CHERN CLASSES OF INTEGRAL SUBMANIFOLDS OF SOME CONTACT MANIFOLDS

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A complex subbundle of the normal bundle to an integral submanifold of the contact distribution in a Sasakian manifold is given. The geometry of this bundle is investigated and some results concerning its Chern classes are obtained.

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1. Introduction. Let $\tilde{M}$ be a $(2m + 1)$ -dimensional manifold endowed with the almost contact metric structure F, ξ, η, g . These tensor fields satisfy the conditions

$$F^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \quad (1.1)$$

for all vector fields X, Y tangent to $\tilde{M}$.

Let $\mathcal{D}$ be the contact distribution of $\tilde{M}$, defined by the equation $\eta = 0$. The study of the integral submanifolds of $\mathcal{D}$ is very difficult for, at least, three reasons: (a) their abundance (see, e.g., [1, 5, 8]), (b) the nonexistence of a natural structure induced on the submanifold M , resulting from the equalities $\eta = 0$, $d\eta = 0$, true along M , and (c) for any vector field X tangent to M , the vector field FX is normal to M and therefore, freely speaking, the geometry of an integral submanifold of $\mathcal{D}$ is normal to the submanifold. However, for maximal integral submanifolds (i.e., $\dim M = m$), we know many properties (see, e.g., [1, Chapter V]); while for nonmaximal integral submanifolds, we have so few results.

In this paper, we associate to each nonmaximal integral submanifold M of $\tilde{M}$ a non-trivial vector bundle $\tau(M)$. The geometry and the topology of this vector bundle are also studied. In Section 2, we give, in an “appropriate” form, the structure equations of an integral submanifold in a Sasakian manifold. In Section 3, we study the geometry of $\tau(M)$, namely, we prove that it has a natural structure of complex symplectic vector bundle.

It is well known that integral submanifolds of an almost contact manifold are anti-invariant, [8]. Thus, such a submanifold is analogous to the isotropic (or totally real) submanifolds of a Kähler manifold, investigated by Chen and Morvan in [2, 4], and we can use some of their technics in order to study Chern classes of the vector bundle $\tau(M)$. In Section 4, by combining these ideas with some Vaisman’s results [7] concerning the characteristic classes of quaternionic bundles, we obtain stronger results than for isotropic submanifolds. Namely, we prove that if $m - n$ is even, then all odd Chern classes of $\tau(M)$ are zero. In absence of this supposition on the dimensions, we prove that the first Chern class of $\tau(M)$ is zero when $\tilde{M}$ is a Sasakian space form.

2. Structure equations of an integral submanifold. Let $\tilde{M}$ be an almost contact metric manifold. Furthermore, we assume that $\tilde{M}$ is Sasakian and let $\mathcal{X}(\tilde{M})$ denote the set of all vector fields tangent to $\tilde{M}$. We have [1, page 73]

$$(\tilde{\nabla}F)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in \mathcal{X}(\tilde{M}), \quad (2.1)$$

where $\tilde{\nabla}$ is the Levi-Civita connection associated to the metric g on $\tilde{M}$. Moreover, we have the well-known equalities

$$F\xi = 0, \quad \eta \circ F = 0, \quad \eta(X) = g(X, \xi), \quad \tilde{\nabla}_X \xi = -FX. \quad (2.2)$$

Now, let M be an n -dimensional submanifold of the Sasakian manifold $\tilde{M}$ and denote by h , $\tilde{\nabla}^\perp$, and A its second fundamental form, normal connection, and Weingarten operator, respectively. It is well known that $n \leq m$ (see [8] or [1, page 36]), and we can consider in $\tilde{M}$ local fields of orthonormal frames $\mathcal{B} = \{e_1, \dots, e_n, e_{n+1}, \dots, e_m, e_{1^*} = Fe_1, \dots, e_{n^*} = Fe_n, e_{(n+1)^*} = Fe_{n+1}, \dots, e_{m^*} = Fe_m, e_{(m+1)^*} = \xi\}$, with the property that the restrictions of $e_1, \dots, e_n$ to the submanifold M are tangent to M , so that $\mathcal{B}$ are local frames such that $TM \oplus \text{span}\{e_{n+1}, \dots, e_m\}$ is a Legendrian subbundle of $T\tilde{M}$.

Afterwards, we will use the following convention on the indices: $j \in \{1, \dots, m\}$; $j^* = j + m$; $a, b, c \in \{1, \dots, n\}$; $a^* = a + m$, $b^* = b + m$, $c^* = c + m$; $\lambda, \mu, \nu \in \{n+1, \dots, m\}$; $\lambda^* = \lambda + m$; $\alpha, \beta, \gamma, \delta \in \{1, \dots, 2m+1\}$.

If $\mathcal{B}^* = \{\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^m, \omega^{1^*}, \dots, \omega^{n^*}, \omega^{(n+1)^*}, \dots, \omega^{m^*}, \omega^{(m+1)^*} = \eta\}$ is the local field of coframes of $\mathcal{B}$, then, at the points of M , we have (locally)

$$\omega^\lambda = \omega^{j^*} = \omega^{(m+1)^*} = 0. \quad (2.3)$$

On the other hand, by computations we prove that if (ω_α^β) is the connection form of $\tilde{\nabla}$, expressed with respect to $\mathcal{B}$, then, on the submanifold M , we have

$$\omega_{(m+1)^*}^a = \omega_{(m+1)^*}^\lambda = \omega_{(m+1)^*}^{\lambda^*} = 0, \quad \omega_{a^*}^{(m+1)^*}(X) = g(X, e_a), \quad (2.4)$$

$$\omega_a^{j^*} = \omega_j^{a^*}, \quad \omega_{a^*}^{j^*} = \omega_a^j, \quad \omega_\lambda^{j^*} = \omega_j^{\lambda^*}, \quad \omega_{\lambda^*}^{j^*} = \omega_\lambda^j. \quad (2.5)$$

The curvature forms of $\tilde{M}$ and M are, respectively,

$$\tilde{\Omega}_\beta^\alpha = \frac{1}{2} \sum_{\alpha, \beta=1}^{2m+1} \tilde{R}_{\beta\gamma\delta}^\alpha \omega^\gamma \wedge \omega^\delta, \quad \Omega_b^a = \frac{1}{2} \sum_{c, d=1}^n R_{bcd}^a \omega^c \wedge \omega^d, \quad (2.6)$$

where $\tilde{R}_{\beta\gamma\delta}^\alpha$ and R_{bcd}^a are the components (with respect to $\mathcal{B}$) of the curvature tensors of $\tilde{M}$ and M , respectively. Then, at the points of M , we have

$$\Omega_b^a = \tilde{\Omega}_b^a - \sum_{\lambda=n+1}^m \omega_\lambda^a \wedge \omega_b^\lambda - \sum_{j=1}^m \omega_{j^*}^a \wedge \omega_b^{j^*}, \quad (2.7)$$

$$\Omega_\mu^\lambda = \tilde{\Omega}_\mu^\lambda - \sum_{a=1}^n \omega_a^\lambda \wedge \omega_\mu^a = \frac{1}{2} \sum_{a, b=1}^n R_{\mu ab}^\lambda \omega^a \wedge \omega^b, \quad (2.8)$$

where $R_{\mu ab}^\lambda$ are the components of the curvature tensor of $\nabla^\perp$. Finally, from (2.3), (2.4), and (2.5) and from the general form of the structure equations (see, e.g., [3, page 121]),

we deduce the structure equations of an integral submanifold of a Sasakian manifold under the form

$$\begin{aligned} d\omega^a &= - \sum_{b=1}^n \omega_b^a \wedge \omega^b, & d\omega_b^a &= - \sum_{c=1}^n \omega_c^a \wedge \omega_b^c + \Omega_b^a, \\ d\omega_\mu^\lambda &= - \sum_{\nu=n+1}^m \omega_\nu^\lambda \wedge \omega_\mu^\nu - \sum_{j=1}^m \omega_{j*}^\lambda \wedge \omega_\mu^{j*} + \Omega_\mu^\lambda. \end{aligned} \quad (2.9)$$

3. Geometry of the maximal invariant normal bundle. The normal space $T_x^\perp M$ at each point $x \in M$ has the following orthogonal decomposition

$$T_x^\perp M = F(T_x M) \oplus \tau_x(M) \oplus \text{span}\{\xi_x\}, \quad (3.1)$$

where $\tau_x(M)$ is the $2(m-n)$ -dimensional subspace of $T_x M$, orthogonal to $F(T_x M) \oplus \text{span}\{\xi_x\}$. Then, $\tau(M) = \bigcup_{x \in M} \tau_x(M)$ is the total space of a subbundle $\tau(M)$ of $T^\perp M$ and $\mathcal{B}_\tau = \{e_\lambda, e_{\lambda^*}\} = \{e_{n+1}, \dots, e_m, e_{(n+1)^*}, \dots, e_{m^*}\}$ is a local basis in the module $\Gamma(\tau)$ of its sections. We also denote this bundle by $\tau(M)$ and call it the *maximal invariant normal bundle* of the integral submanifold M .

THEOREM 3.1. *Let M be an integral submanifold of the Sasakian manifold $\tilde{M}$. Its maximal invariant normal bundle $\tau(M)$ has the following properties:*

- (a) $\tau(M)$ is invariant by F , that is, $F(\tau_x(M)) = \tau_x(M)$ for each $x \in M$;
- (b) $\tau(M)$ has a natural structure of complex vector bundle;
- (c) if $m-n = (\dim \tilde{M} - \dim M)/2$ is even, then $\tau(M)$ has a quaternionic structure.

PROOF. (a) follows easily from (3.1).

(b) Denote by $(n^\lambda, n^{\lambda^*})$ the components of the vector $\vec{n}_x \in \tau_x(M)$, relative to the basis $\mathcal{B}_\tau$, and let $\rho: \tau(M) \rightarrow M$ be the natural projection. Then, using the classical notations, the vector charts

$$\Phi: \rho^{-1}(U) \longrightarrow U \times \mathbb{C}^{m-n}, \quad \Phi(\vec{n}_x) = (x, (n^\lambda + i n^{\lambda^*})), \quad x \in U, \quad (3.2)$$

define on $\tau(M)$ a complex vector bundle structure.

(c) From (a), we deduce that the space $\Gamma(\tau)$ can be considered as a complex space with the following multiplication by complex numbers:

$$(\alpha + i\beta)\vec{n} = \alpha\vec{n} + \beta F\vec{n}, \quad \alpha, \beta \in \mathbb{R}, \quad \vec{n} \in \Gamma(\tau). \quad (3.3)$$

Endowed with this complex structure, $\Gamma(\tau)$ is an $(m-n)$ -dimensional space, denoted by $\Gamma^c(\tau)$. Moreover, we can define the map $F^\tau: \Gamma^c(\tau) \rightarrow \Gamma^c(\tau)$ by $F^\tau(\mathbf{n}) = F\vec{n} - iF\vec{n}^*$ for all $\mathbf{n} = \vec{n} + i\vec{n}^*$, $\vec{n}, \vec{n}^* \in \Gamma(\tau)$, and it has the following properties:

$$F^\tau(\mathbf{n}_1 + \mathbf{n}_2) = F^\tau \mathbf{n}_1 + F^\tau \mathbf{n}_2, \quad F^\tau(\lambda \mathbf{n}) = \bar{\lambda} F^\tau \mathbf{n}, \quad (F^\tau)^2 \mathbf{n} = -\mathbf{n}, \quad (3.4)$$

for $\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \Gamma^c(\tau)$ and $\lambda \in \mathbb{C}$. Hence (see [7, Section 1]), F^τ defines on $\tau(M)$ a quaternionic structure. $\square$

A natural connection can be defined on $\tau(M)$. Firstly, we remark that $g(\nabla_X^\perp \vec{n}, \xi) = 0$ for all $X \in \mathcal{X}(M)$ and $\vec{n} \in \Gamma(\tau)$, hence the normal vector field $\nabla_X^\perp \vec{n}$ has the following decomposition:

$$\nabla_X^\perp \vec{n} = B_{\vec{n}}X + \nabla_X^\tau \vec{n}, \quad (3.5)$$

where $B_{\vec{n}}X \in \Gamma(FTM)$ and $\nabla_X^\tau \vec{n} \in \Gamma(\tau)$. Moreover, the maps $B : \Gamma(\tau) \times \mathcal{X}(M) \rightarrow \Gamma(FTM)$ and $\nabla^\tau : \mathcal{X}(M) \times \Gamma(\tau) \rightarrow \Gamma(\tau)$ have the following properties.

PROPOSITION 3.2. (a) ∇^τ is an almost complex connection on the maximal invariant normal bundle of the integral submanifold M , that is, $(\nabla_X^\tau F)\vec{n} = 0$.

(b) $B_{\vec{n}}X = FA_{F\vec{n}}X$ for all $X \in \mathcal{X}(M)$ and $\vec{n} \in \Gamma(\tau)$.

The proof follows from (3.5) by computation, taking into account (2.1) and (2.2) and using the Weingarten formula for the submanifold M .

Now, if we extend the scalar product g over $\Gamma^c(\tau)$ by

$$g^\tau(\mathbf{n}_1, \lambda \mathbf{n}_2) = \bar{\lambda} g^\tau(\mathbf{n}_1, \mathbf{n}_2), \quad g^\tau(\mathbf{n}_2, \mathbf{n}_1) = \overline{g^\tau(\mathbf{n}_1, \mathbf{n}_2)}, \quad (3.6)$$

for $\lambda \in \mathbb{C}$ and $\mathbf{n}_1, \mathbf{n}_2 \in \Gamma^c(\tau)$, then we have

$$g^\tau(F^\tau \mathbf{n}_1, F^\tau \mathbf{n}_2) = \overline{g^\tau(\mathbf{n}_1, \mathbf{n}_2)}, \quad (3.7)$$

hence g^τ is a Hermitian scalar product on the complex vector bundle $\tau(M)$. Moreover, $\mathcal{B}_\tau^c = \{f_\lambda = (1/\sqrt{2})(e_\lambda + ie_{\lambda^*}), f_{\lambda^*} = (1/\sqrt{2})(e_\lambda - ie_{\lambda^*})\}$ is an orthonormal local basis of $\Gamma^c(\tau)$ with respect to g^τ and $f_{\lambda^*} = -if_\lambda$, $F^\tau f_\lambda = i\overline{f_\lambda} = f_{\lambda^*}$.

For any $\mathbf{n}_1, \mathbf{n}_2 \in \Gamma^c(\tau)$, we put

$$\Omega^\tau(\mathbf{n}_1, \mathbf{n}_2) = -\overline{g^\tau(F^\tau \mathbf{n}_1, \mathbf{n}_2)}, \quad (3.8)$$

and a simple computation shows that Ω^τ is $\mathbb{C}$ -linear with respect to the first argument and

$$\Omega^\tau(\mathbf{n}_1, \mathbf{n}_2) = -\overline{\Omega^\tau(\mathbf{n}_2, \mathbf{n}_1)}, \quad \Omega^\tau(F^\tau \mathbf{n}_1, F^\tau \mathbf{n}_2) = \overline{\Omega^\tau(\mathbf{n}_1, \mathbf{n}_2)}. \quad (3.9)$$

From these relations and because $\mathcal{B}_\tau^c$ is an orthonormal local basis, we deduce that Ω^τ is a nondegenerate skew-symmetric 2-form on the complex vector bundle $\tau(M)$. Hence, we have the following proposition.

PROPOSITION 3.3. For $m - n$ even, the maximal invariant normal bundle $\tau(M)$ of the integral submanifold M of a Sasakian manifold has a structure of complex symplectic vector bundle with the symplectic form Ω^τ .

4. Normal Chern classes of an integral submanifold. As a complex vector bundle, the basic characteristic classes of the maximal invariant normal bundle $\tau(M)$ are the Chern classes $[\gamma_k(\tau)]$, represented by the Chern forms

$$\gamma_k = \frac{i^k}{(2\pi)^k k!} \delta_{\lambda_1 \dots \lambda_k}^{\mu_1 \dots \mu_k} \Omega_{\mu_1}^{\tau \lambda_1} \wedge \dots \wedge \Omega_{\mu_k}^{\tau \lambda_k}, \quad (4.1)$$

where $\Omega_\mu^{\tau\lambda}$ are the curvature forms of ∇^τ and δ_{ijk} is the multiindex Kronecker symbol. We say that $\gamma_k(\tau)$ is the k th normal Chern form of the submanifold M and the purpose of this section is to obtain some results concerning the computation of $\gamma_k(\tau)$ and the k th normal Chern class $[\gamma_k(\tau)]$ of M .

THEOREM 4.1. *Let M be an n -dimensional integral submanifold of a Sasakian manifold of dimension $2m + 1$. If $m - n$ is even, then*

$$[\gamma_{2k+1}(\tau)] = 0 \quad \text{for } k = 0, 1, \dots, \left\lfloor \frac{m-n-1}{2} \right\rfloor. \quad (4.2)$$

PROOF. By Theorem 3.1(c), the maximal invariant normal bundle $\tau(M)$ has a quaternionic structure, and then we can apply [7, Proposition 2.1]. $\square$

Now, we will analyse the first normal Chern form and its associated class in absence of the supposition that $m - n$ is even.

THEOREM 4.2. *The first normal Chern form of the n -dimensional integral submanifold M in a Sasakian manifold of dimension $2m + 1$, $m > n$, is given by*

$$\gamma_1(\tau) = \frac{1}{2\pi} \sum_{\lambda=n+1}^m \Omega_\lambda^{\lambda*}. \quad (4.3)$$

PROOF. Using (3.5), (2.1), and the Weingarten formula, we obtain the components of the curvature tensor $\tilde{R}^\tau$ of ∇^τ under the form

$$\tilde{R}_{\lambda ab}^{\tau\lambda*} = R_{\lambda ab}^{\lambda*} + g(B_{e_\lambda} e_b, B_{e_{\lambda*}} e_a) - g(B_{e_\lambda} e_a, B_{e_{\lambda*}} e_b), \quad (4.4)$$

and then its curvature form is $\tilde{\Omega}_\lambda^{\tau\lambda*} = \Omega_\lambda^{\lambda*}$. On the other hand, from (2.9), it follows the complex form of the second structure equation of $\tau(M)$, namely,

$$\begin{aligned} d\phi_\mu^\lambda &= - \sum_{\nu=n+1}^m \phi_\nu^\lambda \wedge \phi_\mu^\nu + \Phi_\mu^\lambda \quad \text{with } \phi^\lambda = \omega^\lambda + i\omega^{\lambda*}, \\ \phi_\mu^\lambda &= \omega_\mu^\lambda + i\omega_\mu^{\lambda*}, \quad \Phi_\mu^\lambda = \tilde{\Omega}_\mu^{\tau\lambda} + i\tilde{\Omega}_\mu^{\tau\lambda*}. \end{aligned} \quad (4.5)$$

But $\Phi_\lambda^\lambda = i\tilde{\Omega}_\lambda^{\tau\lambda*}$, and then we have

$$\gamma_1(\tau) = \frac{i}{2\pi} \sum_{\lambda=n+1}^m \Phi_\lambda^\lambda = -\frac{1}{2\pi} \sum_{\lambda=n+1}^m \tilde{\Omega}_\lambda^{\tau\lambda*} = -\frac{1}{2\pi} \sum_{\lambda=n+1}^m \Omega_\lambda^{\lambda*} \quad (4.6)$$

and the proof is complete. $\square$

Let $\vec{n}$ be a vector field normal to the integral submanifold M of the Sasakian manifold $\tilde{M}$. For $X \in \mathcal{X}(M)$, the equality $\alpha_{\vec{n}}(X) = g(F\vec{n}, X)$ defines a 1-form $\alpha_{\vec{n}}$ on M . In [6], this form is used for the study of some remarkable vector fields on M (Legendrian, Hamiltonian, and harmonic variations). Another 1-form on M is defined by $\theta = \sum_{a=1}^n \omega_a^{a*}$, and we can state the following proposition.

PROPOSITION 4.3. *The forms $\alpha_{\tilde{n}}$ and θ have the following properties:*

- (a) $\alpha_{\xi} = 0$ and $\theta = -n\alpha_H$, where H is the mean curvature vector of M ;
 (b) $\alpha_{\tilde{n}}$ is closed if and only if

$$g(\nabla_X^\perp \tilde{n}, FY) = g(\nabla_Y^\perp \tilde{n}, FX) \quad (4.7)$$

for all $X, Y \in \mathcal{X}(M)$;

- (c) the exterior derivative of θ is given by

$$d\theta = \sum_{b,c=1}^n \left(\tilde{S}_{bc}^* - \sum_{\lambda} R_{\lambda bc}^* - \frac{1}{2} \sum_{a=1}^n \tilde{R}_{abc}^{a*} \right) \omega^b \wedge \omega^c, \quad (4.8)$$

where $\tilde{S}$ is the Ricci tensor of $\tilde{M}$.

PROOF. (a) We have the well-known equality

$$\tilde{\nabla}_X e_\alpha = \sum_{\beta=1}^m \omega_\alpha^\beta(X) e_\beta \quad (4.9)$$

for any $X \in \mathcal{X}(\tilde{M})$, and, by using (2.4) and (2.5), we obtain

$$\theta(e_b) = \sum_{a=1}^n g(\tilde{\nabla}_{e_b} e_a, e_{a^*}), \quad b \in \{1, 2, \dots, n\}. \quad (4.10)$$

Taking into account (2.1) and the Gauss formula, we deduce

$$\begin{aligned} \theta(e_b) &= \sum_{a=1}^n g(h(e_a, e_b), e_{a^*}) = \sum_{a=1}^n g(\tilde{\nabla}_{e_a} e_b, F e_a) \\ &= - \sum_{a=1}^n g(F(\tilde{\nabla}_{e_a} e_b), e_a) = - \sum_{a=1}^n g(\tilde{\nabla}_{e_a} e_{b^*}, e_a) = \sum_{a=1}^n g(e_{b^*}, \tilde{\nabla}_{e_a} e_a) \\ &= \sum_{a=1}^n g(e_{b^*}, h(e_a, e_a)) = n g(e_{b^*}, H) = -n g(e_b, FH) = -n \alpha_H(e_b). \end{aligned} \quad (4.11)$$

- (b) From the definition of the 1-form $\alpha_{\tilde{n}}$, we obtain

$$d\alpha_{\tilde{n}}(X, Y) = g(\tilde{\nabla}_X(F\tilde{n}), Y) - g(\tilde{\nabla}_Y(F\tilde{n}), X). \quad (4.12)$$

On the other hand, using (2.1), we have

$$\tilde{\nabla}_X(F\tilde{n}) = F(\tilde{\nabla}_X \tilde{n}) - \eta(\tilde{n})X, \quad (4.13)$$

and then, applying the Weingarten formula in (4.12), it follows that

$$d\alpha_{\tilde{n}}(X, Y) = g(\tilde{\nabla}_Y^\perp \tilde{n}, FX) - g(\tilde{\nabla}_X^\perp \tilde{n}, FY). \quad (4.14)$$

- (c) From (2.4) and (2.5) and taking into account the second structure equation of $\tilde{M}$

$$d\omega_\beta^\alpha = - \sum_{\gamma=1}^{2m} \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \tilde{\Omega}_\beta^\alpha, \quad (4.15)$$

we obtain

$$\begin{aligned} d\theta = & - \sum_{\lambda=n+1}^m \sum_{a=1}^n \omega_{\lambda}^{a*} \wedge \omega_a^{\lambda} - \sum_{b=1}^n \sum_{a=1}^n \omega_b^a \wedge \omega_a^{b*} \\ & - \sum_{\lambda=n+1}^m \sum_{a=1}^n \omega_{\lambda}^a \wedge \omega_a^{\lambda*} + \sum_{a=1}^n \tilde{\Omega}_a^{a*}. \end{aligned} \quad (4.16)$$

Using (2.5) again, the above equality becomes

$$d\theta = 2 \sum_{a=1}^n \sum_{\lambda=n+1}^m \omega_a^{\lambda} \wedge \omega_a^{\lambda*} + \sum_{a=1}^n \tilde{\Omega}_a^{a*}. \quad (4.17)$$

Now, applying the Gauss formula for the submanifold M in (4.9), we have

$$\sum_{b=1}^n [\omega_a^b(X)e_b + \omega_a^{b*}(X)e_{b*}] + \sum_{\lambda=n+1}^m [\omega_a^{\lambda}(X)e_{\lambda} + \omega_a^{\lambda*}(X)e_{\lambda*}] = \nabla_X e_a + h(X, e_a) \quad (4.18)$$

for any $X \in \mathcal{X}(M)$. It follows that

$$\begin{aligned} \omega_a^{\mu}(X) &= g(h(X, e_a), e_{\mu}) \\ &= \sum_{b=1}^n X^b g(h(e_a, e_b), e_{\mu}) = \sum_{b=1}^n X^b h_{ba}^{\mu} = \sum_{b=1}^n h_{ba}^{\mu} \omega^b(X), \end{aligned} \quad (4.19)$$

where h_{ac}^{α} are the components of $h(e_a, e_c)$ with respect to the basis $\mathcal{B}_{\tau}$. Therefore, we have

$$\omega_a^{\alpha} = \sum_{c=1}^n h_{ac}^{\alpha} \omega^c \quad (4.20)$$

for any $\alpha = \lambda$ or $\alpha = \lambda^*$. Finally, from (4.17) and (4.20) we deduce

$$d\theta = \sum_{a,b,c=1}^n (h_{ab}^{\lambda} h_{ac}^{\lambda*} - h_{ac}^{\lambda} h_{ab}^{\lambda*}) \omega^b \wedge \omega^c + \sum_{a=1}^n \tilde{\Omega}_a^{a*}. \quad (4.21)$$

Because $\tilde{M}$ is Sasakian, its curvature tensor $\tilde{R}$ satisfies the following equality [1, page 75]:

$$\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in \mathcal{X}(\tilde{M}), \quad (4.22)$$

hence the Ricci tensor $\tilde{S}$ of $\tilde{M}$ is given by

$$\tilde{S}(X, Y) = \sum_{\alpha=1}^{2m} \tilde{\mathcal{R}}(e_{\alpha}, X, e_{\alpha}, Y) - g(X, Y), \quad (4.23)$$

for all $X, Y \in \mathcal{X}(\tilde{M})$ orthogonal to ξ , where $\tilde{\mathcal{R}}$ is the Riemann-Christoffel curvature tensor field of $\tilde{M}$.

Using (2.3), from the first equality in (2.6), we deduce

$$\sum_{a=1}^n \tilde{\Omega}_a^{a*} = \frac{1}{2} \sum_{a,b,c=1}^n \tilde{R}_{abc}^{a*} \omega^b \wedge \omega^c \quad (4.24)$$

at any point of the submanifold M . Moreover, using the first Bianchi identity relative to $\tilde{M}$, we have

$$\tilde{R}_{abc}^{a*} = \tilde{\mathcal{R}}(e_a^*, e_a, e_b, e_c) = \tilde{\mathcal{R}}(e_c, e_a^*, e_a, e_b) + \tilde{\mathcal{R}}(e_c, e_a, e_b, e_c^*). \quad (4.25)$$

On the other hand, on a Sasakian manifold, the following equalities are true [1, page 93]:

$$\tilde{\mathcal{R}}(FX, FY, FZ, FU) = \tilde{\mathcal{R}}(X, Y, Z, U), \quad (4.26)$$

$$\begin{aligned} \tilde{\mathcal{R}}(FX, Y, Z, U) + \tilde{\mathcal{R}}(X, FY, Z, U) &= d\eta(Y, Z)g(U, X) + d\eta(Z, X)g(Y, U) \\ &\quad + d\eta(U, Y)g(X, Z) + d\eta(X, U)g(Y, Z), \end{aligned} \quad (4.27)$$

for all $X, Y, Z, U \in \mathcal{X}(\tilde{M})$ orthogonal to ξ . But $d\eta(e_a, e_b) = 0$, hence, from (4.27), we deduce

$$\tilde{\mathcal{R}}(e_a^*, e_c, e_a, e_b) + \tilde{\mathcal{R}}(e_a, e_c^*, e_a, e_b) = 0 \quad (4.28)$$

and therefore, using (4.23) and (4.26), from (4.25) we obtain

$$\begin{aligned} \sum_{a=1}^n \tilde{R}_{abc}^{a*} &= \sum_{a=1}^n [\tilde{\mathcal{R}}(e_a, e_b, e_a, e_c^*) + \tilde{\mathcal{R}}(e_a^*, e_b, e_a^*, e_c^*)] \\ &= \tilde{S}(e_b, e_c^*) + \sum_{\lambda=n+1}^m [\tilde{\mathcal{R}}(e_c^*, e_\lambda, e_\lambda, e_b) + \tilde{\mathcal{R}}(e_\lambda, e_b^*, e_\lambda, e_c)]. \end{aligned} \quad (4.29)$$

Now, from (4.27), we give

$$\tilde{\mathcal{R}}(e_a^*, e_\lambda, e_\lambda, e_b) + \tilde{\mathcal{R}}(e_a, e_\lambda^*, e_\lambda, e_b) = 0 \quad (4.30)$$

and then

$$\sum_{a=1}^n \tilde{R}_{abc}^{a*} = \tilde{S}(e_b, e_c^*) + \sum_{\lambda=n+1}^m [\tilde{\mathcal{R}}(e_\lambda^*, e_c, e_\lambda, e_b) + \tilde{\mathcal{R}}(e_\lambda^*, e_b, e_c, e_\lambda)]. \quad (4.31)$$

Applying the second Bianchi identity in the above equality, we obtain

$$\sum_{a=1}^n \tilde{R}_{abc}^{a*} = \tilde{S}(e_b, e_c^*) - \sum_{\lambda=n+1}^m \tilde{\mathcal{R}}(e_\lambda^*, e_\lambda, e_b, e_c); \quad (4.32)$$

and taking into account the Ricci equation

$$\tilde{\mathcal{R}}(e_\lambda^*, e_\lambda, e_b, e_c) = \mathcal{R}^\perp(e_\lambda^*, e_\lambda, e_b, e_c) - g([A_{e_\lambda^*}, A_{e_\lambda}]e_c, e_b), \quad (4.33)$$

we deduce

$$\sum_{a=1}^n \tilde{R}_{abc}^{a*} = \tilde{S}_{bc*} - \sum_{\lambda=n+1}^m R_{\lambda bc}^{\lambda*} + \sum_{\lambda=n+1}^m \sum_{d=1}^n (A_{\lambda c}^d A_{\lambda* d}^b - A_{\lambda* c}^d A_{\lambda d}^b), \quad (4.34)$$

where, by $A_{\lambda a}^b$, we denote the components of the Weingarten operator of M , relative to $\mathcal{B}$. Now, (4.8) follows from (4.17), (4.34), and (2.6). $\square$

THEOREM 4.4. *Let M be an integral submanifold of the Sasakian space form $\tilde{M}(c)$.*

- (a) *The first normal Chern class $[\gamma_1(\tau)]$ of M is zero.*
- (b) *If the mean curvature vector of M is parallel, then its first normal Chern form $\gamma_1(\tau)$ is zero.*

PROOF. (a) Recall that in a Sasakian space form $\tilde{M}(c)$, the curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ have the following expressions (see, e.g., [1, pages, 97–98]):

$$\begin{aligned} \tilde{R}(X, Y)Z &= \frac{c+3}{4} [g(Y, Z)X - g(X, Z)Y] \\ &\quad + \frac{c-1}{4} [\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ &\quad + g(Z, FY)FX - g(Z, FX)FY - 2g(X, FY)FZ], \\ \tilde{S}(X, Y) &= \frac{m(c+3)+c-1}{2} g(X, Y) - \frac{(m+1)(c-1)}{2} \eta(X)\eta(Y), \end{aligned} \quad (4.35)$$

for all $X, Y, Z \in \mathcal{X}(\tilde{M})$. From these equalities, we easily deduce $\tilde{R}_{abc}^{a*} = 0$, $\tilde{S}_{bc*} = 0$, and taking into account (2.8) from Proposition 4.3(c), we obtain

$$d\theta = -2 \sum_{\lambda=n+1}^m \Omega_{\lambda}^{\lambda*}. \quad (4.36)$$

From Theorem 4.2 and from Proposition 4.3(a) and (c), it follows that

$$d\alpha_H = -\frac{1}{n} d\theta = \frac{4\pi}{n} \gamma_1(\tau), \quad (4.37)$$

and then the assertion (a) is proved.

- (b) From (4.36) and using Proposition 4.3(b), we obtain $\gamma_1(\tau) = 0$. $\square$

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