SOME THEOREMS OF RANDOM OPERATOR EQUATIONS

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We investigate a class of random operator equations, generalize a famous theorem, and obtain some new results.

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Let $E$ be a separable real Banach space, $(E, B)$ a measurable space, where $B$ denotes the $\sigma$-algebra generated by all open subsets in $E$, let $(\Omega, U, \gamma)$ be a complete probability measure space, where $\gamma(\Omega) = 1$, let $D$ be a bounded open set in $X$ and $\partial D$ the boundary of $D$ in $X$. Let $X$ be a cone in $E$, and let “$\leq$”, “$<$” be partial order of $E$.

**Lemma 1.** When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:

$$(y - 1)^{\alpha + 1} x < y^{\alpha + 1} x - x.$$  \hspace{1cm} (1)

**Proof.** Letting $f(y) = y^{\alpha + 1} - 1 - (y - 1)^{\alpha + 1}$, where $\alpha > 0$, then

$$f'(y) = (\alpha + 1)y^{\alpha} - (\alpha + 1)(y - 1)^{\alpha}$$

$$= (\alpha + 1)[y^{\alpha} - (y - 1)^{\alpha}] > 0$$  \hspace{1cm} (2)

(since $y > 1$, then $0 < y - 1 < y$, and $\alpha > 0$, obtaining $0 < (y - 1)^{\alpha} < y^{\alpha}$, i.e., $y^{\alpha} - (y - 1)^{\alpha} > 0$).

Therefore $f(y)$ is a monotonous increasing function. When $y > 1$, we have $f(y) > f(1)$, and $f(1) = 0$. Hence $f(y) > 0$, that is, $y^{\alpha + 1} - 1 - (y - 1)^{\alpha + 1} > 0$, that is,

$$(y - 1)^{\alpha + 1} < y^{\alpha + 1} - 1.$$  \hspace{1cm} (3)

When $x \in X$, $x \neq \theta$, that is, $x > \theta$, we have $(y - 1)^{\alpha + 1} x < y^{\alpha + 1} x - x$. □

**Lemma 2** (see [1]). Let $X$ be a closed convex subset of $E$, $D$ a bounded open subset in $X$, and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed $1$-set-contractive operator. Meanwhile, such that $x \neq (t/\mu)A(\omega, x)$ a.s., for every $\omega \in \Omega$, for every $x \in \partial D$, where $t \in (0, 1]$, $\mu \geq 1$. Then the random operator equation $A(\omega, x) = \mu x$, (for every $(\omega, x) \in \Omega \times \bar{D}$, $\mu \geq 1$) has a random solution in $D$.

**Theorem 3.** Let $D$ be a bounded open subset in $X$ and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed $1$-set-contractive operator, such that

$$[\lambda\|\mu x\| + \|A(\omega, x) - \mu x\|^{\alpha}]\|A(\omega, x) - \mu x\| x$$

$$\geq [\lambda\|\mu x\| + \|A(\omega, x)\|^{\alpha}]\|A(\omega, x)\| x - \lambda\|\mu x\|^{2} x - \|\mu x\|^{\alpha + 1} x$$  \hspace{1cm} (4)
for every \((\omega, x) \in \Omega \times \partial D, \lambda \geq 0, \mu \geq 1, \alpha > 0\). Then the random operator equation \(A(\omega, x) = \mu x\) (for every \((\omega, x) \in \Omega \times \partial D\), where \(\mu \geq 1\)) has a random solution in \(\bar{D}\).

**Proof.** Assume that \(A(\omega, x) = \mu x\) has no random solution on \(\partial D\) (otherwise, the theorem has obtained proof), that is, \(A(\omega, x) \neq \mu x\) a.s., for every \((\omega, x) \in \Omega \times \partial D\), where \(\mu \geq 1\). That is,

\[
x \neq \frac{1}{\mu} A(\omega, x) \quad \text{a.s.}
\]

We prove that

\[
x \neq t \frac{1}{\mu} A(\omega, x),
\]

where \(\mu \geq 1, t \in (0, 1)\), for every \((\omega, x) \in \Omega \times \partial D\).

Suppose that (6) is not true, that is, there exists a \(t_0 \in (0, 1), \omega_0 \in \Omega, \) and an \(x_0 \in \partial D\), such that \(x_0 = t_0 (1/\mu) A(\omega_0, x_0)\). That is, \(A(\omega_0, x_0) = (\mu/t_0) x_0\), where \(\mu \geq 1, t_0 \in (0, 1), \omega_0 \in \Omega, \) and \(x_0 \in \partial D\).

Inserting \(A(\omega_0, x_0) = (\mu/t_0) x_0\) into (4), obtaining

\[
\left[ \lambda \|x_0\|^\alpha + \left| \frac{\mu}{t_0} x_0 - \mu x_0 \right| \right] \left| \frac{\mu}{t_0} x_0 - \mu x_0 \right| x_0 \geq \left[ \lambda \|x_0\|^\alpha + \left| \frac{\mu}{t_0} x_0 \right| \right] \left| \frac{\mu}{t_0} x_0 - \mu x_0 \right| x_0 - \lambda \|\mu x_0\|^2 x_0 - \|\mu x_0\|^{\alpha + 1} x_0, \]

where \(\lambda \geq 0, \mu \geq 1, \alpha > 0, t_0 \in (0, 1), \) and \(x_0 \in \partial D\). This implies that

\[
\lambda \|x_0\|^\alpha \left| \frac{\mu}{t_0} x_0 - \mu x_0 \right| x_0 + \left| \frac{\mu}{t_0} x_0 - \mu x_0 \right|^{\alpha + 1} x_0 \geq \lambda \|x_0\|^\alpha \left| \frac{\mu}{t_0} x_0 \right| x_0 - \lambda \|\mu x_0\|^2 x_0 - \|\mu x_0\|^{\alpha + 1} x_0, \]

that is,

\[
\lambda \left( \frac{1}{t_0} - 1 \right) \|\mu x_0\|^2 x_0 + \left( \frac{1}{t_0} - 1 \right)^{\alpha + 1} \|\mu x_0\|^{\alpha + 1} x_0 \geq \lambda \left( \frac{1}{t_0} - 1 \right) \|\mu x_0\|^2 x_0 + \frac{1}{t_0^{\alpha + 1}} \|\mu x_0\|^{\alpha + 1} x_0 - \|\mu x_0\|^{\alpha + 1} x_0
\]

since \(\mu \geq 1, x_0 \in \partial D\), thus \(\mu x_0 \neq 0\).

Therefore \(\|\mu x_0\|^{\alpha + 1} \neq 0\), by (9), we obtain

\[
\left( \frac{1}{t_0} - 1 \right)^{\alpha + 1} x_0 \geq \frac{1}{t_0^{\alpha + 1}} x_0 - x_0. \]
Letting $y = 1/t_0$, by (10), we have

$$(y - 1)^{\alpha + 1}x_0 \geq y^{\alpha + 1}x_0 - x_0,$$  \hspace{1cm} (11)

where $y > 1$, $\alpha > 0$, $x_0 \in X$, and $x_0 \neq \theta$.

This is in contradiction with Lemma 1. Hence

$$x \neq \frac{1}{\mu}A(\omega, x)$$  \hspace{1cm} (12)

for every $(\omega, x) \in \Omega \times \partial D$, where $t \in (0, 1)$, $\mu \geq 1$.

By (5) and (12), we know that

$$x \neq \frac{1}{\mu}A(\omega, x) \text{ a.s.},$$  \hspace{1cm} (13)

where $\mu \geq 1$, $t \in (0, 1)$, for every $(\omega, x) \in \Omega \times \partial D$.

According to Lemma 2, we obtain that the random operator equation $A(\omega, x) = \mu x$ (where $\mu \geq 1$, for every $(\omega, x) \in \Omega \times \bar{D}$) has a random solution in $D$.

**Remark 4.** In Theorem 3, when $\lambda = 0$, $\alpha = 1$, $\mu = 1$, and $A(\omega, \cdot) = A$, (4) is that $\|Ax - x\|^2 \geq \|Ax\|^2 - \|x\|^2$. Thus, Theorem 3 is a generalization of the famous Altman theorem.

We can see that Lemma 5 holds easily.

**Lemma 5.** When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:

$$(y + 1)^{\alpha + 1}x > y^{\alpha + 1}x + x.$$  \hspace{1cm} (14)

**Theorem 6.** Let $D$ be a bounded open subset in $X$ and $\theta \in D$. Suppose that $A : \Omega \times \bar{D} \to X$ is a random semiclosed 1-set-contractive operator, such that

$$\left[\lambda \|\mu x\| + \|A(\omega, x) + \mu x\|^{\alpha}\right]A(\omega, x) + \mu x\|x \leq \left[\lambda \|\mu x\| + \|A(\omega, x)\|^{\alpha}\right]A(\omega, x)\|x + \lambda \|\mu x\|^2x + \|\mu x\|^{\alpha + 1}x,$$  \hspace{1cm} (15)

where $\lambda \geq 0$, $\mu \geq 1$, $\alpha > 0$, for every $(\omega, x) \in \Omega \times \partial D$. Then the random operator equation $A(\omega, x) = \mu x$ (where $\mu \geq 1$, for every $(\omega, x) \in \Omega \times \bar{D}$) has a random solution in $D$.

**Proof.** From (15), we can easily prove that $A(\omega, x) = \mu x$ has no random solution on $\partial D$, by virtue of Lemma 5, see Theorem 3 for other section.

**Lemma 7.** When $y > 1$, $\alpha > 0$, $x \in X$, and $x \neq \theta$, the following inequality holds:

$$(y + 1)^{\alpha + 1}x - (y - 1)^{\alpha + 1}x > 2x.$$  \hspace{1cm} (16)

**Proof.** By Lemmas 1 and 5, we have

$$(y - 1)^{\alpha + 1}x < y^{\alpha + 1}x - x,$$  \hspace{1cm} (17)

$$y^{\alpha + 1}x + x < (y + 1)^{\alpha + 1}x,$$  \hspace{1cm} (18)
summing (17) and (18) we obtain

\[(y - 1)^{\alpha + 1} x + y^{\alpha + 1} x + x < y^{\alpha + 1} x - x + (y + 1)^{\alpha + 1} x.\]  \hspace{1cm} (19)

That is,

\[(y + 1)^{\alpha + 1} x - (y - 1)^{\alpha + 1} x > 2x,\]  \hspace{1cm} (20)

where \(\alpha > 0, y > 1, x \in X,\) and \(x \neq \theta.\)

**Theorem 8.** Let \(D\) be a bounded open subset in \(X\) and \(\theta \in D.\) Suppose that \(A : \Omega \times \bar{D} \to X\) is a random semiclosed \(1\)-set-contractive operator, such that

\[||A(\omega, x) + \mu x||^{\alpha + 1} x - ||A(\omega, x) - \mu x||^{\alpha + 1} x \leq 2||\mu x||^{\alpha + 1} x,\]  \hspace{1cm} (21)

where \(\alpha > 0, \mu \geq 1,\) for every \((\omega, x) \in \Omega \times \bar{D}.\) Then the random operator equation \(A(\omega, x) = \mu x\) (where \(\mu \geq 1,\) for every \((\omega, x) \in \Omega \times \bar{D})\) has a random solution in \(D.\)

**Proof.** The theorem can be proved using Lemma 7, see also Theorems 3 and 6.

**Remark 9.** Since \(X\) is a cone in \(E,\) then \(X\) is a closed convex subset of \(E.\)

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**References**
