PYTHAGOREAN IDENTITY FOR POLYHARMONIC POLYNOMIALS

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Received 13 May 1999

Polyharmonic polynomials in n variables are shown to satisfy a Pythagorean identity on the unit hypersphere. Application is made to establish the convergence of series of polyharmonic polynomials.

2000 Mathematics Subject Classification: 31B99.

1. Introduction. Let L_n^k denote the vector space of real homogeneous polynomial solutions of degree k of Laplace's equation

$$\Delta u = 0, \quad (1.1)$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \quad (1.2)$$

Such polynomials are called spherical harmonics. As shown in [9, pages 140-141],

$$\dim L_n^k = d_n^k = (n+k-2) \frac{(n+2k-3)!}{k!(n-2)!}. \quad (1.3)$$

Suppose that $\{y_j^k(x)\}_{j=1}^{d_n^k}$ is an orthonormal basis for L_n^k , where orthonormality is with respect to the inner product

$$\langle f, g \rangle = \int_{\Sigma_1} f(x)g(x)dx \quad (1.4)$$

on the unit sphere $\Sigma_1 : x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. It is well known (cf. [9, page 144]) that for all $s \in \Sigma_1$,

$$\sum_{j=1}^{d_n^k} [y_j^k(s)]^2 = \omega_n d_n^k, \quad (1.5)$$

where ω_n is the surface area of the unit sphere Σ_1 in $\mathbb{R}^n$. We call (1.5) the Pythagorean identity for spherical harmonics, since it generalizes the Pythagorean theorem

$$\sin^2 \theta + \cos^2 \theta = 1. \quad (1.6)$$

Solutions of partial differential equation

$$\Delta^m u = 0, \quad (1.7)$$

where Δ is the Laplacian (1.2) and m is a positive integer, are called polyharmonic functions. In the case $m = 2$, such functions are called biharmonic and are used to model the bending of thin plates (for a brief history of this application, see [7, pages 416 and 432–443]).

We show here that homogeneous polyharmonic polynomials satisfy a Pythagorean identity on Σ_1 and use this identity to establish the convergence of polyharmonic polynomial series.

2. Pythagorean identity. Let J_n^k denote the vector space of real homogeneous polynomial solutions of the partial differential equation (1.7). Since Δ^m is a homogeneous differential operator of order $2m$, using a standard argument (cf. [5, Theorem 1]) we find that

$$\dim J_n^k = b_n^k = \binom{n-1+k}{k} - \binom{n-1+k-2m}{k-2m}. \quad (2.1)$$

In the vector space J_n^k , we introduce the Calderón inner product [1]

$$(p, q) = p\left(\frac{\partial}{\partial x}\right)q(x), \quad (2.2)$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right), \quad p\left(\frac{\partial}{\partial x}\right) = p\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right). \quad (2.3)$$

THEOREM 2.1. *Suppose that $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ is an orthonormal basis for the vector space J_n^k of homogeneous polyharmonic polynomials of degree k , where orthonormality is with respect to the inner product (2.2). Then for all $s = (s_1, s_2, \dots, s_n) \in \Sigma_1$, the unit sphere in $\mathbb{R}^n$,*

$$\sum_{j=1}^{b_n^k} [Q_k^j(s)]^2 = y_n^k, \quad (2.4)$$

where y_n^k is a constant depending only on n and k .

PROOF. A modification in the argument used for spherical harmonics suffices: fix a point $y \in \mathbb{R}^n$ and consider the linear functional $L : J_n^k \rightarrow \mathbb{R}$ defined by

$$L(p) = p(y). \quad (2.5)$$

Since J_n^k is a finite-dimensional inner product space, there exists a unique $Z_y \in J_n^k$ such that

$$L(p) = (p(x), Z_y(x)), \quad (2.6)$$

for all $p \in J_n^k$ (i.e., all finite-dimensional inner product spaces are self-dual). Further, since $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ is an orthonormal basis for J_n^k ,

$$Z_y(x) = \sum_{j=1}^{b_n^k} (Z_y(x), Q_k^j(x)) Q_k^j(x). \quad (2.7)$$

But, by the defining property of Z_y ,

$$(Z_y(x), Q_k^j(x)) = Q_k^j(y). \quad (2.8)$$

Hence

$$Z_y(x) = \sum_{j=1}^{b_n^k} Q_k^j(y) Q_k^j(x). \quad (2.9)$$

Since the choice of $y \in \mathbb{R}^n$ was arbitrary, $Z_y(x)$ is a function of the two variables $x, y \in \mathbb{R}^n$; thus, we write

$$Z(x, y) = Z_y(x) = \sum_{j=1}^{b_n^k} Q_k^j(x) Q_k^j(y). \quad (2.10)$$

The Calderón inner product (2.2) is invariant with respect to rotations; that is, if $O : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rotation, then $(f(x), g(Ox)) = (f(O^{-1}x), g(x))$. Suppose $p(x) \in J_n^k$. Then

$$(p(x), Z(Ox, Oy)) = (p(O^{-1}x), Z(x, Oy)) = (q(x), Z(x, Oy)), \quad (2.11)$$

where $q(x) = p(O^{-1}x)$. Since rotations are invariant transformations for the Laplacian, it follows that $q(x) \in J_n^k$. Thus, by the defining property of $Z(x, y)$,

$$(q(x), Z(x, Oy)) = q(Oy). \quad (2.12)$$

But $q(Oy) = p(O^{-1}Oy) = p(y)$. Thus, we have shown that

$$(p(x), Z(Ox, Oy)) = p(y). \quad (2.13)$$

From the uniqueness of the representation of linear functionals, it follows that

$$Z(Ox, Oy) = Z(x, y), \quad (2.14)$$

for all $x, y \in \mathbb{R}^n$. In particular,

$$Z(Ox, Ox) = Z(x, x), \quad (2.15)$$

for every rotation O . Since every point on the unit sphere $\sum_1$ is the image under rotation for some fixed point on $\sum_1$, the equality (2.15) implies that $Z(x, x)$ is constant on $\sum_1$. That is,

$$\sum_{j=1}^{b_n^k} Q_k^j(s) Q_k^j(s) = C, \quad (2.16)$$

a constant, for all $s \in \sum_1$. □

3. Polyharmonic polynomial series. Pythagorean identities have been used to establish the convergence of series of spherical harmonics [4], as well as series of orthonormal homogeneous polynomials in several real variables in general [3]. We obtain here convergence for series of polyharmonic polynomials.

THEOREM 3.1. *Suppose that $\{Q_k^j(x)\}_{j=1}^{b_n^k}$ are sets of orthonormal polyharmonic polynomials in $\mathbb{R}^n$ of degree k , $k = 0, 1, 2, \dots$. Then the series*

$$\sum_{k=0}^{\infty} \sum_{j=1}^{b_n^k} a_{kj} Q_k^j(x) \quad (3.1)$$

converges absolutely and uniformly on compact subsets of the open ball $|x| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2} < R$, where

$$R^{-1} = \limsup_{k \rightarrow \infty} \left(\sqrt{y_n^k} \|a_k\| \right)^{1/k}, \quad \|a_k\| = \left(\sum_{j=1}^{b_n^k} a_{kj}^2 \right)^{1/2}, \quad (3.2)$$

and y_n^k is the Pythagorean constant appearing in (2.4).

PROOF. Since each of the polynomials Q_k^j is homogeneous of degree k , we have $Q_k^j(x) = r^k Q_k^j(x/r)$, where $r = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$. Thus

$$\begin{aligned} \left| \sum_{k=0}^{\infty} \sum_{j=1}^{b_n^k} a_{kj} Q_k^j(x) \right| &= \left| \sum_{k=0}^{\infty} r^k \sum_{j=1}^{b_n^k} a_{kj} Q_k^j\left(\frac{x}{r}\right) \right| \\ &\leq \sum_{k=0}^{\infty} r^k \left| \sum_{j=1}^{b_n^k} a_{kj} Q_k^j\left(\frac{x}{r}\right) \right|, \end{aligned} \quad (3.3)$$

by the Cauchy-Schwarz inequality

$$\left| \sum_{k=0}^{\infty} \sum_{j=1}^{b_n^k} a_{kj} Q_k^j(x) \right| \leq \sum_{k=0}^{\infty} r^k \left(\sum_{j=1}^{b_n^k} a_{kj}^2 \right)^{1/2} \left(\sum_{j=1}^{b_n^k} Q_k^j\left(\frac{x}{r}\right)^2 \right)^{1/2}. \quad (3.4)$$

Appealing now to the Pythagorean identity (2.4), we find that

$$\left| \sum_{k=0}^{\infty} \sum_{j=1}^{b_n^k} a_{kj} Q_k^j(x) \right| = \sum_{k=0}^{\infty} r^k \|a_k\| \sqrt{y_n^k}, \quad (3.5)$$

from which the desired result is immediate. $\square$

Let H_n^k denote the vector space of homogeneous polynomials of degree k in $\mathbb{R}^n$. Since every orthonormal basis of J_n^k be extended to an orthonormal basis of H_n^k , it follows from [2, Theorem 3] that

$$y_n^k \leq \frac{1}{k!}. \quad (3.6)$$

Thus,

$$R^{-1} = \limsup_{k \rightarrow \infty} \left(\sqrt{y_n^k} \|a_k\| \right)^{1/2} \leq \limsup_{k \rightarrow \infty} \left(\frac{\|a_k\|}{\sqrt{k!}} \right)^{1/k} = \rho^{-1}, \quad (3.7)$$

and appealing to the result of [Theorem 3.1](#) we find that the polyharmonic polynomial series (3.1) converges absolutely and uniformly on compact subsets of the open ball $|x| < \rho$. We predict that the evaluation of the Pythagorean constant γ_n^k will show that such convergence actually obtains within a somewhat larger ball.

In [11], it was shown that, in the space of homogeneous harmonic polynomials L_n^k , the Calderón inner product (2.2) is a constant multiple of the inner product (1.4). That is,

$$(p, q) = c_n^k(p, q), \quad (3.8)$$

for all $p, q \in L_n^k$, where c_n^k is a constant depending only on n and k . Thus, the Pythagorean identity for spherical harmonics (1.5) is a special case ($m = 1$) of the result of [Theorem 2.1](#).

The Pythagorean identity for spherical harmonics is also a special case of the addition formula for spherical harmonics [9, page 149] and [8, page 268]. This leads us to conjecture that the homogeneous polyharmonic polynomials satisfy a similar addition formula, from which [Theorem 2.1](#) might follow as an immediate consequence. Such a development could include a significant generalization of the ultraspherical polynomials [6, 10].

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