

$$a(5^\alpha n) \equiv 0 \pmod{5^\alpha} \quad \forall n \leq \frac{((b-1)/2 + 2\varepsilon \cdot 5^{\alpha-1})(16 \cdot 5^\alpha)}{12} \prod_{p \mid 10} \left(1 + \frac{1}{p}\right). \quad (2.13)$$

Therefore, $c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$ for all n if and only if the congruence holds for all $n \leq C(\alpha)$. $\square$

For certain values of α , it is not difficult to make modest improvements to [Theorem 1.2](#). In the case $\alpha = 4$, this modest improvement will bring $C(\alpha)$ more comfortably within the realm of computational feasibility.

THEOREM 2.3. *Let*

$$C(4) := 198745. \quad (2.14)$$

Then

$$c\phi_2(625n + 573) \equiv 0 \pmod{625} \quad \forall n \quad (2.15)$$

if and only if

$$c\phi_2(625n + 573) \equiv 0 \pmod{625} \quad \forall n \leq C(4). \quad (2.16)$$

PROOF. Let

$$f(z) = \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)} \eta^{44}(625z) \eta^7(1250z) \eta^{10}(2500z) \left(\frac{\eta^5(z)}{\eta(5z)}\right)^{250} = \sum_{n=0}^{\infty} a(n) q^n, \quad (2.17)$$

where $q = e^{2\pi iz}$. We find that $f(z)$ is a holomorphic modular form of weight 530 and character χ_0 , the trivial character, with respect to $\Gamma_0(2500)$.

Notice that

$$\left(\frac{\eta^5(z)}{\eta(5z)}\right)^{250} = 1 + 625 \sum_{n=1}^{\infty} h(n)q^n, \quad (2.18)$$

where the $h(n)$ are integers, and thus the Fourier coefficients of $f(z)$ are congruent to the Fourier coefficients of

$$\frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}\eta^{44}(625z)\eta^7(1250z)\eta^{10}(2500z) \pmod{625}. \quad (2.19)$$

Recalling that

$$\sum_{n \geq 0} c\phi_2(n)q^n = q^{1/12} \frac{\eta^5(2z)}{\eta^4(z)\eta^2(4z)}, \quad (2.20)$$

if we let

$$q^{-61250/24}\eta^{44}(625z)\eta^7(1250z)\eta^{10}(2500z) = \sum_{n=0}^{\infty} d(625n)q^{625n}, \quad (2.21)$$

then

$$a\left(625n + 573 + \frac{61250-2}{24}\right) \equiv \sum_{m=0}^{\infty} d(625m)c\phi_2(625n + 573 - 625m) \pmod{625}. \quad (2.22)$$

Since $d(0) = 1$, this becomes

$$\begin{aligned} a(625n + 573 + 2552) &\equiv c\phi_2(625n + 573) \\ &+ \sum_{m=1}^{\infty} d(625m)c\phi_2(625n + 573 - 625m) \pmod{625}. \end{aligned} \quad (2.23)$$

By induction, it is easy to see that $c\phi_2(625n + 573) \equiv 0 \pmod{625}$ for all $n \leq C(4)$ if and only if $a(625n + 573 + 2552) \equiv 0 \pmod{625}$ for all $n \leq C(4)$. Hence, we also have that $c\phi_2(625n + 573) \equiv 0 \pmod{625}$ for all n if and only if $a(625n + 573 + 2552) \equiv 0 \pmod{625}$ for all n .

Now notice that $573 + 2552 \equiv 0 \pmod{625}$, so consider

$$f_1(z) = f(z) \mid T_{625} = \sum_{n=0}^{\infty} a(625n)q^n, \quad (2.24)$$

which is also a holomorphic modular form of weight 530 and character χ_0 with respect to $\Gamma_0(2500)$. We find by Sturm's theorem that $a(625n) \equiv 0 \pmod{625}$ for all n if and only if

$$a(625n) \equiv 0 \pmod{625} \quad \forall n \leq \frac{(530)(2500)}{12} \prod_{p \mid 10} \left(1 + \frac{1}{p}\right). \quad (2.25)$$

Therefore, $c\phi_2(625n + 573) \equiv 0 \pmod{625}$ for all n if and only if the congruence holds for all $n \leq C(4)$. $\square$

3. Calculating the needed values of $c\phi_2$. From the above discussion, we can prove the congruences desired for all n after calculating the first M values of $c\phi_2$, for any $M > 5^\alpha C(\alpha) + \lambda_\alpha$. We calculate the necessary terms using recurrences.

The recurrences needed for $c\phi_2(m)$ are easily developed. Recurrences are suitable for calculating the values of $c\phi_2(m)$ for *small* m . This, of course, is the historical approach to the calculation of partition function values. For example, this was the technique used by MacMahon to compute the first 200 values of $p(m)$ [5, Table IV]. This same table was used by Ramanujan [8] in conjecturing several of the congruences in (1.5).

We now prove a result from which the necessary recurrences follow.

THEOREM 3.1.

$$\left[\sum_{n \geq 0} c\phi_2(n)q^n \right] \left[\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \right] = \left[\sum_{n \geq 0} p(n)q^{2n} \right] \left[\sum_{n \in \mathbb{Z}} q^{n^2} \right]. \quad (3.1)$$

PROOF. From Jacobi's triple product identity [1, Theorem 2.8], we see that

$$\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} = (q^2; q^2)_\infty (q; q^2)_\infty^2, \quad \sum_{n \in \mathbb{Z}} q^{n^2} = \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2}. \quad (3.2)$$

Also, since

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_\infty}, \quad (3.3)$$

it is clear that

$$\sum_{n \geq 0} p(n)q^{2n} = \frac{1}{(q^2; q^2)_\infty}. \quad (3.4)$$

Then

$$\begin{aligned} \left[\sum_{n \geq 0} c\phi_2(n)q^n \right] \left[\sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} \right] &= \frac{(q^2; q^4)_\infty}{(q; q^2)_\infty^4 (q^4; q^4)_\infty} \cdot (q^2; q^2)_\infty (q; q^2)_\infty^2 \\ &= \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty^2 (q^4; q^4)_\infty^2} \\ &= \frac{1}{(q^2; q^2)_\infty} \cdot \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2} \\ &= \left[\sum_{n \geq 0} p(n)q^{2n} \right] \left[\sum_{n \in \mathbb{Z}} q^{n^2} \right]. \end{aligned} \quad (3.5)$$

□

From this theorem, we have the following recurrences:

$$\begin{aligned} c\phi_2(2k) &= p(k) + 2 \sum_{m \geq 1} (-1)^{m+1} c\phi_2(2k - m^2) + 2 \sum_{m \geq 1} p(k - 2m^2), \\ c\phi_2(2k+1) &= 2 \sum_{m \geq 1} (-1)^{m+1} c\phi_2(2k+1 - m^2) + 2 \sum_{m \geq 0} p(k - 2m(m+1)). \end{aligned} \quad (3.6)$$

Since $p(n)$ satisfies $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \dots$, where the values in question are the pentagonal numbers, the above recurrences can easily be implemented to calculate several values of $c\phi_2$.

Using these recurrences, we have calculated the necessary 124, 216, 198 values of $c\phi_2$ on a Linux PC with 768MB of RAM and a 600Mhz Pentium III processor. The calculations, all performed modulo 625, were completed in approximately 147 hours of computing time.

With these calculations complete and the congruences checked modulo 625, [Theorem 1.2](#) has been proven.

4. Closing remarks. While it would be nice to prove additional cases of [\(1.4\)](#) using this technique, it is clear that $C(\alpha)$ grows too rapidly to make such an approach feasible. For example, the proof of the $\alpha = 5$ case of [\(1.4\)](#) would require the calculation of $C(5) = 11279958$ values of $c\phi_2$ in the arithmetic progression $5^5n + \lambda_5$. Hence, we would have to calculate the first 3.5×10^{10} values of $c\phi_2$ (approximately).

Certainly, a proof of [Conjecture 1.1](#) via modular forms or generating function manipulations is still desired. This was originally requested in [\[9\]](#), and we renew that request here, given the new computational information that is now known about this partition function and the fact that [Theorem 1.2](#) is proven.

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REFERENCES

- [1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications, vol. 2, Addison-Wesley, Massachusetts, 1976.
- [2] ———, *Generalized Frobenius partitions*, Mem. Amer. Math. Soc. **49** (1984), no. 301, iv+44.
- [3] D. Eichhorn and K. Ono, *Congruences for partition functions*, Analytic Number Theory, vol. 1 (Allerton Park, IL, 1995), Progr. Math., vol. 138, Birkhäuser, Massachusetts, 1996, pp. 309–321.
- [4] B. Gordon and K. Hughes, *Multiplicative properties of η -products. II*, A Tribute to Emil Grosswald: Number Theory and Related Analysis, Contemp. Math., vol. 143, American Mathematical Society, Rhode Island, 1993, pp. 415–430.
- [5] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. **17** (1918), no. 2, 75–115.
- [6] M. D. Hirschhorn and D. C. Hunt, *A simple proof of the Ramanujan conjecture for powers of 5*, J. Reine Angew. Math. **326** (1981), 1–17.
- [7] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, 2nd ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York, 1993.

- [8] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n* , Proc. Cambridge Philos. Soc. **19** (1919), 207–210.
- [9] J. Sellers, *Congruences involving F -partition functions*, Int. J. Math. Math. Sci. **17** (1994), no. 1, 187–188.
- [10] J. Sturm, *On the congruence of modular forms*, Number Theory (New York, 1984–1985), Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 275–280.

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