Let $\Omega$ be a relatively compact subdomain of a complex manifold, exhaustible by Stein open sets. We give a necessary and sufficient condition for $\Omega$ to be Stein, in terms of $L^2$-estimates for the $\bar{\partial}$-operator, equivalent to the condition of Markoe (1977) and Silva (1978).

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1. Introduction. As indicated in [7], from the beginning of the theory of Stein spaces, the following question has held great interest: is a complex space, which is exhaustible by a sequence $X_1 \subset X_2 \subset \cdots$ of Stein subspaces, itself Stein?

In [1], the following is proved: every domain in $\mathbb{C}^m$ which is exhaustible by a sequence of Stein domains $B_1 \subset B_2 \subset \cdots$ is itself Stein, and this is shown to hold more generally for unramified Riemann domain $\mathcal{R}$ over $\mathbb{C}^m$ in [6]. In [11], the following is proved: let $X$ be a reduced complex space and $X_1 \subset X_2 \subset \cdots$ be an exhaustion of $X$ by Stein domains, if every pair $(X_j, X_{j+1})$ is Runge then $X = UX_j$ is Stein. Recently, Markoe [9] and Silva [10] proved the following: let $X$ be reduced and $X_1 \subset X_2 \subset \cdots$ be an exhaustion of $X$ by Stein domains. Then $X$ is Stein if and only if $H^1(X, \mathcal{O}) = 0$ ($\mathcal{O}$ being the structure sheaf of $X$).

More recently the following has been proved in [12]: let $\Omega_1 \subset \Omega_2 \subset \cdots$ be a sequence of open Stein subsets of a Stein space $X$, $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, and $\dim H^1(\Omega, \mathcal{O}) < \infty$. Then $\Omega$ is Stein.

Fornæss [4] produced an example to show that if $X_1 \subset X_2 \subset \cdots$ is a sequence of Stein manifolds, the limit manifold $X = \bigcup X_j$, in which each $X_j$ is an open submanifold, need not be Stein. But it is known that if the limit manifold is itself an open submanifold of a Stein manifold then the limit manifold is necessarily Stein.

This led Fornæss and Narasimhan to pose the following problem [5]: let $X$ be a Stein space and $\Omega_1 \subset \Omega_2 \subset \cdots$ an increasing sequence of Stein open sets in $X$. Is $\bigcup \Omega_j$ Stein? As indicated above this is the case when $X$ is a Stein manifold, but this question remains open in the general case.

In this paper, we consider the case where $X$ is a general complex manifold and $\Omega_1 \subset \Omega_2 \subset \cdots$ an increasing sequence of open Stein manifolds in $X$ such that $\Omega = \bigcup \Omega_j$ is relatively compact in $X$. We give a condition for $\Omega$ to be Stein, equivalent to Markoe’s and Silva’s condition and involving $L^2$-estimates for the $\bar{\partial}$ operator.

2. Preliminaries. Let $X$ be an $n$-dimensional complex manifold with a $C^\infty$ Hermitian metric. The space $L^2_{(p,q)}(X)$ of square integrable differential forms of type $(p, q)$ on $X$
is a Hilbert space under the scalar product,

\[ (f, g) = \int_X f^\ast \, \bar{\partial} g, \]

where \(*\) is the Hodge \(*\)-operator associated with the metric and orientation of \(X\).

Let \(\Omega_1 \Subset \Omega_2 \Subset \cdots\) be an increasing sequence of Stein open sets in \(X\) such that their union \(\Omega = \bigcup_{j=1}^\infty \Omega_j\) is relatively compact in \(X\).

The following theorem is our main result.

**Theorem 2.1.** The union \(\Omega\) is Stein if and only if given an \(f \in L^2_{(p,q)}(\Omega)\), which is \(\delta\)-closed in the sense of distributions, there is a \(u \in L^2_{(p,q-1)}(\Omega)\) such that \(\delta u = f\) in the sense of distributions and

\[ \|u\|_{L^2_{(p,q-1)}(\Omega)} \leq K \|f\|_{L^2_{(p,q)}(\Omega)}, \quad q > 0, \]

where \(K\) depends on \(\Omega\).

Let \(U\) be a bounded open set in \(\mathbb{C}^n\), and \(\mathcal{O}\) the structure sheaf of \(\mathbb{C}^n\). A section \(f = (f_1, \ldots, f_p) \in \Gamma(U, \mathcal{O}^p)\), where \(p > 0\) is an integer, is \(L^2\)-bounded if

\[ \|f\|_{L^2(U)} = \|f_1\|_{L^2(U)} + \cdots + \|f_p\|_{L^2(U)} < \infty. \]

We then denote all sections of \(\mathcal{O}^p\) over \(U\) that are \(L^2\)-bounded by \(\Gamma_2(U, \mathcal{O}^p)\).

For the definition of \(L^2\)-bounded sections of coherent analytic sheaves, we require the coherent analytic sheaf \(\mathcal{F}\) to be defined on a simply connected polycylinder neighborhood \(V\) of the closure of \(U\). Then by [8, Theorem 5, Section F, Chapter VI], there is an \(\mathcal{O}\)-homomorphic in another simply connected polycylinder neighborhood \(\hat{V}'\) of the closure of \(U\),

\[ \mathcal{O}^p \xrightarrow{\lambda} \mathcal{F} \rightarrow 0, \]

where \(p > 0\) is some integer; and \(f \in \Gamma(U, \mathcal{F})\) is \(L^2\)-bounded if \(f \in \Gamma_2(U, \mathcal{F}) := \lambda(\Gamma_2(U, \mathcal{O}^p))\). It can be shown that \(\Gamma_2(U, \mathcal{F})\) is independent of \(\lambda\) and \(p\), so that \(\Gamma_2(U, \mathcal{F})\) is well defined.

Now let \(\Omega\) be a relatively compact subdomain of an \(n\)-dimensional complex manifold \(X\). An open subset \(Y\) of \(\Omega\) is said to be admissible for the coherent analytic sheaf \(\mathcal{F}\) defined in the neighborhood of the closure of \(\Omega\) in \(X\), if \(Y\) is Stein. There is a coordinate neighborhood \(V\) in \(X\) of the closure, \(\hat{Y}\) of \(Y\) such that \(V\) is biholomorphic to a simply connected polycylinder \(\hat{V}'\) in \(\mathbb{C}^n\), and \(\hat{Y}\) is contained in the neighborhood of \(\hat{\Omega}\) where \(\hat{\mathcal{F}}\) is defined as \(f \in \Gamma(Y, \hat{\mathcal{F}})\) which is \(L^2\)-bounded if

\[ f \in \Gamma_2(Y, \mathcal{F}) := \{g \in \Gamma(Y, \hat{\mathcal{F}}) : \eta^* (g) \in \Gamma_2(\eta(Y), \eta^* (\hat{\mathcal{F}}))\}, \]

where \(\eta\) is the restriction of the biholomorphic map \(V \rightarrow V'\) to \(Y\), and \(\eta^* (\mathcal{F})\) is the zero direct image of \(\mathcal{F}\) on \(Y\).

Let \(\Omega\) be as in Theorem 2.1 (then clearly \(\Omega\) is locally Stein). Let \(\mathcal{F}\) be a coherent analytic sheaf in a neighborhood of the closure of \(\Omega\). Then it is clear that \(\Omega\) is a finite union, \(\Omega = \bigcup_{j=1}^m U_j\), where each \(U_j\) is admissible for \(\mathcal{F}\). If \(\mathcal{V} = \{U_j\}_{j \in I}, I = \{1, \ldots, m\},\)
where the $U_j$’s are as above, we say that $\mathcal{V}$ is a finite admissible cover of $\Omega$ for $\mathcal{F}$ and we define the $L^2$ (alternate) $q$-cochains of $\mathcal{V}$ with values in $\mathcal{F}$ as those cochains,

$$c = (c_\alpha) \in C^q(\mathcal{V}, \mathcal{F}) = \prod_{\alpha \in I^q + 1} \Gamma(U_\alpha, \mathcal{F}),$$

$$U_\alpha = U_{i_0} \cap \cdots \cap U_{i_q}, \quad \alpha = (i_0, \ldots, i_q),$$

which are alternate and satisfy $c_\alpha \in \Gamma^2(U_\alpha, \mathcal{F})$ for all $\alpha \in I^q + 1$. We denote by $C^q_2(\mathcal{V}, \mathcal{F})$ the space of $L^2$-bounded cochains.

The coboundary operator,

$$\delta : C^q(\mathcal{V}, \mathcal{F}) \rightarrow C^{q+1}(\mathcal{V}, \mathcal{F}),$$

maps $C^q_2(\mathcal{V}, \mathcal{F})$ into $C^{q+1}_2(\mathcal{V}, \mathcal{F})$. If $Z^q_2(\mathcal{V}, \mathcal{F}) = \{ c \in C^q_2(\mathcal{V}, \mathcal{F}) : \delta c = 0 \}$ and $B^q_2(\mathcal{V}, \mathcal{F}) = \delta C^{q-1}_2(\mathcal{V}, \mathcal{F})$, then as usual $B^q_2(\mathcal{V}, \mathcal{F}) \subseteq Z^q_2(\mathcal{V}, \mathcal{F})$ and we define $H^q_2(\mathcal{V}, \mathcal{F}) := Z^q_2(\mathcal{V}, \mathcal{F}) / B^q_2(\mathcal{V}, \mathcal{F})$ and call it the $L^2$-bounded cohomology of $\mathcal{V}$ with values in $\mathcal{F}$. We then have the following theorem.

**Theorem 2.2.** For any $q > 0$, the natural map

$$H^q_2(\mathcal{V}, \mathcal{F}) \rightarrow H^q(\Omega, \mathcal{F})$$

(2.8)

is an isomorphism.

We use Theorem 2.2 as a pivot to prove Theorem 2.1, but the proof of Theorem 2.2 is not given here, since it is similar to that of [2, Theorem].

**3. A triangle of isomorphisms.** Let $\Omega$ be as in Theorem 2.1. By the end of the section Theorem 2.1 will be proved. If $U \neq \emptyset$ is an open set in $\tilde{\Omega}$, then $\mathcal{B}^p_\Omega(U)$ is the Hilbert space of holomorphic $p$-forms $h$ on $\Omega \cap U$ such that

$$\|h\|_{L^2_2(p,0)(\Omega \cap U)} < \infty.$$  

(3.1)

If $V$ is open in $\tilde{\Omega}$ with $\emptyset \neq V \subset U$, the restriction map $\gamma^V_U : \mathcal{B}^p_\Omega(U) \rightarrow \mathcal{B}^p_\Omega(V)$ is defined. Then $\mathcal{B}^p_\Omega = \{ \mathcal{B}^p_\Omega(U), \gamma^V_U \}$ is the canonical presheaf of $L^2$-holomorphic $p$-forms on $\tilde{\Omega}$. The associated sheaf $\mathcal{B}^p_\Omega$ is the sheaf of germs of $L^2$-holomorphic $p$-forms on $\tilde{\Omega}$. We then have the following lemma.

**Lemma 3.1.** Let $\mathcal{D}^p$ be the sheaf of germs of holomorphic $p$-forms on $X$, and $\mathcal{V}$ a finite admissible cover of $\Omega$ for $\mathcal{D}^p$. Then the following diagram is an isomorphism triangle of cohomology groups:

$$
\begin{array}{ccc}
H^q_2(\mathcal{V}, \mathcal{D}^p) & \longrightarrow & H^q(\Omega, \mathcal{D}^p) \\
& \searrow & \downarrow \\
& & H^q(\tilde{\Omega}, \mathcal{D}^p)
\end{array}
$$

(3.2)

for $q \geq 1$ and $p \geq 0$. 

**Proof.** From Theorem 2.2 and the fact that any finite cover of $\tilde{\Omega}$ has a refinement $\mathcal{U} = \{V_j\}_{j \in J}$ such that $\mathcal{R}_\mathcal{U} = \{V_j \cap \Omega\}_{j \in J}$ is a finite admissible cover of $\Omega$ for $\mathcal{D}^p$, the lemma follows.

Now, using Hörmander’s $L^2$-estimates locally we get the following lemma.

**Lemma 3.2.** The cohomology group $H^q(\tilde{\Omega}, \mathcal{R}^p_\mathcal{U})$ is isomorphic to the quotient space

$$\{g : g \in L^2_{(p,q)}(\Omega) \text{ and } \partial g = 0\} / \{\partial h : h \in L^2_{p,q-1}(\Omega) \text{ and } \partial h \in L^2_{(p,q)}(\Omega)\},$$

where $\Omega$ as in Theorem 2.1.

Also the following lemma is proved in [3].

**Lemma 3.3.** If $\Omega \Subset X$ is Stein, where $X$ is a complex manifold, then given $f \in L^2_{(p,q)}(\Omega)$ with $\partial f = 0$, there is $u \in L^2_{(p,q-1)}(\Omega)$ such that

$$\partial u = f, \quad \|u\|_{L^2_{(p,q-1)}(\Omega)} \leq K \|f\|_{L^2_{(p,q)}(\Omega)},$$

where $K$ depends on $\Omega$.

To finish with the proof of Theorem 2.1 we remark that $\mathcal{D}^0 = \mathcal{C}$ is the structure sheaf of $X$ (as in Theorem 2.1), therefore Theorem 2.1 follows from Lemmas 3.1, 3.2, and 3.3, and from Markoe’s and Silva’s condition.

**References**


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