CONDITION R AND FEFFERMAN'S MAPPING THEOREM

PATRICK DARKO

Received 14 May 2001

It is shown that Fefferman's mapping theorem extends to the case of manifolds, that is a biholomorphic map between two strictly pseudoconvex manifolds extends smoothly to their boundaries.

2000 Mathematics Subject Classification: 32H40, 32T15.

1. Introduction. A central question in complex analysis is “does every proper holomorphic mapping $f : D \rightarrow D'$ of bounded domains D, D' with smooth boundaries in $\mathbb{C}^n$ extend smoothly to the boundary of D ?”

The answer has been known to be “yes” in dimension one for a long time. In higher dimensions, in case D and D' are strictly pseudoconvex and f is biholomorphic, Fefferman's famous mapping theorem [8] answers the question in the affirmative.

Bell and Ligocka [4] simplified the proof of Fefferman's mapping theorem and extended the theorem to a wide class of pseudoconvex domains.

In [3], Fefferman's mapping theorem was extended to smoothly bounded pseudoconvex subdomains of Stein manifolds that satisfy condition R . Thereafter, the question was asked whether all smoothly bounded pseudoconvex domains satisfy condition R . Recently, Barrett [2] and Christ [5] have shown that this question has an answer in the negative. But that was not the end of condition R , because the case of strictly pseudoconvex manifolds that are not Stein had not been determined. At first it was thought (because of the work of Barrett [1]) that one could not do without the assumption of Steinness.

In this note, we show that a strictly pseudoconvex manifold need not be Stein before it satisfies condition R ; and following the work of Bedford et al. [3], we extend Fefferman's mapping theorem to all strictly pseudoconvex manifolds.

2. Preliminaries. Let Ω be a relatively compact domain in an n -dimensional complex manifold X . The space $L^2_{(n,0)}(\Omega)$ is defined to be the set of $(n,0)$ forms ω such that

$$\|\omega\|^2 = (\sqrt{-1})^{n^2} \int_{\Omega} \omega \wedge \bar{\omega} \quad (2.1)$$

is finite. The space $L^2_{(n,0)}(\Omega)$ is a Hilbert space with inner product given by

$$(\omega, \eta) = (\sqrt{-1})^{n^2} \int_{\Omega} \omega \wedge \bar{\eta}. \quad (2.2)$$

The Bergman-Kobayashi projection P_Ω associated to Ω is the orthogonal projection of $L^2_{(n,0)}(\Omega)$ onto $H_{(n,0)}(\Omega)$, the closed subspace of $L^2_{(n,0)}(\Omega)$ consisting of holomorphic $(n,0)$ forms. If Ω has a smooth boundary, Ω satisfies condition R if the Bergman-Kobayashi projection associated to Ω maps $C^\infty_{(n,0)}(\bar{\Omega})$ into $C^\infty_{(n,0)}(\bar{\Omega})$.

To make use of the proof in [3], we show that if Ω above has smooth boundary and it is strictly pseudoconvex, then Ω satisfies condition R ; and, in addition, if p_0 is a point in X near the boundary $\partial\Omega$ of Ω , then there are n functions $g_1, \dots, g_n$ that are holomorphic in a neighborhood of $\bar{\Omega}$ and that form a coordinate system at p_0 .

Our main result is the following theorem.

THEOREM 2.1. *Let X_1 and X_2 be n -dimensional complex manifolds and $\Omega_1 \Subset X_1$, $\Omega_2 \Subset X_2$ strictly pseudoconvex subdomains with smooth boundaries. Let $f : \Omega_1 \rightarrow \Omega_2$ be a biholomorphic mapping between Ω_1 and Ω_2 . Then f extends smoothly to a C^∞ diffeomorphism of $\bar{\Omega}_1$ and $\bar{\Omega}_2$.*

3. Condition R . To establish condition R for smoothly bounded strictly pseudoconvex subdomains of complex manifolds, we need a result of Gunning and Rossi [9] which we met on the way to proving theorems in [6, 7]. Their result is the following theorem.

THEOREM 3.1. *Let Ω be a strictly pseudoconvex domain in a complex manifold Y . There are a Stein manifold X and a proper holomorphic mapping $\pi : \Omega \rightarrow X$ with the following properties:*

- (i) $\pi : \mathbb{O}_X \cong \mathbb{O}_\Omega$;
- (ii) *there are finitely many points $x_1, \dots, x_z$ in X such that $\pi^{-1}(x_j)$ is a compact subvariety of Ω of positive dimension, and $\pi : \Omega \setminus \bigcup \pi^{-1}(x_j) \cong X \setminus \{x_1, \dots, x_z\}$.*

The first statement means that the rings of holomorphic functions $\mathbb{O}_X$ and $\mathbb{O}_\Omega$ on X and Ω , respectively, are isomorphic under the map induced by π . The second means that $\Omega \setminus \bigcup \pi^{-1}(x_j)$ and $X \setminus \{x_1, \dots, x_z\}$ are biholomorphic.

Now from the proof of [Theorem 3.1](#) as given in [9], it is clear that there is a strictly pseudoconvex neighborhood Ω' of $\bar{\Omega}$ such that Ω' can replace Ω in [Theorem 3.1](#) so that the compact set $\bigcup \pi^{-1}(x_j)$ corresponding to Ω' is contained in Ω .

If X' corresponds to Ω' in [Theorem 3.1](#) and $X = \pi(\Omega)$, then clearly if Ω has a smooth boundary then X is a Stein strictly pseudoconvex manifold with a smooth boundary, and therefore, as is well-known, X satisfies condition R .

We can regard Ω as imbedded in X . Then it is clear that $L^2_{(n,0)}(\Omega) = L^2_{(n,0)}(X)$ and $H_{(n,0)}(\Omega) = H_{(n,0)}(X)$. Therefore the Bergman-Kobayashi projections P_X and P_Ω are equal, and it is not difficult to see (using Sobolev spaces) that Ω satisfies condition R .

4. Local coordinates near the boundary. Again from [Theorem 3.1](#) we get the last theorem that we need in the proof of [Theorem 2.1](#).

THEOREM 4.1. *Let Ω be a strictly pseudoconvex subdomain of a complex manifold Y . Then near the boundary $\partial\Omega$ of Ω , local coordinates are given by holomorphic functions in a neighborhood of $\bar{\Omega}$.*

PROOF. As indicated in Section 3, from the proof of Theorem 3.1 as given in [9] it is clear that there is a strictly pseudoconvex neighborhood Ω' of $\bar{\Omega}$ such that Ω' can replace Ω in Theorem 3.1 so that the compact set $\cup \pi^{-1}(x_j)$ corresponding to Ω' is contained in Ω . Now if p_0 is a point in Ω' near the boundary $\partial\Omega$ of Ω , let $\pi(p_0)$ have holomorphic functions $g_1, \dots, g_n$ on the Stein manifold X that form local coordinates at $\pi(p_0)$. Then $g_1 \circ \pi, \dots, g_n \circ \pi$ form local coordinates at p_0 , which are holomorphic in a neighborhood of $\bar{\Omega}$. $\square$

5. Proof of Theorem 2.1. The proof of Theorem 2.1 relies on the following two lemmas whose proofs are in [3].

LEMMA 5.1. *If ω is a holomorphic $(n, 0)$ form in $C_{(n,0)}^\infty(\bar{\Omega}_2)$, then $f^*\omega$ is in $C_{(n,0)}^\infty(\bar{\Omega}_1)$.*

LEMMA 5.2. *If ω is a holomorphic $(n, 0)$ form in $C_{(n,0)}^\infty(\bar{\Omega}_2)$ that vanishes to at most finite order at any boundary point of Ω_2 , then $f^*\omega$ vanishes to at most finite order at any boundary point of Ω_1 .*

Now to prove Theorem 2.1, we initiate the proof of Theorem 2.1 in [3]:

Let p_0 be a boundary point of Ω_1 and let $z_1, \dots, z_n$ be holomorphic coordinates near p_0 . We show that f extends smoothly to $\partial\Omega_1$ near p_0 . Let $\{p_i\}$ be a sequence of points in Ω_1 that converges to p_0 . Then $\{f(p_i)\}$ converges to a point q_0 in $\partial\Omega_2$. Let $g_1, \dots, g_n$ be n functions on Ω_2 that extend to be holomorphic in a neighborhood of $\bar{\Omega}_2$ in X_2 and that form a coordinate chart at q_0 . Define a holomorphic function u near p_0 via

$$udz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = f^*(dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n). \quad (5.1)$$

By Lemmas 5.1 and 5.2 u extends smoothly to $\partial\Omega_1$ near p_0 and u vanishes to a finite order near p_0 .

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, then we define $g^\alpha = \prod_{i=1}^n g_i^{\alpha_i}$. Lemma 5.1 implies that the form $f^*(g^\alpha dg_1 \wedge \cdots \wedge dg_n)$ extends smoothly to $\partial\Omega_1$ near p_0 for each α . Hence, u and $u(g^\alpha \circ f)$ extend smoothly to $\partial\Omega_1$ near p_0 for each α , and u vanishes to at most finite order at p_0 . By the division theorem cited in [3], $g_i \circ f$ extends smoothly to $\partial\Omega_1$ near p_0 for each i . Hence f extends smoothly to $\partial\Omega_1$ near p_0 . Since p_0 was arbitrarily chosen, we conclude that f extends smoothly to all of $\partial\Omega_1$. Now we can replace f by f^{-1} and then the theorem follows.

REFERENCES

- [1] D. E. Barrett, *Biholomorphic domains with inequivalent boundaries*, Invent. Math. **85** (1986), 373–377.
- [2] ———, *Behavior of the Bergman projection on the Diederich-Fornæss worm*, Acta Math. **168** (1992), 1–10.
- [3] E. Bedford, S. Bell, and D. Catlin, *Boundary behavior of proper holomorphic mappings*, Michigan Math. J. **30** (1983), 107–111.
- [4] S. Bell and E. Ligočka, *A simplification and extension of Fefferman's theorem on biholomorphic mappings*, Invent. Math. **57** (1980), 283–289.
- [5] M. Christ, *Global C^∞ irregularity of the $\bar{\partial}$ -Neumann problem for worm domains*, J. Amer. Math. Soc. **9** (1996), 1171–1185.
- [6] P. W. Darko, *The L^2 - $\bar{\partial}$ -problem on manifolds with piecewise strictly pseudoconvex boundaries*, Math. Proc. Cambridge Philos. Soc. **116** (1994), 147–149.

- [7] ———, *Corrigendum: "The $L^2 \bar{\partial}$ -problem on manifolds with piecewise strictly pseudoconvex boundaries"*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 191–192.
- [8] C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.
- [9] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, New Jersey, 1965.

PATRICK DARKO: DEPARTMENT OF MATHEMATICS, LINCOLN UNIVERSITY, LINCOLN, PA 19352,
USA