IJMMS 30:9 (2002) 511–514
PII. S0161171202111252
http://ijmms.hindawi.com

SOME THEOREMS OF RANDOM OPERATOR EQUATIONS

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Received 17 January 2002

We investigate a class of random operator equations, generalize a famous theorem, and obtain some new results.

2000 Mathematics Subject Classification: 60H25, 47H10.

Let E be a separable real Banach space, $(\mathbf{E},\mathbf{B})$ a measurable space, where $\mathbf{B}$ denotes the σ-algebra generated by all open subsets in E, let (Ω, U, γ) be a complete probability measure space, where $\gamma(\Omega) = 1$, let D be a bounded open set in X and ∂D the boundary of D in X. Let X be a cone in E, and let "$\leq$", "$<$" be partial order of E.

Lemma 1. *When $\gamma > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(\gamma-1)^{\alpha+1}x < \gamma^{\alpha+1}x - x. \tag{1}$$

Proof. Letting $f(\gamma) = \gamma^{\alpha+1} - 1 - (\gamma-1)^{\alpha+1}$, where $\alpha > 0$, then

$$\begin{aligned} f'(\gamma) &= (\alpha+1)\gamma^{\alpha} - (\alpha+1)(\gamma-1)^{\alpha} \\ &= (\alpha+1)\left[\gamma^{\alpha} - (\gamma-1)^{\alpha}\right] > 0 \end{aligned} \tag{2}$$

(since $\gamma > 1$, then $0 < \gamma - 1 < \gamma$, and $\alpha > 0$, obtaining $0 < (\gamma-1)^{\alpha} < \gamma^{\alpha}$, i.e., $\gamma^{\alpha} - (\gamma-1)^{\alpha} > 0$).

Therefore $f(\gamma)$ is a monotonous increasing function. When $\gamma > 1$, we have $f(\gamma) > f(1)$, and $f(1) = 0$. Hence $f(\gamma) > 0$, that is, $\gamma^{\alpha+1} - 1 - (\gamma-1)^{\alpha+1} > 0$, that is,

$$(\gamma-1)^{\alpha+1} < \gamma^{\alpha+1} - 1. \tag{3}$$

When $x \in X$, $x \neq \theta$, that is, $x > \theta$, we have $(\gamma-1)^{\alpha+1}x < \gamma^{\alpha+1}x - x$. □

Lemma 2 (see [1]). *Let X be a closed convex subset of E, D a bounded open subset in X, and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator. Meanwhile, such that $x \neq (t/\mu)A(\omega, x)$ a.s., for every $\omega \in \Omega$, for every $x \in \partial D$, where $t \in (0,1]$, $\mu \geq 1$. Then the random operator equation $A(\omega, x) = \mu x$, (for every $(\omega, x) \in \Omega \times \bar{D}$, $\mu \geq 1$) has a random solution in D.*

Theorem 3. *Let D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\begin{aligned} &\left[\lambda\|\mu x\| + \|A(\omega,x) - \mu x\|^{\alpha}\right]\|A(\omega,x) - \mu x\|x \\ &\quad \geq \left[\lambda\|\mu x\| + \|A(\omega,x)\|^{\alpha}\right]\|A(\omega,x)\|x - \lambda\|\mu x\|^2 x - \|\mu x\|^{\alpha+1}x \end{aligned} \tag{4}$$

for every $(\omega,x) \in \Omega\times\partial D$, $\lambda \geq 0$, $\mu \geq 1$, $\alpha > 0$. *Then the random operator equation* $A(\omega,x) = \mu x$ *(for every* $(\omega,x) \in \Omega\times\bar{D}$, *where* $\mu \geq 1$*) has a random solution in* $\bar{D}$.

Proof. Assume that $A(\omega,x) = \mu x$ has no random solution on ∂D (otherwise, the theorem has obtained proof), that is, $A(\omega,x) \neq \mu x$ a.s., for every $(\omega,x) \in \Omega\times\partial D$, where $\mu \geq 1$. That is,

$$x \neq \frac{1}{\mu}A(\omega,x) \quad \text{a.s.} \tag{5}$$

We prove that

$$x \neq t\frac{1}{\mu}A(\omega,x), \tag{6}$$

where $\mu \geq 1$, $t \in (0,1)$, for every $(\omega,x) \in \Omega\times\partial D$.

Suppose that (6) is not true, that is, there exists a $t_0 \in (0,1)$, an $\omega_0 \in \Omega$, and an $x_0 \in \partial D$, such that $x_0 = t_0(1/\mu)A(\omega_0,x_0)$. That is, $A(\omega_0,x_0) = (\mu/t_0)x_0$, where $\mu \geq 1$, $t_0 \in (0,1)$, $\omega_0 \in \Omega$, and $x_0 \in \partial D$.

Inserting $A(\omega_0,x_0) = (\mu/t_0)x_0$ into (4), obtaining

$$\begin{aligned}&\left[\lambda\|\mu x_0\| + \left\|\frac{\mu}{t_0}x_0 - \mu x_0\right\|^{\alpha}\right]\left\|\frac{\mu}{t_0}x_0 - \mu x_0\right\|x_0\\ &\quad\geq \left[\lambda\|\mu x_0\| + \left\|\frac{\mu}{t_0}x_0\right\|^{\alpha}\right]\left\|\frac{\mu}{t_0}x_0\right\|x_0 - \lambda\|\mu x_0\|^2 x_0 - \|\mu x_0\|^{\alpha+1}x_0,\end{aligned} \tag{7}$$

where $\lambda \geq 0$, $\mu \geq 1$, $\alpha > 0$, $t_0 \in (0,1)$, and $x_0 \in \partial D$. This implies that

$$\begin{aligned}&\lambda\|\mu x_0\|\left\|\frac{\mu}{t_0}x_0 - \mu x_0\right\|x_0 + \left\|\frac{\mu}{t_0}x_0 - \mu x_0\right\|^{\alpha+1}x_0\\ &\quad\geq \lambda\|\mu x_0\|\left\|\frac{\mu}{t_0}x_0\right\|x_0 + \left\|\frac{\mu}{t_0}x_0\right\|^{\alpha+1}x_0 - \lambda\|\mu x_0\|^2 x_0 - \|\mu x_0\|^{\alpha+1}x_0,\end{aligned} \tag{8}$$

that is,

$$\begin{aligned}&\lambda\left(\frac{1}{t_0} - 1\right)\|\mu x_0\|^2 x_0 + \left(\frac{1}{t_0} - 1\right)^{\alpha+1}\|\mu x_0\|^{\alpha+1}x_0\\ &\quad\geq \lambda\left(\frac{1}{t_0} - 1\right)\|\mu x_0\|^2 x_0 + \frac{1}{t_0^{\alpha+1}}\|\mu x_0\|^{\alpha+1}x_0 - \|\mu x_0\|^{\alpha+1}x_0\end{aligned} \tag{9}$$

since $\mu \geq 1$, $x_0 \in \partial D$, thus $\mu x_0 \neq 0$.

Therefore $\|\mu x_0\|^{\alpha+1} \neq 0$, by (9), we obtain

$$\left(\frac{1}{t_0} - 1\right)^{\alpha+1}x_0 \geq \frac{1}{t_0^{\alpha+1}}x_0 - x_0. \tag{10}$$

Letting $y = 1/t_0$, by (10), we have

$$(y-1)^{\alpha+1}x_0 \geq y^{\alpha+1}x_0 - x_0, \tag{11}$$

where $y > 1$, $\alpha > 0$, $x_0 \in X$, and $x_0 \neq \theta$.

This is in contradiction with Lemma 1. Hence

$$x \neq t\frac{1}{\mu}A(\omega,x) \tag{12}$$

for every $(\omega,x) \in \Omega\times\partial D$, where $t \in (0,1)$, $\mu \geq 1$.

By (5) and (12), we know that

$$x \neq t\frac{1}{\mu}A(\omega,x) \quad \text{a.s.}, \tag{13}$$

where $\mu \geq 1$, $t \in (0,1)$, for every $(\omega,x) \in \Omega\times\partial D$.

According to Lemma 2, we obtain that the random operator equation $A(\omega,x) = \mu x$ (where $\mu \geq 1$, for every $(\omega,x) \in \Omega\times\bar{D}$) has a random solution in D. □

Remark 4. In Theorem 3, when $\lambda = 0$, $\alpha = 1$, $\mu = 1$, and $A(\omega,\cdot) = A$, (4) is that $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$. Thus, Theorem 3 is a generalization of the famous Altman theorem.

We can see that Lemma 5 holds easily.

Lemma 5. *When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(y+1)^{\alpha+1}x > y^{\alpha+1}x + x. \tag{14}$$

Theorem 6. *Let D be a bounded open subset in X and $\theta \in D$. Suppose that $A : \Omega\times\bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\begin{aligned}&[\lambda\|\mu x\| + \|A(\omega,x)+\mu x\|^{\alpha}]\|A(\omega,x)+\mu x\|x \\ &\quad\leq [\lambda\|\mu x\| + \|A(\omega,x)\|^{\alpha}]\|A(\omega,x)\|x + \lambda\|\mu x\|^2 x + \|\mu x\|^{\alpha+1}x,\end{aligned} \tag{15}$$

where $\lambda \geq 0$, $\mu \geq 1$, $\alpha > 0$, for every $(\omega,x) \in \Omega\times\partial D$. Then the random operator equation $A(\omega,x) = \mu x$ (where $\mu \geq 1$, for every $(\omega,x) \in \Omega\times\bar{D}$) has a random solution in D.

Proof. From (15), we can easily prove that $A(\omega,x) = \mu x$ has no random solution on ∂D, by virtue of Lemma 5, see Theorem 3 for other section. □

Lemma 7. *When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:*

$$(y+1)^{\alpha+1}x - (y-1)^{\alpha+1}x > 2x. \tag{16}$$

Proof. By Lemmas 1 and 5, we have

$$(y-1)^{\alpha+1}x < y^{\alpha+1}x - x, \tag{17}$$

$$y^{\alpha+1}x + x < (y+1)^{\alpha+1}x, \tag{18}$$

summing (17) and (18) we obtain

$$(y-1)^{\alpha+1}x+y^{\alpha+1}x+x<y^{\alpha+1}x-x+(y+1)^{\alpha+1}x. \tag{19}$$

That is,

$$(y+1)^{\alpha+1}x-(y-1)^{\alpha+1}x>2x, \tag{20}$$

where $\alpha>0$, $y>1$, $x\in X$, and $x\neq\theta$. □

Theorem 8. *Let D be a bounded open subset in X and $\theta\in D$. Suppose that $A:\Omega\times\bar{D}\to X$ is a random semiclosed 1-set-contractive operator, such that*

$$\|A(\omega,x)+\mu x\|^{\alpha+1}x-\|A(\omega,x)-\mu x\|^{\alpha+1}x\leq 2\|\mu x\|^{\alpha+1}x, \tag{21}$$

where $\alpha>0$, $\mu\geq 1$, for every $(\omega,x)\in\Omega\times\partial D$. Then the random operator equation $A(\omega,x)=\mu x$ (where $\mu\geq 1$, for every $(\omega,x)\in\Omega\times\bar{D}$) has a random solution in D.

Proof. The theorem can be proved using Lemma 7, see also Theorems 3 and 6. □

Remark 9. Since X is a cone in E, then X is a closed convex subset of E.

Acknowledgment. This work was funded by the Natural Science Foundation of China.

References

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