IMPULSIVE FUNCTIONAL-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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The existence, uniqueness, and continuous dependence of a mild solution of an impulsive functional-differential evolution nonlocal Cauchy problem in general Banach spaces are studied. Methods of fixed point theorems, of a C_0 semigroup of operators and the Banach contraction theorem are applied.

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1. Introduction. In this paper, we study the existence, uniqueness, and continuous dependence of a mild solution of a nonlocal Cauchy problem for impulsive functional-differential evolution equation. Such problems arise in some physical applications as a natural generalization of the classical initial value problems. The results for semilinear functional-differential evolution nonlocal problem [2] are extended for the case of impulse effect. We consider the nonlocal Cauchy problem in the form

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t, u_t), \quad t \in (0, a], \quad t \neq \tau_k, \\ u(\tau_k + 0) &= Q_k u(\tau_k) \equiv u(\tau_k) + I_k u(\tau_k), \quad k = 1, 2, \dots, \kappa, \\ u(t) + (g(u_{t_1}, \dots, u_{t_p}))(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \quad (1.1)$$

where $0 < t_1 < \dots < t_p \leq a$, $p \in \mathbb{N}$, A and I_k ($k = 1, 2, \dots, \kappa$) are linear operators acting in a Banach space E ; f , g , and ϕ are given functions satisfying some assumptions, $u_t(s) := u(t + s)$ for $t \in [0, a]$, $s \in [-r, 0]$, $I_k u(\tau_k) = u(\tau_k + 0) - u(\tau_k - 0)$ and the impulsive moments τ_k are such that $0 < \tau_1 < \tau_2 < \dots < \tau_\kappa < a$, $\kappa \in \mathbb{N}$.

Theorems about the existence, uniqueness, and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied in [1, 2, 3]. The results presented in this paper are a generalization and a continuation of some results reported in [1, 2, 3]. We consider classical impulsive functional-differential equation in the case of nonlocal condition, reduced to the classical initial functional value problem.

As usual, in the theory of impulsive differential equations [4, 5] at the points of discontinuity τ_i of the solution $t \mapsto u(t)$ we assume that $u(\tau_i) \equiv u(\tau_i - 0)$. It is clear that, in general, the derivatives $\dot{u}(\tau_i)$ do not exist. On the other hand, according to the first equality of (1.1) there exist the limits $\dot{u}(\tau_i \mp 0)$. According to the above convention, we assume $\dot{u}(\tau_i) \equiv \dot{u}(\tau_i - 0)$.

Throughout, we assume that E is a Banach space with norm $\|\cdot\|$, A is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t \geq 0}$ on E , $D(A)$ is the domain of A , and

$$M := \sup_{t \in [0, a]} \{ \|T(t)\|_{BL(E, E)} \}. \quad (1.2)$$

Let $f : [0, a] \times C([-r, 0], E) \rightarrow E$. Introduce the following assumptions:

- (H1) for every $w \in C([-r, a], E)$ and $t \in [0, a]$, $f(\cdot, w_t) \in C([0, a], E)$;
 (H2) there exists a constant $L > 0$ such that

$$\begin{aligned} & \|f(t, w_t) - f(t, \tilde{w}_t)\|_E \\ & \leq L_1 \|w - \tilde{w}\|_{C([-r, t], E)} \quad \text{for } w, \tilde{w} \in C([-r, a], E), t \in [0, a], \\ & \|I_k v\|_E \leq L_2 \|v\|_E \quad \text{for } v \in E, k = 1, 2, \dots, \kappa, \\ & L = \max \{L_1, L_2\}. \end{aligned} \quad (1.3)$$

Let $g : [C([-r, 0], E)]^p \rightarrow C([-r, 0], E)$. Then we have the following assumptions:

- (H3) there exists a constant $K > 0$ such that

$$\| (g(w_{t_1}, \dots, w_{t_p}))(t) - (g(\tilde{w}_{t_1}, \dots, \tilde{w}_{t_p}))(t) \| \leq K \|w - \tilde{w}\|_{C([-r, a], E)} \quad (1.4)$$

for $w, \tilde{w} \in C([-r, a], E)$, $t \in [-r, 0]$;

- (H4) assume that $\phi \in C([-r, 0], E)$.

A function $u \in C([-r, a], E)$ satisfying the following conditions:

$$\begin{aligned} u(t) &= T(t)\phi(0) - T(t)[(g(u_{t_1}, \dots, u_{t_p}))(0)] \\ &+ \int_0^t T(t-s)f(s, u_s)ds + \sum_{0 < \tau_k < t} T(t-\tau_k)I_k u(\tau_k), \quad t \in [0, a], \\ u(t) &+ (g(u_{t_1}, \dots, u_{t_p}))(t) = \phi(t), \quad t \in [-r, 0], \end{aligned} \quad (1.5)$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1).

2. Existence and uniqueness of a mild solution

THEOREM 2.1. *Suppose that assumptions (H1)–(H4) are satisfied and*

$$M[K + L(a + 1)] < 1. \quad (2.1)$$

Then the impulsive nonlocal Cauchy problem (1.1) has a unique mild solution.

PROOF. The mild solution of the impulsive system (1.1) with nonlocal condition can be written in the form

$$u(t; \phi) = (Fu)(t), \quad (2.2)$$

where

$$(Fw)(t) := \begin{cases} \phi(t) - (g(w_{t_1}, \dots, w_{t_p}))(t), & t \in [-r, 0), \\ T(t)\phi(0) - T(t)[(g(w_{t_1}, \dots, w_{t_p}))(0)] \\ \quad + \int_0^t T(t-s)f(s, w_s)ds + \sum_{0 < \tau_k < t} T(t-\tau_k)I_k w(\tau_k), & t \in [0, a], \end{cases} \quad (2.3)$$

such that $w \in C([-r, a], E)$ and $F: C([-r, a], E) \rightarrow C([-r, a], E)$. Now, we show that F is a contraction mapping on $C([-r, a], E)$. Therefore,

$$(Fw)(t) - (F\tilde{w})(t) := \begin{cases} (g(\tilde{w}_{t_1}, \dots, \tilde{w}_{t_p}))(t) - (g(w_{t_1}, \dots, w_{t_p}))(t) \\ \quad \text{for } w, \tilde{w} \in C([-r, a], E), t \in [-r, 0), \\ T(t)[(g(\tilde{w}_{t_1}, \dots, \tilde{w}_{t_p}))(0) - (g(w_{t_1}, \dots, w_{t_p}))(0)] \\ \quad + \int_0^t T(t-s)[f(s, w_s) - f(s, \tilde{w}_s)]ds \\ \quad + \sum_{0 < \tau_k < t} T(t-\tau_k)[I_k w(\tau_k) - I_k \tilde{w}(\tau_k)] \\ \quad \text{for } w, \tilde{w} \in C([-r, a], E), t \in [0, a]. \end{cases} \quad (2.4)$$

From (2.4), we have

$$\begin{aligned} \|(Fw)(t) - (F\tilde{w})(t)\| &\leq \|T(t)\| \cdot \|(g(\tilde{w}_{t_1}, \dots, \tilde{w}_{t_p}))(0) - (g(w_{t_1}, \dots, w_{t_p}))(0)\| \\ &\quad + \int_0^t \|T(t-s)\| \cdot \|f(s, w_s) - f(s, \tilde{w}_s)\| ds \\ &\quad + \sum_{0 < \tau_k < t} \|T(t-\tau_k)\| \cdot \|I_k w(\tau_k) - I_k \tilde{w}(\tau_k)\| \end{aligned} \quad (2.5)$$

for $w, \tilde{w} \in C([-r, a], E)$, $t \in [0, a]$. Because of (2.5), in view of (1.2), and applying assumptions (H1)–(H4) we obtain

$$\begin{aligned} \|(Fw)(t) - (F\tilde{w})(t)\| &\leq MK\|w - \tilde{w}\|_{C([-r, a], E)} \\ &\quad + ML_1 \int_0^t \|w - \tilde{w}\|_{C([-r, a], E)} ds + ML_2 \|w(\tau_k) - \tilde{w}(\tau_k)\|_E \\ &\leq (MK + MaL_1 + ML_2) \|w - \tilde{w}\|_{C([-r, a], E)} \\ &\leq M[K + L(a+1)] \cdot \|w - \tilde{w}\|_{C([-r, a], E)} \end{aligned} \quad (2.6)$$

for $w, \tilde{w} \in C([-r, a], E)$, $t \in [0, a]$, which implies that

$$\|Fw - F\tilde{w}\|_{C([-r, a], E)} \leq \beta \|w - \tilde{w}\|_{C([-r, a], E)}, \quad w, \tilde{w} \in C([-r, a], E), \quad (2.7)$$

where $\beta := M[K + L(a+1)]$. The operator F satisfies all the assumptions of the Banach contraction theorem, and therefore, in the space $C([-r, a], E)$ there is only one fixed point of F and this is the mild solution of the nonlocal Cauchy problem with impulse effect. This completes the proof of the theorem. $\square$

3. Continuous dependence of a mild solution

THEOREM 3.1. Suppose that the functions f, g , and $I_k(u)$, $k = 1, 2, \dots, \kappa$, satisfy the assumptions (H1)–(H4) and $M[K + L(a + 1)] < 1$. Then, for each $\phi_1, \phi_2 \in C([-r, a], E)$, and for the corresponding mild solutions u_1, u_2 of the problems,

$$\begin{aligned} \dot{u}(t) &= Au(t) + f(t, u_t), \quad t \in (0, a], \quad t \neq \tau_k, \\ u(\tau_k + 0) &= Q_k u(\tau_k) \equiv u(\tau_k) + I_k u(\tau_k), \quad k = 1, 2, \dots, \kappa, \\ u(t) + (g(u_{t_1}, \dots, u_{t_p}))(t) &= \phi_i(t) \quad (i = 1, 2), \quad t \in [-r, 0], \end{aligned} \quad (3.1)$$

the following inequality holds:

$$\|u_1 - u_2\|_{C([-r, a], E)} \leq M e^{aML} (1 + ML)^\kappa \left\{ \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + K \|u_1 - u_2\|_{C([-r, a], E)} \right\}. \quad (3.2)$$

Additionally, if

$$K < \frac{e^{-aML} (1 + ML)^{-\kappa}}{M}, \quad (3.3)$$

then

$$\|u_1 - u_2\|_{C([-r, a], E)} \leq \frac{M e^{aML} (1 + ML)^\kappa}{1 - K M e^{aML} (1 + ML)^\kappa} \|\phi_1 - \phi_2\|_{C([-r, 0], E)}. \quad (3.4)$$

PROOF. Assume that $\phi_i \in C([-r, 0], E)$ ($i = 1, 2$) are arbitrary functions and let u_i ($i = 1, 2$) be the mild solutions of problem (3.1). Then

$$\begin{aligned} u_1(t) - u_2(t) &= T(t)[\phi_1(0) - \phi_2(0)] \\ &\quad - T(t)\{[g((u_1)_{t_1}, \dots, (u_1)_{t_p}))(0) - [g((u_2)_{t_1}, \dots, (u_2)_{t_p}))(0)]\} \\ &\quad + \int_0^t T(t-s)[f(s, (u_1)_s) - f(s, (u_2)_s)]ds \\ &\quad + \sum_{0 < \tau_k < t} T(t - \tau_k)[I_k u_1(\tau_k) - I_k u_2(\tau_k)] \end{aligned} \quad (3.5)$$

for $t \in [0, a]$ and

$$u_1(t) - u_2(t) = \phi_1(t) - \phi_2(t) - \{[g((u_2)_{t_1}, \dots, (u_2)_{t_p}))(t) - [g((u_1)_{t_1}, \dots, (u_1)_{t_p}))(t)]\} \quad (3.6)$$

for $t \in [-r, 0]$. From (3.5), (1.2), and using (H2) we get

$$\begin{aligned} \|u_1(\xi) - u_2(\xi)\| &\leq M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + MK \|u_1 - u_2\|_{C([-r, a], E)} \\ &\quad + ML_1 \int_0^\xi \|u_1 - u_2\|_{C([-r, s], E)} ds + ML_2 \sum_{0 < \tau_k < \xi} \|u_1(\tau_k) - u_2(\tau_k)\|_E \\ &\leq M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + MK \|u_1 - u_2\|_{C([-r, a], E)} \\ &\quad + ML_1 \int_0^t \|u_1 - u_2\|_{C([-r, s], E)} ds + ML_2 \sum_{0 < \tau_k < t} \|u_1(\tau_k) - u_2(\tau_k)\|_E \end{aligned} \quad (3.7)$$

for $0 \leq \xi \leq t \leq a$. With this result, by virtue of (H3) it follows that

$$\begin{aligned} & \sup_{\xi \in [0, t]} \|u_1(\xi) - u_2(\xi)\| \\ & \leq M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + MK \|u_1 - u_2\|_{C([-r, a], E)} \\ & \quad + ML_1 \int_0^t \|u_1 - u_2\|_{C([-r, s], E)} ds + ML_2 \sum_{0 < \tau_k < t} \|u_1(\tau_k) - u_2(\tau_k)\|_E \end{aligned} \quad (3.8)$$

for $t \in [0, a]$. At the same time, by (3.6) and (H3) we have

$$\|u_1(t) - u_2(t)\| \leq M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + MK \|u_1 - u_2\|_{C([-r, a], E)} \quad (3.9)$$

for $t \in [-r, 0]$. Formulas (3.8) and (3.9) imply that

$$\begin{aligned} \|u_1(t) - u_2(t)\| & \leq M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} + MK \|u_1 - u_2\|_{C([-r, a], E)} \\ & \quad + ML \left\{ \int_0^t \|u_1 - u_2\|_{C([-r, s], E)} ds + \sum_{0 < \tau_k < t} \|u_1(\tau_k) - u_2(\tau_k)\|_E \right\}. \end{aligned} \quad (3.10)$$

Applying Gronwall's inequality for discontinuous functions (see [5]), from (3.10) it follows that

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{C([-r, a], E)} & \leq \left\{ M \|\phi_1 - \phi_2\|_{C([-r, 0], E)} \right. \\ & \quad \left. + MK \|u_1 - u_2\|_{C([-r, a], E)} \right\} e^{aML} (1 + ML)^\kappa \end{aligned} \quad (3.11)$$

and therefore, (3.2) holds. Inequality (3.4) is a consequence of (3.2). This completes the proof of the theorem. $\square$

REMARK 3.2. If $K = \kappa = 0$, then (3.2) is reduced to the classical inequality

$$\|u_1(t) - u_2(t)\|_{C([-r, a], E)} \leq Me^{aML} \|\phi_1 - \phi_2\|_{C([-r, 0], E)}, \quad (3.12)$$

which is characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

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REFERENCES

- [1] H. Akça and V. Covachev, *Periodic solutions of impulsive systems with delay*, Funct. Differ. Equ. **5** (1998), no. 3-4, 275-286.
- [2] L. Byszewski and H. Akça, *On a mild solution of a semilinear functional-differential evolution nonlocal problem*, J. Appl. Math. Stochastic Anal. **10** (1997), no. 3, 265-271.
- [3] ———, *Existence of solutions of a semilinear functional-differential evolution nonlocal problem*, Nonlinear Anal. **34** (1998), no. 1, 65-72.
- [4] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Modern Applied Mathematics, vol. 6, World Scientific Publishing, New Jersey, 1989.

- [5] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientific Publishing, New Jersey, 1995.

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