ON THE EXTENSIONS OF INFINITE-DIMENSIONAL REPRESENTATIONS OF LIE SEMIGROUPS

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The necessary and sufficient conditions have been obtained for extendability of a Banach representation of a generating Lie semigroup S to a local representation of the Lie group G generated by S when the tangent wedge of S is a Lie semialgebra. The most convenient conditions we obtain correspond to the case of unitary representations. In this case, we give a criterion of global extendability if G is exponential and solvable.

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1. Introduction. If S is a subsemigroup of a topological group G with interior points and G is a left quotient group for S , it is easy to prove that every representation of S by invertible operators on a Banach space $\mathcal{V}$ may be extended, in a unique manner, to the representation of G on $\mathcal{V}$ (see [Proposition 6.1](#) below). It is easy to prove that every finite-dimensional representation of a generating Lie semigroup S can be extended to the local representation of the Lie group G generated by S , if G is connected and the tangent wedge of S is a Lie semialgebra [[14](#)]. In this paper, using the infinitesimal method, we study the problem of extendability of a Banach representations π of a generating Lie semigroup S to the connected Lie group G generated by S when the tangent wedge of S is a Lie semialgebra. We show that the differential $d\pi$ extends to the representation of the Lie algebra of G if at least one operator $\pi(s_0)$, $s_0 \in \text{int} S$, is invertible. In this way, the necessary and sufficient conditions of a local extendability of π have been obtained in terms of the tangent objects. The most convenient conditions we obtain, correspond to the case of unitary representations. Thus, the problem of extendability of a representation of S to G has been reduced to the (group theoretical) problem of extendability of a local representation of G to a global one (see [Corollary 2.2](#) below). For exponential and solvable G , we give also necessary and sufficient conditions of global extendability of unitary representations.

The main results of this paper appeared earlier in [[12](#)]. One-dimensional and positive representations of S were studied in detail in [[13](#)].

Throughout, unless otherwise stated, G denotes a connected Lie group with unit e , $L(G)$ its Lie algebra, and $\exp : L(G) \rightarrow G$ its exponential function. For a closed subsemigroup S of G its *tangent wedge* is defined by

$$L(S) := \{X \in L(G) : \exp(\mathbb{R}^+ X) \subseteq S\}. \quad (1.1)$$

A closed subsemigroup $S \subseteq G$ is called *Lie semigroup* if S is the closure in G of $\langle \exp L(S) \rangle$, the semigroup, generated by

$$\exp L(S) := \{\exp X : X \in L(S)\}. \quad (1.2)$$

After shrinking G we may assume that S is a *generating* Lie semigroup, that is, $L(G)$ is the smallest Lie algebra containing $L(S)$ (see [6]).

DEFINITION 1.1. A Lie subsemigroup $S \subseteq G$ is *quasi-invariant* if its tangent wedge $L(S)$ is a Lie semialgebra, that is, for some Campbell-Hausdorff-neighborhood B in $L(G)$

$$(L(S) \cap B) * (L(S) \cap B) \subseteq L(S), \quad (1.3)$$

where $*$ denotes the Campbell-Hausdorff multiplication in $L(G)$

$$X * Y = (X + Y) + \frac{1}{2}[X, Y] + \cdots + H_n(X, Y) + \cdots. \quad (1.4)$$

Note that every *invariant* (with respect to all inner automorphisms of the group G) generating Lie semigroup S is quasi-invariant [16, Proposition IV.7], [6, Theorem III.2.15], but the converse is false. As an example, take the subsemigroup $S^+ \subset \mathrm{GL}(2, \mathbb{R})$ consisting of all matrices with nonnegative entries [8, Example 6.3, page 182].

For a generating Lie semigroup S in a connected Lie group G the interior $\mathrm{int} S$ of S (with respect to G) is a dense ideal of S , and $\mathrm{int} S \subseteq \langle \exp L(S) \rangle$ [16, Proposition IV.6], [6, Theorem V.1.10]. If, in addition, S is quasi-invariant then, according to [6, Theorem II.2.13], we have

$$L(G) = L(S) - L(S). \quad (1.5)$$

The basic reference in Lie semigroups is [6].

We will employ some preliminary results to prove our main theorems.

2. The Gårding space. Let π be a representation of S on a Banach space $\mathcal{V}$, that is, a homomorphism of S into the multiplicative semigroup $\mathrm{gl}(\mathcal{V})$ of all bounded (linear) operators on $\mathcal{V}$ which is continuous with respect to the strong operator topology and satisfies $\pi(e) = I$, the identity operator on $\mathcal{V}$. For every $X \in L(S)$ the map $t \mapsto \pi(\exp tX) : \mathbb{R}^+ \rightarrow \mathrm{gl}(\mathcal{V})$ is a one-parameter C_0 -semigroup of operators. We denote by $A(X)$ or $d\pi(X)$ the generator of this semigroup.

LEMMA 2.1. *The representation π of a Lie semigroup S is uniquely determined by $d\pi$.*

PROOF. Let π and π_1 be two representations of S in a Banach space $\mathcal{V}$ and $d\pi(X) = d\pi_1(X)$ for all $X \in L(S)$. Then C_0 -semigroups $t \mapsto \pi(\exp tX)$ and $t \mapsto \pi_1(\exp tX)$ coincide ($X \in L(S)$). Therefore, $\pi|_{\langle \exp L(S) \rangle} = \pi_1|_{\langle \exp L(S) \rangle}$, and $\pi = \pi_1$ by continuity. $\square$

COROLLARY 2.2. *Let a Banach representation π of a Lie semigroup S extend to a local representation T of the Lie group G generated by S . If, in turn, T extends to a representation $\hat{\pi}$ of the whole G , then $\hat{\pi}$ is an extension of π to G .*

Indeed, set $\pi_1 := \hat{\pi}|_S$. Then π_1 and π coincide on the set $U \cap S$, where U is an e -neighborhood in G , and so $d\pi_1 = d\pi$.

Let $C_0^\infty(S)$ be the space of all compactly supported functions in $C^\infty(G)$ concentrated in $\mathrm{int} S$. For $\phi \in C_0^\infty(S)$ and $u \in \mathcal{V}$, set

$$u(\phi) := \int_G \phi(x) \pi(x) u \, d_l x \quad (2.1)$$

($d_l x$ denotes the left Haar measure in G), and denote by $\mathfrak{D}_S$ the linear subspace of $\mathcal{V}$ generated by $\{u(\phi) : \phi \in C_0^\infty(S), u \in \mathcal{V}\}$. For the case $S = G$ we get the Gårding space $\mathfrak{D}_G$ (cf. [1, Chapter 11, Section 1]). We denote also by $\mathcal{V}^\infty$ the set of such $u \in \mathcal{V}$ that the function $s \mapsto \pi(s)u$ is in $C^\infty(\text{int } S)$.

PROPOSITION 2.3. *Let S be a quasi-invariant generating Lie semigroup in a connected Lie group G and let π be a representation of S on a Banach space $\mathcal{V}$. Then the following assertions hold:*

- (i) $\mathfrak{D}_S \subseteq \mathcal{V}^\infty$.
- (ii) $\mathfrak{D}_S \subseteq \mathfrak{D}(A(X))$, the domain of $A(X)$, and $\mathfrak{D}_S$ is $A(X)$ -invariant and π -invariant ($X \in L(S)$).
- (iii) $\mathfrak{D}_S$ is dense in $\mathcal{V}$.
- (iv) The map $X \mapsto A(X) | \mathfrak{D}_S$ extends uniquely to a linear mapping $\hat{A}$ between $L(G)$ and the linear space of operators on $\mathfrak{D}_S$.
- (v) If $\pi(s_0)$ is invertible for some $s_0 \in \text{int } S$, then $\mathcal{V}^\infty \subseteq \mathfrak{D}(A(X))$ ($X \in L(S)$), and the map $X \mapsto A(X) | \mathcal{V}^\infty$ extends uniquely to a linear mapping $\hat{A}$ between $L(G)$ and the linear space of operators $\mathcal{V}^\infty \rightarrow \mathcal{V}$.

PROOF. (i) Following the proof of Theorem 1 in [1, Chapter 11], let $\gamma(t) = \exp tY$ ($Y \in L(G)$, $t \in \mathbb{R}$) be a one-parameter subgroup in G , $\phi \in C_0^\infty(S)$, $s_0 \in \text{int } S$. Then (cf. [1, formula (14)])

$$t^{-1}(\pi(s_0\gamma(t)) - \pi(s_0))u(\phi) = \pi(s_0) \int_G t^{-1}(\phi(\gamma^{-1}(t)x) - \phi(x))\pi(x)u d_l x. \quad (2.2)$$

Setting in (2.2) $t \rightarrow 0$, we get for the derivative of the function $s \mapsto \pi(s)u(\phi)$ at s_0 along the vector Y

$$Y\pi(s_0)u(\phi) = \pi(s_0)u(\tilde{Y}\phi), \quad (2.3)$$

where the function

$$(\tilde{Y}\phi)(x) := \lim_{t \rightarrow 0} \frac{\phi(\gamma(t)^{-1}x) - \phi(x)}{t} \quad (2.4)$$

belongs to $C_0^\infty(S)$. Indeed, we know from the proof of Theorem 1 in [1, Chapter 11] that $\tilde{Y}\phi \in C_0^\infty(G)$. Thus, it is sufficient to prove that the support $\text{supp } \tilde{Y}\phi \subseteq \text{int } S$ if $K := \text{supp } \phi \subset \text{int } S$. Choose $U \subset G$ open such that $K \subset U \subset U^- \subset \text{int } S$ (U^- is the closure of U). Then there is the neighborhood N of e in G with $NK \subseteq U$ and $\delta > 0$ such that $\gamma(t) \in N$ for $|t| < \delta$. Since the support of the function $x \mapsto \phi(\gamma(t)^{-1}x)$ is $\gamma(t)K \subseteq U$ for $|t| < \delta$, the support of the function $x \mapsto t^{-1}(\phi(\gamma(t)^{-1}x) - \phi(x))$ is contained in U , too. It follows that $\text{supp } \tilde{Y}\phi \subseteq U^- \subset \text{int } S$. Now from (2.3) we conclude that $u(\phi) \in \mathcal{V}^\infty$.

(ii) Suppose that $Y \in L(S)$ and $t > 0$. Then the preceding arguments are true for all $s_0 \in S$. In particular, setting in (2.2) $s_0 = e$ and $t \rightarrow 0+$ we conclude that $u(\phi) \in \mathfrak{D}(A(X))$ and

$$A(Y)u(\phi) = u(\tilde{Y}\phi). \quad (2.5)$$

Equation (2.5) shows that $\mathcal{D}_S$ is $A(Y)$ -invariant. The invariance of $\mathcal{D}_S$ with respect to $\pi(s)$ ($s \in S$) follows from the equality

$$\pi(s)u(\phi) = \int_G \phi(x)\pi(sx)u d_l x = \int_G \phi(s^{-1}x)\pi(x)u d_l x. \quad (2.6)$$

(iii) Since e is an adherent point for $\text{int} S$, there is a net of compact subsets $K \subset \text{int} S$, which shrinks to e . By an argument similar to that given in the proof of Theorem 1 in [1, Chapter 11], $\mathcal{D}_S$ is dense in $\mathcal{V}$.

(iv) Let $X_1, X_2 \in L(S)$ and $u \in \mathcal{D}_S$. Then

$$\begin{aligned} A(X_1 + X_2)u &= \frac{d}{dt} \pi(\exp t(X_1 + X_2))u|_{t=0} \\ &= \frac{d}{dt} \pi(\exp tX_1 \exp tX_2)|_{t=0} \\ &= A(X_1)u + A(X_2)u. \end{aligned} \quad (2.7)$$

The proof of the equality $A(cX_1)u = cA(X_1)u$ ($c \geq 0$) is similar to the preceding one. Now let, by definition,

$$\hat{A}(X_1 - X_2)u := A(X_1)u - A(X_2)u. \quad (2.8)$$

This definition is correct, $\hat{A}$ is linear on $L(G)$ (see (1.5)) and $\hat{A}|L(S) = A$ on $\mathcal{D}_S$. The uniqueness of such an extension follows from (1.5), too.

(v) For $X \in L(S)$ and $s_0 \in \text{int} S$ the function $t \mapsto s_0 \exp tX : \mathbb{R}^+ \rightarrow \text{int} S$ is differentiable. Therefore, for each $u \in \mathcal{V}^\omega$ and $t \geq 0$ the function $t \mapsto \pi(s_0 \exp tX)u = \pi(s_0)\pi(\exp tX)u$ is differentiable, as well. Since $\pi(s_0)$ is invertible, it follows that the function $t \mapsto \pi(\exp tX)u$ is differentiable at $t = 0$ and so $u \in \mathcal{D}(A(X))$. The last assertion follows as in (iv). $\square$

3. Analytical vectors. The vector $u \in \mathcal{V}$ is called *analytic* for the representation π if the map $s \mapsto \pi(s)u$ is analytic on $\text{int} S$. Let $\mathcal{V}^\omega$ or $\mathcal{V}^\omega(\pi)$ be the space of all analytic vectors for π . It is obvious that $\mathcal{V}^\omega \subseteq \mathcal{V}^\infty$. Note that $\mathcal{V}^\omega$ can be trivial even for Hilbert $\mathcal{V}$ and isometric π [3, Example 3.1.18].

PROPOSITION 3.1. *Let S be a quasi-invariant generating Lie semigroup in a connected Lie group G and let $\pi(s_0)$ be invertible for some $s_0 \in \text{int} S$. The space $\mathcal{V}^\omega$ is π -invariant and $A(X)$ -invariant for $X \in L(S)$.*

PROOF. Let $x \in S$. Since the mapping $s \mapsto sx : \text{int} S \rightarrow \text{int} S$ is analytic, the same is $s \mapsto \pi(sx)u = \pi(s)\pi(x)u$ for $u \in \mathcal{V}^\omega$. Therefore $\pi(x)u \in \mathcal{V}^\omega$, that is, $\mathcal{V}^\omega$ is $\pi(x)$ -invariant.

To prove the second statement, first note that $\mathcal{V}^\omega \subseteq \mathcal{D}(A(X))$ ($X \in L(S)$) by Proposition 2.3(v). Let $X \in L(S)$, $u \in \mathcal{V}^\omega$, and $v := A(X)u$. If $s \in \text{int} S$, then

$$\pi(s)v = \pi(s) \frac{d}{dt} \pi(\exp tX)u|_{t=0} = \frac{d}{dt} \pi(s \exp tX)u|_{t=0}. \quad (3.1)$$

Since $s \mapsto \pi(s \exp tX)u$ is analytic on $\text{int} S$, it follows that $s \mapsto \pi(s)v$ is analytic on $\text{int} S$ by Vitali theorem. Indeed,

$$\frac{d}{dt} \pi(s \exp tX)u|_{t=0} = \lim_{h \rightarrow 0} \frac{\pi(s \exp hX)u - \pi(s)u}{h}. \quad (3.2)$$

For sufficiently small h and compact $C \subset \text{int } S$ we have ($s \in C$)

$$\left\| \frac{\pi(s \exp hX)u - \pi(s)u}{h} \right\| \leq \sup_{s \in C} \|\pi(s)\| \left\| \frac{\pi(\exp hX)u - u}{h} \right\| \leq \text{const}, \quad (3.3)$$

because

$$\lim_{h \rightarrow 0} \left\| \frac{\pi(\exp hX)u - u}{h} \right\| = \|A(X)u\| < \infty. \quad (3.4)$$

Thus, the family of analytic functions $s \mapsto h^{-1}(\pi(\exp hX)u - \pi(s)u)$ ($s \in \text{int } S$) is uniformly bounded for $s \in C$ and small h , and all the conditions of the Vitali theorem are satisfied. $\square$

4. The main lemma. Recall that $\hat{A}$ denotes the linear continuation from $L(S)$ to $L(G)$ of the map $A : X \mapsto (d/dt)\pi(\exp tX)|_{t=0}$ (the right side is an operator on $\mathcal{V}^\infty$ whenever at least one operator $\pi(s)$, $s \in \text{int } S$, is invertible).

LEMMA 4.1. *Let S be a quasi-invariant generating Lie semigroup in a connected Lie group G and let π be a representation of S on a Banach space $\mathcal{V}$ such that the operator $\pi(s_0)$ is invertible for some $s_0 \in \text{int } S$. Then $\hat{A}$ is a representation of the Lie algebra $L(G)$ by operators on $\mathcal{D}_S$ or $\mathcal{V}^\omega$.*

PROOF. There exists a star-shaped neighborhood $B_0 \subset L(G)$ of 0 such that $s_0 \exp rX \in \text{int } S$ for $X \in B_0$ and $|r| \leq 2$. Fix vectors $u \in \mathcal{D}_S$ (or $\mathcal{V}^\omega$) and $X \in B_0 \cap L(S)$. Applying the Taylor formula

$$f(1) = f(0) + f'(0) + \frac{1}{2!}f''(0) + \frac{1}{2!} \int_0^1 f'''(\xi)(1-\xi)^2 d\xi \quad (4.1)$$

to the smooth function $f(r) = \pi(s_0 \exp rX)u$ ($|r| \leq 2$), we have

$$\begin{aligned} \pi(s_0 \exp X)u &= \pi(s_0)u + \frac{d}{dr} \pi(s_0 \exp rX)u|_{r=0} + \frac{1}{2} \frac{d^2}{dr^2} \pi(s_0 \exp rX)u|_{r=0} \\ &\quad + \frac{1}{2} \int_0^1 \frac{d^3}{dr^3} \pi(s_0 \exp rX)u|_{r=\xi} (1-\xi)^2 d\xi. \end{aligned} \quad (4.2)$$

Since for all $k \in \mathbb{N}$ and $r \geq 0$

$$\frac{d^k}{dr^k} \pi(s_0 \exp rX)u = \pi(s_0) \pi(\exp rX) (A(X))^k u \quad (4.3)$$

and $\pi(s_0)$ is invertible, it follows from (4.2) that

$$\pi(\exp X)u = u + A(X)u + \frac{1}{2}(A(X))^2 u + \frac{1}{2} \int_0^1 \pi(\exp \xi X) (A(X))^3 u (1-\xi)^2 d\xi. \quad (4.4)$$

Now fix $X_1, X_2 \in L(S)$ and choose $\delta > 0$ such that tX_1, tX_2 , and $tX_1 * tX_2$ belong to $B_0 \cap B$ for $|t| < \delta$, where B is a Campbell-Hausdorff-neighborhood in $L(G)$ and (1.3) holds. Then $tX_1 * tX_2 \in L(S)$ for $t \in [0, \delta)$ and the Campbell-Hausdorff formula implies that

$$tX_1 * tX_2 = t(X_1 + X_2) + \frac{1}{2}t^2[X_1, X_2] + t^3H_3(X_1, X_2) + \dots \quad (4.5)$$

As $\exp(tX_1 * tX_2) = \exp tX_1 \exp tX_2$, we have for $t \in [0, \delta)$, $u \in \mathcal{D}_S$

$$\pi(\exp tX_1)\pi(\exp tX_2)u = \pi(\exp(tX_1 * tX_2))u. \quad (4.6)$$

If we substitute $X = tX_1 * tX_2$ ($t \in [0, \delta)$) to (4.4), then

$$\begin{aligned} \pi(\exp(tX_1 * tX_2))u &= u + A(tX_1 * tX_2)u + \frac{1}{2}(A(tX_1 * tX_2))^2u \\ &\quad + \frac{1}{2} \int_0^1 \pi(\exp \xi(tX_1 * tX_2))(A(tX_1 * tX_2))^3u(1-\xi)^2 d\xi. \end{aligned} \quad (4.7)$$

In view of (4.5) and by the continuity of the linear map $X \mapsto \hat{A}(X)u$,

$$\begin{aligned} A(tX_1 * tX_2)u &= t(A(X_1) + A(X_2))u + \frac{1}{2}t^2\hat{A}([X_1, X_2])u + t^3\hat{A}(H_3(X_1, X_2))u + \dots \\ &= \left(t(A(X_1) + A(X_2)) + \frac{1}{2}t^2\hat{A}([X_1, X_2]) + t^3R_1(t, X_1, X_2)\right)u. \end{aligned} \quad (4.8)$$

Thus,

$$\begin{aligned} (A(tX_1 * tX_2))^2u &= (t^2(A(X_1) + A(X_2))^2 + t^3R_2(t, X_1, X_2))u, \\ (A(tX_1 * tX_2))^3u &= t^3R_3(t, X_1, X_2)u, \end{aligned} \quad (4.9)$$

where the functions $t \mapsto R_n(t, X_1, X_2)u$ ($n = 1, 2, 3$) are bounded for $t \rightarrow 0$. From (4.7) it follows, and since (4.8), (4.9), that for $t \in [0, \delta)$

$$\begin{aligned} \pi(\exp(tX_1 * tX_2))u &= u + t(A(X_1) + A(X_2))u + \frac{1}{2}t^2(\hat{A}([X_1, X_2]) + (A(X_1) + A(X_2))^2)u \\ &\quad + \frac{1}{2}t^3 \int_0^1 \pi(\exp \xi(tX_1 * tX_2))R_3(t, X_1, X_2)u(1-\xi)^2 d\xi \\ &= u + t(A(X_1) + A(X_2))u + \frac{1}{2}t^2(\hat{A}([X_1, X_2]) \\ &\quad + (A(X_1) + A(X_2))^2)u + o(t^2), \end{aligned} \quad (4.10)$$

because $\|\pi(\exp \xi(tX_1 * tX_2))\|$ is bounded for t sufficiently small and $\xi \in [0, 1]$ by the uniform boundedness principle.

By (4.6) and $u \in \mathcal{D}_S$, the left-hand side in (4.10) is a smooth function of t . By the uniqueness of the Taylor polynomial, (4.10) implies

$$\frac{d^2}{dt^2} \pi(\exp(tX_1 * tX_2))u|_{t=0} = \hat{A}([X_1, X_2])u + (A(X_1) + A(X_2))^2u. \quad (4.11)$$

On the other hand, let

$$F(t)u = \pi(\exp tX_1)\pi(\exp tX_2)u, \quad t \geq 0, u \in \mathcal{D}_S. \quad (4.12)$$

Then, for each $\Delta t > 0$

$$\begin{aligned} \Delta F(t)u &= \pi(\exp tX_1)\pi(\exp \Delta tX_1)\pi(\exp tX_2)\pi(\exp \Delta tX_2)u \\ &\quad - \pi(\exp tX_1)\pi(\exp tX_2)u \\ &= \pi(\exp tX_1)(\pi(\exp \Delta tX_1)\pi(\exp \Delta tX_2)v - v), \end{aligned} \quad (4.13)$$

where $v := \pi(\exp tX_2)u \in \mathcal{D}_S$ ($\pi(\exp tX)$ commutes with $\pi(\exp sX)$ for all $X \in L(S)$, $t, s \geq 0$).

Therefore,

$$\begin{aligned} \frac{d}{dt}F(t)u &= \lim_{\Delta t \rightarrow 0} \pi(\exp tX_1) \left(\pi(\exp \Delta tX_1) \frac{\pi(\exp \Delta tX_2)v - v}{\Delta t} + \frac{\pi(\exp \Delta tX_1)v - v}{\Delta t} \right) \\ &= \pi(\exp tX_1)(A(X_2)v + A(X_1)v) \\ &= \pi(\exp tX_1)A(X_1)\pi(\exp tX_2)u + \pi(\exp tX_1)A(X_2)\pi(\exp tX_2)u. \end{aligned} \quad (4.14)$$

Since $\pi(\exp tX_i)$ commutes with $A(X_i)$, $i = 1, 2$,

$$\frac{d}{dt}F(t)u = A(X_1)F(t)u + F(t)A(X_2)u. \quad (4.15)$$

Because $A(X_1)$ is close, it follows that

$$\begin{aligned} \frac{d^2}{dt^2}F(t)u|_{t=0} &= A(X_1)(A(X_1)u + A(X_2)u) + A(X_1)A(X_2)u + A(X_2)A(X_2)u \\ &= (A(X_1))^2u + 2A(X_1)A(X_2)u + (A(X_2))^2u. \end{aligned} \quad (4.16)$$

Now differentiating (4.6) twice at $t = 0$ we obtain for $X_1, X_2 \in L(S)$ by virtue of (4.11)

$$(A(X_1))^2u + 2A(X_1)A(X_2)u + (A(X_2))^2u = \hat{A}([X_1, X_2])u + (A(X_1) + A(X_2))^2u, \quad (4.17)$$

or

$$\hat{A}([X_1, X_2])u = [\hat{A}(X_1), \hat{A}(X_2)]u. \quad (4.18)$$

Since both sides of the last equality are bilinear, it is valid for all $X_1, X_2 \in L(G)$ by (1.5). This completes the proof. $\square$

EXAMPLE 4.2. Let S be as in Lemma 4.1 and $\mathcal{F}$ a finite-dimensional subspace of $L^p(S)$ ($1 \leq p < \infty$) which is invariant under left translations $\lambda(s)f(x) = f(sx)$ ($s, x \in S$). Then λ is a left regular representation of S on $\mathcal{F}$. Since $t \mapsto \lambda(t \exp X)$ ($t \geq 0$, $X \in L(S)$) is a one-parameter C_0 -semigroup on $\mathcal{F}$, the operator $\lambda(\exp X)$ is invertible. Thus $\lambda(s)$ is invertible for all $s \in \langle \exp L(S) \rangle \supseteq \text{int } S$ and $d\lambda$ extends to a representation $\hat{A}$ of $L(G)$ on $\mathcal{F}$ by Lemma 4.1. Then the local representation T of G on $\mathcal{F}$ with $dT = \hat{A}$ extends λ . The representation T is global for simply connected G .

5. Extension of unitary representations. Our first statement deals with local extensions.

THEOREM 5.1. *Let π be a unitary representation on a Hilbert space $\mathcal{H}$ of a quasi-invariant generating Lie semigroup S in a connected Lie group G and let one of the following conditions holds:*

- (1) $\mathcal{H}^w$ is dense in $\mathcal{H}$;
- (2) for some linear independent $X_1, \dots, X_d \in L(S)$ ($d = \dim L(G)$) the Nelson operator

$$\Delta = \sum_{j=1}^d (A(X_j))^2 \quad (5.1)$$

is essentially selfadjoint on $\mathcal{D}_S$.

Then π extends to a local unitary representation of G on $\mathcal{H}$. Moreover, if S algebraically generates G , for some e -neighborhood $U \subset G$ the extension of π to a local representation of U is unique.

PROOF. (1) First, prove that every vector $u \in \mathcal{H}^\omega$ is analytic for the operator $A(X)$ if $X \in L(S)$ satisfies $\exp X \in \text{int} S$. Indeed, $\mathcal{H}^\omega \subseteq \mathcal{D}(A(X))$ and since the function $t \mapsto \exp tX : \mathbb{R} \rightarrow G$ is analytic, the function $t \mapsto \pi(\exp tX)u$ has the same property in the same ϵ -neighborhood of the point $t = 1$. Therefore, u is an analytic vector of the one-parameter unitary group $T_1(t)$ which agrees with $\pi(\exp tX)$ for $t \geq 0$ and has the generator $A(X)$ (in the sense of semigroup theory). Standard Cauchy estimates for the disc $\{|t - 1| < \epsilon\}$ give

$$\|(A(X))^n u\| = \|T_1(1)(A(X))^n u\| = \left\| \frac{d^n}{dt^n} T_1(t)u \Big|_{t=1} \right\| \leq \frac{n!M}{\epsilon^n} \quad (5.2)$$

for some $M > 0$, and so u is an analytic vector for $A(X)$.

Being the generator of $T_1(t)$, operator $A(X)$ is antiselfadjoint and so, it is antisymmetric on $\mathcal{H}^\omega$ ($X \in L(S)$). We claim that $\exp X_j \in \text{int} S$ for some linear independent $X_j \in L(S)$, $j = 1, \dots, d$. Since $L(S)$ is generating in $L(G)$, zero is in the closure of $\text{int} L(S)$ (with respect to $L(G)$). Let B_1 be a neighborhood of zero in $L(G)$ such that $\exp|_{B_1}$ is a homomorphism on its image. Then $\exp(B_1 \cap \text{int} L(S))$ is a nonvoid open subset of S , and we may choose linear independent $X_j \in B_1 \cap \text{int} L(S)$, $j = 1, \dots, d$.

Now if (1) holds, then for the representation $\hat{A}$ of the Lie algebra $L(G)$ by operators on $\mathcal{H}^\omega$ (see Lemma 4.1) all the conditions of the FS^3 -criterion (cf. [1, Chapter 11, Section 6, Theorem 5]), except the simply connectedness of G , are satisfied. But the proof of the FS^3 -criterion in [1] shows that for connected G we get a unitary representation T of a local group W , where W is a neighborhood of unit in G , and a T -invariant dense linear subspace $\mathcal{H}_\infty \subseteq \mathcal{H}$ such that for all $X \in L(G)$ we have $\mathcal{H}^\omega \subseteq \mathcal{H}_\infty \subseteq \mathcal{D}(T(X))$ and $dT(X)|_{\mathcal{H}_\infty} = \hat{A}(X)|_{\mathcal{H}_\infty}$ (the bar denotes the closure of an operator; see Lemmas 3, 4, and formula (11) therein). Since the one-parameter C_0 -semigroup $T_1(t) = \pi(\exp tX)$, $X \in L(S)$, leaves $\mathcal{H}^\omega$ invariant, $\mathcal{H}^\omega$ is an essential domain for $A(X)$ by [3, Corollary 3.1.7]. Hence, $\hat{A}(X) = \overline{A(X)|_{\mathcal{H}^\omega}} = A(X)$ for $X \in L(S)$, and the definition of $\mathcal{H}_\infty$ in [1, Chapter 11, Section 6, Lemma 3] entails that $\mathcal{H}_\infty \subseteq \mathcal{D}(A(X))$. The first assertion follows now from the next proposition.

PROPOSITION 5.2. *Let π be a representation on a Banach space $\mathcal{V}$ of a quasi-invariant generating Lie semigroup S in a connected Lie group G .*

(i) *Let T be a local representation of G on $\mathcal{V}$ and let $\mathcal{V}_0 \subseteq \mathcal{V}$ be a T -invariant dense linear subspace such that $\mathcal{V}_1 \subseteq \mathcal{V}_0 \subseteq \mathcal{D}(T(X)) \cap \mathcal{D}(A(X))$ for a some π -invariant dense linear subspace $\mathcal{V}_1 \subseteq \mathcal{V}$ and $dT(X)|_{\mathcal{V}_0} = A(X)|_{\mathcal{V}_0}$ for all $X \in L(S)$. Then $T|_{U \cap S} = \pi|_{U \cap S}$ for some e -neighborhood $U \subseteq G$, and $\pi(s)$ is invertible for all $s \in S$.*

(ii) *If S algebraically generates G , then for some e -neighborhood $U \subseteq G$ the extension of π to a local representation of U (if such exists) is unique.*

PROOF. (i) Let T be a representation of a local group W , where W is a neighborhood of e in G . Fix $X \in L(S)$ and pick $\delta > 0$ such that $\exp tX \in W$ for $t \in \mathbb{R}$ with $|t| < \delta$. The

map $t \mapsto T(\exp tX)$ is a representation of the additive local group $(-\delta, \delta)$ and for δ sufficiently small it extends to the one-parameter group T_X on $\mathcal{V}$ with the generator $dT(X)$ (cf. [2, Chapter 3, Section 6, Lemma 1]), and $\mathcal{V}_0$ is T_X -invariant. Consider also the C_0 -semigroup $T_1(t) = \pi(\exp tX)$, $t \in \mathbb{R}^+$, with generator $A(X)$. Note that generators of T_X and T_1 coincide on $\mathcal{V}_0$ and that $\mathcal{V}_0$ and $\mathcal{V}_1$ are essential domains for $dT(X)$ and $A(X)$, respectively, by [3, Corollary 3.1.7]. Since $\mathcal{V}_1 \subseteq \mathcal{V}_0 \subseteq \mathcal{D}(A(X))$, $\mathcal{V}_0$ is an essential domain for $A(X)$, too. Therefore, $dT(X) = A(X)$ and hence $T_X(t) = T_1(t)$ for all $t \in \mathbb{R}^+$. In particular, $T(\exp tX) = \pi(\exp tX)$ for $0 \leq t < \delta$. Let B_0 be a star-shaped neighborhood of zero in $L(G)$ such that $\exp B_0 \subset W$. Then the preceding equality implies that $T(\exp X) = \pi(\exp X)$ for $X \in B_0 \cap L(S)$. But $\exp(B_0 \cap L(S)) \supset U \cap S$ for some e -neighborhood $U \subset W$, because $L(S)$ is a semialgebra [15, Theorem III.9], and hence $T|U \cap S = \pi|U \cap S$.

The equality $T_X(t) = T_1(t)$, $t \geq 0$, shows also that $\pi(\exp X)$ is invertible for $X \in L(S)$. It follows that $\pi(s_0)$ is invertible for $s_0 \in \langle \exp L(S) \rangle \supseteq \text{int} S$, and so is $\pi(s) = \pi(ss_0)\pi(s_0)^{-1}$ for all $s \in S$ ($s_0 \in \text{int} S$).

(ii) As before, for an arbitrary small e -neighborhood U in G we have $U \cap S = \exp(B \cap L(S))$ ($B \subset L(G)$ is a neighborhood of zero) by [15, Theorem III.9]. Then the set $U \cap \sigma_X$ algebraically generates the full one-parameter semigroup σ_X corresponding to $X \in L(S)$. Therefore, the set $U \cap S$ algebraically generates the semigroup $\langle \exp L(S) \rangle \supseteq \text{int} S$. Being an ideal in S , $\text{int} S$ algebraically generates G , too. Thus $U \cap S$ algebraically generates G and the local representation T is completely determined by its restriction $T|U \cap S$. This completes the proof of Proposition 5.2. $\square$

(2) Let condition (2) of Theorem 5.1 be satisfied. As it was mentioned above, $A(X)$ is an antiselfadjoint operator and $\mathcal{D}_S$ is an essential domain for $A(X)$, $X \in L(S)$, by [3, Corollary 3.1.7]. Therefore, operator $iA(X)|\mathcal{D}_S$ has a selfadjoint closure. Thus for the representation $\hat{A}$ of the Lie algebra $L(G)$ by operators on $\mathcal{D}_S$ all the conditions of the Nelson criterion (cf. [1, Chapter 11, Section 5, Theorem 2]), excepting the simply connectedness of G , are satisfied. The proof of this criterion (see especially Lemma 1 therein) shows that for connected G we get a local unitary representation T of some neighborhood N of unit in G such that for $X \in L(G)$ with $\exp X \in N$ we have $T(\exp X) = \overline{e^{\hat{A}(X)}} := T_X(1)$, where T_X is a one-parameter unitary group on $\mathcal{H}$ with the generator $\hat{A}(X)$ (in the sense of semigroup theory). Choose, as in the proof of Proposition 5.2, a neighborhood of zero $B_0 \subset L(G)$ such that $\exp B_0 \subset N$ and $\exp(B_0 \cap L(S))$ contains $U \cap S$ for some e -neighborhood $U \subset G$. Since $\hat{A}(X) = A(X)$ for $X \in L(S)$, $T_X(t) = T_1(t) := \pi(\exp tX)$ for $X \in L(S)$, $t \geq 0$. Thus for $X \in B_0 \cap L(S)$ we have $T(\exp X) = \pi(\exp X)$ and $T|U \cap S = \pi|U \cap S$. In other words, T is an extension of π to U . The uniqueness of such an extension follows from Proposition 5.2 immediately. $\square$

We call a Lie group G *exponential* if $\exp L(G) = G$.

THEOREM 5.3. *Let G be a connected exponential and solvable Lie group, S a quasi-invariant generating Lie subsemigroup of G . A unitary representation π of S on a Hilbert space $\mathcal{H}$ extends to the whole G if and only if condition (2) of Theorem 5.1 and the following condition (3) hold:*

(3) if $X_{1,2}, Y_{1,2} \in L(S)$ satisfy $\exp(X_1 - X_2) = \exp(Y_1 - Y_2)$, then

$$\lim_{n \rightarrow \infty} \left(\pi \left(\exp \frac{1}{n} X_1 \right) \left(\pi \left(\exp \frac{1}{n} X_2 \right) \right)^{-1} \right)^n = \lim_{n \rightarrow \infty} \left(\pi \left(\exp \frac{1}{n} Y_1 \right) \left(\pi \left(\exp \frac{1}{n} Y_2 \right) \right)^{-1} \right)^n \quad (5.3)$$

(the strong operator convergence). Moreover, such an extension is unique.

PROOF. The existence of a basis $X_1, \dots, X_d \in L(S)$ has been proved above (Theorem 5.1), and the necessity of (2) follows from the Nelson-Stinespring theorem (cf. [1, Chapter 11, Section 2, Theorem 2]).

Next we have

$$\exp(X_1 - X_2) = \lim_{n \rightarrow \infty} \left(\exp \frac{1}{n} X_1 \left(\exp \frac{1}{n} X_2 \right)^{-1} \right)^n, \quad (5.4)$$

and the same formula for $\exp(Y_1 - Y_2)$. The necessity of (3) follows.

Now let (2) and (3) hold. By Theorem 5.1, there is a local unitary representation T which extends π to some e -neighborhood $N \subseteq G$. For $X \in L(G)$ denote by T_X the unitary one-parameter group in $\mathcal{H}$ such that $T_X(t) = T(\exp tX)$ for $t \in \mathbb{R}$ with $\exp tX \in N$. The proof of Theorem 5.1 yields that the antiselfadjoint operator $\hat{A}(X)$ is the generator of T_X . We extend T to the full G setting $\hat{\pi}(\exp X) := T_X(1)$. To prove the correctness of this definition one should show that $\exp X = \exp Y$ implies $T_X(1) = T_Y(1)$ for $X, Y \in L(G)$. Let $X = X_1 - X_2$ where $X_1, X_2 \in L(S)$. We claim that $\overline{A(X_1) - A(X_2)} = \hat{A}(X)$ and that the Trotter multiplicative formula can be applied to the unitary group T_X (cf. [17, Theorem VIII.31]). Indeed, by definition $\hat{A}(X) = (A(X_1) - A(X_2)) \upharpoonright \mathcal{D}_S$. Since the operator $A(X_1) - A(X_2) \supseteq \hat{A}(X)$ is antisymmetric, it has an antisymmetric closure $\overline{A(X_1) - A(X_2)} \supseteq \hat{A}(X)$, and the hypermaximality property of a selfadjoint operator implies $\overline{A(X_1) - A(X_2)} = \hat{A}(X)$, an antiselfadjoint operator. Now by the Trotter formula mentioned above

$$\begin{aligned} T_X(1) &= e^{\overline{A(X_1) - A(X_2)}} = \lim_{n \rightarrow \infty} (e^{(1/n)A(X_1)} e^{-(1/n)A(X_2)})^n \\ &= \lim_{n \rightarrow \infty} \left(\pi \left(\exp \frac{1}{n} X_1 \right) \left(\pi \left(\exp \frac{1}{n} X_2 \right) \right)^{-1} \right)^n, \end{aligned} \quad (5.5)$$

so the mapping $\hat{\pi}$ is well defined by virtue of (3).

Let G_1 be the simply connected Lie group with $L(G_1) = L(G)$. Then there exist e -neighborhoods $U \subseteq N$ and $U_1 \subseteq G_1$ and a local Lie group morphism $\varphi : U \rightarrow U_1$ such that φ and its reciprocal φ^{-1} both are analytic. Write $\pi_1(x) := T(\varphi^{-1}(x))$, $x \in U_1$, and extend π_1 to a (unique) unitary representation π_1 of G_1 on $\mathcal{H}$. Then for each $X \in L(G)$ and sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned} \hat{\pi}(\exp X) &= T_X(1) = \left(T_X \left(\frac{1}{n} \right) \right)^n = \left(T \left(\exp \frac{1}{n} X \right) \right)^n \\ &= \left(\pi_1 \left(\varphi \left(\exp \frac{1}{n} X \right) \right) \right)^n = \pi_1 \left(\left(\varphi \left(\exp \frac{1}{n} X \right) \right)^n \right). \end{aligned} \quad (5.6)$$

Hence, the mapping $f(X) := \hat{\pi}(\exp X)v$ is analytic on $L(G)$ for every v in $\mathcal{H}^\omega(\pi_1)$, which is a π_1 -invariant dense linear subspace of $\mathcal{H}$, and thus the map $(X, Y) \mapsto$

$\hat{\pi}(\exp X)\hat{\pi}(\exp Y)v$ is separately analytic on $L(G) \times L(G)$ (formula (5.6) shows that $\mathcal{H}^\omega(\pi_1)$ is $\hat{\pi}(\exp X)$ -invariant).

Since G is exponential and solvable, for each $x_0 \in G$ there is an $X_0 \in L(G)$ such that $x_0 = \exp X_0$ and $\exp$ is regular at X_0 [18, Theorem IV.2.44] (see also [4]). So the mapping $\exp^{-1} = \log$ is analytic on the neighborhood $U := \exp B$ of x_0 for some neighborhood B of X_0 in $L(G)$. Therefore the function $x \mapsto \hat{\pi}(x)v = f(\log x)$ ($x \in U$) is analytic for $v \in \mathcal{H}^\omega(\pi_1)$ too, and hence the map $(X, Y) \mapsto \hat{\pi}(\exp X \exp Y)v$ is analytic on $L(G) \times L(G)$.

Because T is a local representation, the equality

$$\hat{\pi}(\exp X \exp Y)v = \hat{\pi}(\exp X)\hat{\pi}(\exp Y)v \quad (5.7)$$

holds for all $X, Y \in L(G)$ with sufficiently small norms, and $v \in \mathcal{H}^\omega(\pi_1)$. The separate analyticity of both sides of (5.7) implies that this formula is valid for all $X, Y \in L(G)$, $v \in \mathcal{H}^\omega(\pi_1)$. By continuity, (5.7) holds for all $v \in \mathcal{H}$ so $\hat{\pi}$ is a unitary representation of G (the continuity of $\hat{\pi}$ on G is a consequence of its continuity on N). The application of Corollary 2.2 shows that $\hat{\pi}$ is an extension of π to G .

Finally, let π' be a unitary extension of π to G . Note that $\mathcal{H}^\omega(\pi') \subseteq \mathcal{H}^\omega(\pi)$. Since for all $X_1, X_2 \in L(S)$, $u \in \mathcal{H}^\omega(\pi')$,

$$d\pi'(X_1 - X_2)u = d\pi'(X_1)u - d\pi'(X_2)u = d\pi(X_1)u - d\pi(X_2)u \quad (5.8)$$

and (1.5) hold, π' is completely determined by π , which completes the proof. $\square$

6. The case of Banach representations. We begin with a very simple case when G is the left quotient group for S [10, Chapter 1, Section 1.10], that is, $S \subset G$ and $G = S^{-1}S$.

PROPOSITION 6.1. *Let G be a topological group and a group of left quotients for S , $\text{int} S \neq \emptyset$. The representation π of S on a Banach space $\mathcal{V}$ is extendable to a representation of G on $\mathcal{V}$ if and only if all operators $\pi(s)$, $s \in S$, are invertible. Moreover, such an extension is unique.*

PROOF. To prove the sufficiency, for every $x \in G$, $x = a^{-1}b$ ($a, b \in S$), set $\hat{\pi}(x) := \pi(a)^{-1}\pi(b)$. Let, in addition, $x = c^{-1}d$ ($c, d \in S$). Then $ac^{-1} = p^{-1}q$ for some $p, q \in S$. Since $pa = qc$, $\pi(p)\pi(a) = \pi(q)\pi(c)$. On the other hand, $a^{-1}b = c^{-1}d$ implies $pb = qd$. Therefore, $\pi(p)\pi(b) = \pi(q)\pi(d)$ or

$$\pi(p)\pi(a)\hat{\pi}(a^{-1}b) = \pi(q)\pi(c)\hat{\pi}(c^{-1}d). \quad (6.1)$$

Thus, we proved that $\hat{\pi}(a^{-1}b) = \hat{\pi}(c^{-1}d)$ and the definition of $\hat{\pi}$ is consistent.

Now let $x = a^{-1}b$ and $y = c^{-1}d$ ($a, b, c, d \in S$) are arbitrary elements of G , and let $bc^{-1} = r^{-1}s$, where $r, s \in S$. Then $xy = (ra)^{-1}sd$ and we have

$$\hat{\pi}(xy) = (\pi(ra))^{-1}\pi(sd) = \pi(a)^{-1}\pi(r)^{-1}\pi(s)\pi(d). \quad (6.2)$$

Since $rb = sc$, $\pi(r)^{-1}\pi(s) = \pi(b)\pi(c)^{-1}$. Therefore

$$\hat{\pi}(xy) = \pi(a)^{-1}\pi(b)\pi(c)^{-1}\pi(d) = \hat{\pi}(x)\hat{\pi}(y). \quad (6.3)$$

Finally, $\hat{\pi}$ is continuous on G , because it is continuous on $\text{int} S$. The uniqueness of $\hat{\pi}$ is obvious. $\square$

COROLLARY 6.2. *Let S be a generating Lie subsemigroup in a connected Lie group G and let one of the following conditions be satisfied:*

- (1) G is nilpotent.
- (2) $L(G)$ is a compact Lie algebra.
- (3) S is invariant in G .

Then every representation of S by invertible operators on a Banach space $\mathcal{V}$ extends uniquely to the representation of G on $\mathcal{V}$.

Indeed, by [7, Theorem 3.46], G is a left quotient group for S .

The general case is more complicated. The following theorem describes the tangent map of a Banach representation of S which extends to a local representation of G . We assume below that $\mathcal{D}(\hat{A}(X)) = \mathcal{D}_S$.

THEOREM 6.3. *Let S be a quasi-invariant generating Lie subsemigroup in a connected Lie group G . The representation π of S on a Banach space $\mathcal{V}$ is extendable to a local representation of G on $\mathcal{V}$ which leaves $\mathcal{D}_S$ invariant if and only if the following conditions hold:*

- (a) *the operators $\hat{A}(X)$ are closable for all $X \in L(G)$;*
- (b) *there exist constants C and $\epsilon > 0$ such that for all $X \in L(G)$ with the norms $|X| < 1$ and $|\operatorname{Re} \lambda| > \epsilon$ the resolvents $R(\lambda, \hat{A}(X))$ are defined and*

$$\|R^n(\lambda, \hat{A}(X))\| \leq \frac{C}{(|\operatorname{Re} \lambda| - \epsilon)^n}, \quad n \in \mathbb{N}; \quad (6.4)$$

- (c) $\mathcal{D}_S$ is invariant with respect to $R(\lambda, \hat{A}(X))$, $X \in L(G)$.

In this case, all operators $\pi(s)$, $s \in S$, are invertible. Moreover, if S algebraically generates G , for some e -neighborhood U in G the extension of π to a local representation of U is unique.

PROOF. Let T be a representation of a local Lie group $U \subset G$ (U is a neighborhood of unit) which extends π and leaves $\mathcal{D}_S$ invariant. The fact that $T|_{U \cap S} = \pi|_{U \cap S}$ entails that $dT(X)u = \hat{A}(X)u (= A(X)u)$ for all $X \in L(S)$, $u \in \mathcal{D}_S$. In view of formula (1.5), the last equality is valid for all $X \in L(G)$ by linearity, and conditions (a), (b), (c) follow from the Krein-Shihvatov theorem [11] in the Kirillov formulation [9, Section 10.5, Theorem 2], for $\mathcal{V}^0 = \mathcal{D}_S$.

Now let π satisfies (a), (b), and (c). We claim that $\pi(s)$ is invertible for each $s \in S$. Indeed, since $\mathcal{D}_S$ is an essential domain for $A(X)$, $X \in L(S)$, condition (b) implies that $A(X)$ is a generator of a C_0 -group by Gelfand theorem [5]. Thus operators $\pi(\exp tX)$, $t \geq 0$, are invertible and the invertibility of $\pi(s)$ follows as in Proposition 5.2. According to Lemma 4.1, $\hat{A}$ is a representation of $L(G)$ by operators on $\mathcal{D}_S$. Again, by the Krein-Shihvatov theorem in the Kirillov formulation for $\mathcal{V}^0 = \mathcal{D}_S$, $T^0 = \hat{A}$, there exists a representation T of some local Lie group W with $L(W) = L(G)$ such that $dT(X)|_{\mathcal{D}_S} = \hat{A}(X)$, $X \in L(G)$, and $\mathcal{D}_S$ is T -invariant. After shrinking W we may assume that W is an e -neighborhood in G and then apply Proposition 5.2 with $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{D}_S$. $\square$

COROLLARY 6.4. *Let S be a quasi-invariant generating Lie semigroup in a simply connected Lie group G . The representation π of S on a Banach space $\mathcal{V}$ extends to a representation of the full G on $\mathcal{V}$ which leaves $\mathcal{D}_S$ invariant if and only if conditions (a), (b), and (c) of Theorem 6.3 hold. Moreover, such an extension is unique.*

PROOF. To demonstrate the sufficiency, note that for simply connected G the local representation T constructed in Theorem 6.3 extends uniquely to the representation $\hat{\pi}$ of the whole G . Since every symmetric e -neighborhood generates G as a semigroup, the extension $\hat{\pi}$ leaves $\mathcal{D}_S$ invariant, as well. Finally, $\hat{\pi}|_S = \pi$ by Lemma 2.1.

The uniqueness of the extension can be proved as in Theorem 5.3. This completes the proof. $\square$

REMARK 6.5. The results in Sections 4, 5, and 6 are true with any $A(X)$ - and π -invariant dense linear subspace $\mathcal{D} \subseteq \mathcal{V}^\infty \cap \mathcal{D}(A(X))$, $X \in L(S)$, in place of $\mathcal{D}_S$.

REFERENCES

- [1] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications*, PWN—Polish Scientific Publishers, Warsaw, 1977.
- [2] N. Bourbaki, *Groupes et Algèbres de Lie*, Masson, Paris, 1990 (French).
- [3] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics. Vol. 1*, Springer-Verlag, New York, 1979.
- [4] D. Z. Dokovic and K. H. Hofmann, *The exponential map in real Lie algebras*, J. Lie Theory 7 (1997), 177–199.
- [5] I. Gelfand, *On one-parametrical groups of operators in a normed space*, CR (Doklady) Acad. Sci. URSS (NS) 25 (1939), 713–718.
- [6] J. Hilgert, K. H. Hofmann, and J. D. Lawson, *Lie Groups, Convex Cones, and Semigroups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1989, Oxford Science Publications.
- [7] J. Hilgert and K. H. Neeb, *Lie Semigroups and Their Applications*, Lecture Notes in Math., vol. 1552, Springer-Verlag, Berlin, 1993.
- [8] K. H. Hofmann and J. D. Lawson, *Foundations of Lie semigroups*, Recent Developments in the Algebraic, Analytical, and Topological Theory of Semigroups (Oberwolfach, 1981), Lecture Notes in Math., vol. 998, Springer, Berlin, 1983, pp. 128–201.
- [9] A. A. Kirillov, *Elements of Representation Theory*, Springer-Verlag, Berlin, 1976.
- [10] A. Klifford and G. Preston, *Algebraicheskaya Teoriya Polugrupp. Tom 1 [The Algebraic Theory of Semigroups. Vol. 1]*, Izdat. “Mir”, Moscow, 1972 (Russian), translated from the English by V. A. Baranovskii and V. G. Zitomirskii. Edited by L. N. Sevrin.
- [11] S. G. Krein and A. M. Shihvatov, *Linear differential equations on a Lie group*, Funkcional. Anal. i Priložen. 4 (1970), no. 1, 52–61.
- [12] A. R. Mirotin, *On the infinite-dimensional representations of Lie semigroups*, preprint 89, Skoryna Gomel State University, 1999.
- [13] ———, *Positive semicharacters of Lie semigroups*, Positivity 3 (1999), no. 1, 23–31.
- [14] ———, *On the extension of finite-dimensional representations of Lie semigroups*, Vestsi Nats. Akad. Navuk Belarusi Ser. Fiz.-Mat. Navuk (2001), no. 1, 18–21, 140.
- [15] K. H. Neeb, *On the foundations of Lie semigroups*, J. Reine Angew. Math. 431 (1992), 165–189.
- [16] ———, *Invariant subsemigroups of Lie groups*, Mem. Amer. Math. Soc. 104 (1993), no. 499, viii+193.
- [17] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators*, Academic Press, New York, 1978.
- [18] M. Wustner, *Contributions to the Structure Theory of Solvable Lie Groups*, Dissertation (German). Technische Hochschule Darmstadt, Germany, 1995.

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